# On the Group $H^3(F(\psi, D)/F)$

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Abstract. Let $F$ be a field of characteristic different from 2, $\psi$ a quadratic $F$-form of dimension $\geq 5$, and $D$ a central simple $F$-algebra of exponent 2. We denote by $F(\psi, D)$ the function field of the product $X_\psi \times X_D$, where $X_\psi$ is the projective quadric determined by $\psi$ and $X_D$ is the Severi-Brauer variety determined by $D$. We compute the relative Galois cohomology group $H^3(F(\psi, D)/F, \mathbb{Z}/2\mathbb{Z})$ under the assumption that the index of $D$ goes down when extending the scalars to $F(\psi)$. Using this, we give a new, shorter proof of the theorem [23, Th. 1] originally proved by A. Laghribi, and a new, shorter, and more elementary proof of the assertion [2, Cor. 9.2] originally proved by H. Esnault, B. Kahn, M. Levine, and E. Viehweg.

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Let $\psi$ be a quadratic form and $D$ be an exponent 2 central simple algebra over a field $F$ (always assumed to be of characteristic not 2). Let $X_\psi$ be the projective quadric determined by $\psi$, $X_D$ the Severi-Brauer variety determined by $D$, and $F(\psi, D)$ the function field of the product $X_\psi \times X_D$.

A computation of the relative Galois cohomology group

$$H^3(F(\psi, D)/F) \stackrel{\text{def}}{=} \ker\left(H^3(F, \mathbb{Z}/2\mathbb{Z}) \to H^3(F(\psi, D), \mathbb{Z}/2\mathbb{Z})\right)$$

plays a crucial role in obtaining the results of [8] and [10] concerning the problem of isotropy of quadratic forms over the function fields of quadrics.

The group $H^3(F(\psi, D)/F)$ is closely related to the Chow group $\mathrm{CH}^2(X_\psi \times X_D)$ of 2-codimensional cycles on the product $X_\psi \times X_D$. The main result of this paper is the following theorem, where both groups are computed assuming $\dim \psi \geq 5$ and the index of $D$ goes down when extending the scalars to the function field of $\psi$:

Theorem 0.1. *Let $D$ be a central simple $F$-algebra of exponent 2. Let $\psi$ be a quadratic form of dimension $\geq 5$. Suppose that $\operatorname{ind} D_{F(\psi)} < \operatorname{ind} D$. Then $\operatorname{Tors}\mathrm{CH}^2(X_\psi \times X_D) = 0$ and $H^3(F(\psi, D)/F) = [D] \cup H^1(F)$.*

A proof is given in §8. The essential part of the proof is Theorem 6.9, dealing with the special case where $D$ is a division algebra of degree 8. This theorem has two applications in the theory of quadratic forms. The first one is a new, shorter proof of the following assertion, originally proved by A. Laghribi ([23, Th. 1]):

Corollary 0.2. *Let* $\phi \in I^2(F)$ *be an* 8*-dimensional quadratic form such that* $\operatorname{ind} C(\phi) = 8$. *Let* $\psi$ *be a quadratic form of dimension* $\geq 5$ *such that* $\phi_{F(\psi)}$ *is isotropic. Then there exists a half-neighbor* $\phi^*$ *of* $\phi$ *such that* $\psi \subset \phi^*$.

The other application we demonstrate is a new, shorter, and more elementary proof of the assertion, originally proved by H. Esnault, B. Kahn, M. Levine, and E. Viehweg ([2, Cor. 9.2]):

Corollary 0.3. *Let* $\phi \in I^2(F)$ *be any quadratic form such that* $\operatorname{ind} C(\phi) \geq 8$. *Let* $A$ *be a central simple* $F$*-algebra Brauer equivalent to* $C(\phi)$ *and let* $F(A)$ *be the function field of the Severi-Brauer variety of* $A$. *Then* $\phi_{F(A)} \notin I^4(F(A))$. *In particular,* $\phi_{F(A)}$ *is not hyperbolic. Moreover, if* $\dim \phi = 8$ *then* $\phi_{F(A)}$ *is anisotropic.*

Our proofs of Corollaries 0.2 and 0.3 are given in §7.

An important part in the proof of Theorem 6.9 is played by the formula of Proposition 4.5, which is in fact applicable to a wide class of algebraic varieties.

A computation of the group $H^3(F(\psi, D)/F)$ in some cases not covered by Theorem 0.1 is given in [8] and [10].

## 1. Terminology, notation, and backgrounds

1.1. Quadratic forms. Mainly, we use notation of [24] and [30]. However there is a slight difference: we denote by $\langle\langle a_1, \ldots, a_n \rangle\rangle$ the $n$-fold Pfister form

$$\langle 1, -a_1 \rangle \otimes \cdots \otimes \langle 1, -a_n \rangle .$$

The set of all $n$-fold Pfister forms over $F$ is denoted by $P_n(F)$; $GP_n(F)$ is the set of forms similar to a form from $P_n(F)$.

We recall that a quadratic form $\psi$ is called a *(Pfister) neighbor* (of a Pfister form $\pi$), if it is similar to a subform in $\pi$ and $\dim \phi > \frac{1}{2} \dim \pi$. Two quadratic forms $\phi$ and $\phi^*$ are *half-neighbors*, if $\dim \phi = \dim \phi^*$ and there exists $s \in F^*$ such that the sum $\phi \perp s\phi^*$ is similar to a Pfister form.

For a quadratic form $\phi$ of dimension $\geq 3$, we denote by $X_\phi$ the projective variety given by the equation $\phi = 0$ and we set $F(\phi) = F(X_\phi)$.

1.2. Generic splitting tower. Let $\gamma$ be a non-hyperbolic quadratic form over $F$. Put $F_0 \overset{\text{def}}{=} F$ and $\gamma_0 \overset{\text{def}}{=} \gamma_{an}$. For $i \geq 1$ let $F_i \overset{\text{def}}{=} F_{i-1}(\gamma_{i-1})$ and $\gamma_i \overset{\text{def}}{=} ((\gamma_{i-1})_{F_i})_{an}$. The smallest $h$ such that $\dim \gamma_h \leq 1$ is called the *height* of $\gamma$. The sequence $F_0, F_1, \ldots, F_h$ is called the *generic splitting tower* of $\gamma$ ([21]). We need some properties of the fields $F_s$:

Lemma 1.3 ([22]). *Let* $M/F$ *be a field extension such that* $\dim(\gamma_M)_{an} = \dim \gamma_s$. *Then the field extension* $MF_s/M$ *is purely transcendental.*

The following proposition is a consequence of the index reduction formula [25].

Proposition 1.4 (see [6, Th. 1.6] or [5, Prop. 2.1]). *Let* $\phi \in I^2(F)$ *be a quadratic form with* $\operatorname{ind}(C(\phi)) \geq 2^r > 1$. *Then there is* $s$ $(0 \leq s \leq h(\phi))$ *such that* $\dim \phi_s = 2r + 2$ *and* $\operatorname{ind} C(\phi_s) = 2^r$.

Corollary 1.5. *Let* $\phi \in I^2(F)$ *be a quadratic form with* $\operatorname{ind}(C(\phi)) \geq 8$. *Then there is* $s$ $(0 \leq s \leq h(\phi))$ *such that* $\dim \phi_s = 8$ *and* $\operatorname{ind} C(\phi_s) = 8$. □

1.6. Central simple algebras. We are working with finite-dimensional associative algebras over a field. Let $D$ be a central simple $F$-algebra. We denote by $X_D$ the Severi-Brauer variety of $D$ and by $F(D)$ the function field $F(X_D)$.

For another central simple $F$-algebra $D'$ and for a quadratic $F$-form $\psi$ of dimension $\geq 3$, we set $F(D',D) \stackrel{\text{def}}{=} F(X_{D'} \times X_D)$ and $F(\psi,D) \stackrel{\text{def}}{=} F(X_\psi \times X_D)$.

1.7. Galois cohomology. By $H^*(F)$ we denote the graded ring of Galois cohomology

$$H^*(F,\mathbb{Z}/2\mathbb{Z}) = H^*(\mathrm{Gal}(F_{\mathrm{sep}}/F),\mathbb{Z}/2\mathbb{Z}).$$

For any field extension $L/F$, we set $H^*(L/F) \stackrel{\text{def}}{=} \ker(H^*(F) \to H^*(L))$.

We use the standard canonical isomorphisms $H^0(F) = \mathbb{Z}/2\mathbb{Z}$, $H^1(F) = F^*/F^{*2}$, and $H^2(F) = \mathrm{Br}_2(F)$.

We also work with the cohomology groups $H^n(F,\mathbb{Q}/\mathbb{Z}(i))$, $i = 0,1,2$ (see e.g. [12] for the definition). For any field extension $L/F$, we set

$$H^*(L/F,\mathbb{Q}/\mathbb{Z}(i)) \stackrel{\text{def}}{=} \ker\left(H^*(F,\mathbb{Q}/\mathbb{Z}(i)) \to H^*(L,\mathbb{Q}/\mathbb{Z}(i))\right).$$

For $n = 1,2,3$, the group $H^n(F)$ is naturally identified with

$$\mathrm{Tors}_2\, H^n(F,\mathbb{Q}/\mathbb{Z}(n-1)).$$

1.8. K-theory and Chow groups. We are mainly working with smooth algebraic varieties over a field, although the smoothness assumption is not always essential.

Let $X$ be a smooth algebraic $F$-variety. The Grothendieck ring of $X$ is denoted by $K(X)$. This ring is supplied with the filtration "by codimension of support" (which respects multiplication); the adjoint graded ring is denoted by $G^*K(X)$. There is a canonical surjective homomorphism of the graded Chow ring $\mathrm{CH}^*(X)$ onto $G^*K(X)$; its kernel consists only of torsion elements and is trivial in the 0-th, 1-st and 2-nd graded components ([32, §9]). In particular we have the following

Lemma 1.9. *The homomorphism* $\mathrm{CH}^i(X) \to G^iK(X)$ *is bijective if at least one of the following conditions holds:*

- $i = 0$, $1$, *or* $2$,
- $\mathrm{CH}^i(X)$ *is torsion-free.*

Let $X$ be a variety over $F$ and $E/F$ be a field extension. We denote by $i_{E/F}$ the restriction homomorphism $K(X) \to K(X_E)$. We use the same notation for the restriction homomorphisms $\mathrm{CH}^*(X) \to \mathrm{CH}^*(X_E)$ and $G^*K(X) \to G^*K(X_E)$. Note that for any projective homogeneous variety $X$, the homomorphism $i_{E/F}\colon K(X) \to K(X_E)$ is injective by [27].

1.10. Other notations. We denote by $\bar{F}$ a separable closure of the field $F$. The order of a set $S$ is denoted by $|S|$ (if $S$ is infinite, we set $|S| \stackrel{\text{def}}{=} \infty$).

## 2. The group $\operatorname{Tors} G^*K(X)$

Lemma 2.1. *Let $X$ be a variety over $F$ and $E/F$ be a field extension such that the homomorphism $i_{E/F} : K(X) \to K(X_E)$ is injective and the factor group $K(X_E)/i_{E/F}(K(X))$ is finite. Then*

$$|\ker(G^*K(X) \to G^*K(X_E)| = \frac{|G^*K(X_E)/i_{E/F}(G^*K(X))|}{|K(X_E)/i_{E/F}(K(X))|}$$

*Proof.* The proof is the same as the proof of [15, Prop. 2]. □

Lemma 2.2. *Let $X$ be a variety, $i$ be an integer, and $E/F$ be a field extension such that the group $G^iK(X_E)$ is torsion-free. Then*

$$\ker(G^iK(X) \to G^iK(X_E)) = \operatorname{Tors} G^iK(X)\ .$$

*Proof.* Since $G^iK(X_E)$ is torsion-free, one has $\ker(G^iK(X) \to G^iK(X_E)) \supset \operatorname{Tors} G^iK(X)$.

To prove the inverse inclusion, let us take an intermediate field $E_0$ such that the extension $E_0/F$ is purely transcendental while the extension $E/E_0$ is algebraic. The specialization argument shows that the homomorphism $G^iK(X) \to G^iK(X_{E_0})$ is injective; the transfer argument shows that $\ker(G^iK(X_{E_0}) \to G^iK(X_E)) \subset \operatorname{Tors} G^iK(X_{E_0})$. Therefore $\ker(G^iK(X) \to G^iK(X_E)) \subset \operatorname{Tors} G^iK(X)$. □

Lemma 2.3. *Let $X$ be a smooth variety, $i$ be an integer, and $E/F$ be a field extension such that the group $\mathrm{CH}^i(X_E)$ is torsion-free. Then*

- $\mathrm{CH}^i(X_E) \simeq G^iK(X_E)$ *(and hence the group $G^iK(X_E)$ is torsion-free),*
- $\mathrm{CH}^i(X_E)/i_{E/F}(\mathrm{CH}^i(X)) \simeq G^iK(X_E)/i_{E/F}(G^iK(X))$.

*Proof.* The first assertion is contained in Lemma 1.9. The canonical homomorphism $\mathrm{CH}^i(X_E) \to G^iK(X_E)$ induces a homomorphism

$$\mathrm{CH}^i(X_E)/i_{E/F}(\mathrm{CH}^i(X)) \to G^iK(X_E)/i_{E/F}(G^iK(X))$$

which is bijective since $\mathrm{CH}^i(X_E) \to G^iK(X_E)$ is bijective and $\mathrm{CH}^i(X) \to G^iK(X)$ is surjective. □

Proposition 2.4. *Suppose that a smooth $F$-variety $X$ and a field extension $E/F$ satisfy the following three conditions:*

- *the homomorphism $i_{E/F} : K(X) \to K(X_E)$ is injective,*
- *the factor group $K(X_E)/i_{E/F}(K(X))$ is finite,*
- *the group $\mathrm{CH}^*(X_E)$ is torsion-free.*

*Then*

$$|\operatorname{Tors} G^*K(X)| = \frac{|G^*K(X_E)/i_{E/F}(G^*K(X))|}{|K(X_E)/i_{E/F}(K(X))|} = \frac{|\mathrm{CH}^*(X_E)/i_{E/F}(\mathrm{CH}^*K(X))|}{|K(X_E)/i_{E/F}(K(X))|}$$

*Proof.* It is an obvious consequence of Lemmas 2.1, 2.2, and 2.3. □

## 3. Auxiliary lemmas

For an Abelian group $A$ we use the notation $\operatorname{rk}(A)=\dim_{\mathbb{Q}}(A\otimes_{\mathbb{Z}}\mathbb{Q})$.

Lemma 3.1. *Let $A_0\subset A$, $B_0\subset B$ be free Abelian groups such that* $\operatorname{rk}A_0=\operatorname{rk}A=r_A$, $\operatorname{rk}B_0=\operatorname{rk}B=r_B$. *Then*

$$\left|\frac{A\otimes_{\mathbb{Z}}B}{A_0\otimes_{\mathbb{Z}}B_0}\right|=\left|\frac{A}{A_0}\right|^{r_B}\cdot\left|\frac{B}{B_0}\right|^{r_A}.$$

*Proof.* One has

$$(A\otimes B)/(A_0\otimes B)\simeq(A/A_0)\otimes B\simeq(A/A_0)\otimes\mathbb{Z}^{r_B}\simeq(A/A_0)^{r_B},$$
$$(A_0\otimes B)/(A_0\otimes B_0)\simeq A_0\otimes(B/B_0)\simeq\mathbb{Z}^{r_A}\otimes(B/B_0)\simeq(B/B_0)^{r_A}.$$

Therefore,

$$\left|\frac{A\otimes B}{A_0\otimes B_0}\right|=\left|\frac{A\otimes B}{A_0\otimes B}\right|\cdot\left|\frac{A_0\otimes B}{A_0\otimes B_0}\right|=\left|\frac{A}{A_0}\right|^{r_B}\cdot\left|\frac{B}{B_0}\right|^{r_A}.$$

□

The following lemma is well-known.

Lemma 3.2. *Let $A$ be an Abelian group with a finite filtration $A=\mathcal{F}^0A\supset\mathcal{F}^1A\supset\cdots\supset\mathcal{F}^kA=0$. Let $B$ be a subgroup of $A$ with the filtration $\mathcal{F}^pB=B\cap\mathcal{F}^pA$. Let $G^*A=\bigoplus_{p\geq 0}\mathcal{F}^pA/\mathcal{F}^{p+1}A$ and $G^*B=\bigoplus_{p\geq 0}\mathcal{F}^pB/\mathcal{F}^{p+1}B$. Then*

- $|A/B|=|G^*A/G^*B|$,
- *if $A$ is a finitely generated group then* $\operatorname{rk}G^*A=\operatorname{rk}A$. □

In the following lemma the term "ring" means a *commutative ring with unit.*

Lemma 3.3. *Let $A$ and $B$ be rings whose additive groups are finitely generated Abelian groups. Let $I$ be a nilpotent ideal of $A$ such that $A/I\simeq\mathbb{Z}$. Let $R$ be a subring of $A\otimes_{\mathbb{Z}}B$ and $A_R$ be a subring of $A$ such that $A_R\otimes 1\subset R$. Then the following inequality holds*

$$\left|\frac{A\otimes_{\mathbb{Z}}B}{R}\right|\leq\left|\frac{A}{A_R}\right|^{r_B}\cdot\left|\frac{A\otimes_{\mathbb{Z}}B}{R+(I\otimes_{\mathbb{Z}}B)}\right|^{r_A}$$

*where $r_A=\operatorname{rk}A$ and $r_B=\operatorname{rk}B$.*

*Proof.* Let us denote by $B_R$ the image of $R$ under the following composition $A\otimes B\to(A/I)\otimes B\simeq\mathbb{Z}\otimes B\simeq B$. Obviously,

$$\left|\frac{A\otimes_{\mathbb{Z}}B}{R+(I\otimes_{\mathbb{Z}}B)}\right|=\left|\frac{B}{B_R}\right|.$$

For any $p\geq 0$ we set $\mathcal{F}^pA=\{a\in A\,|\,\exists m\in\mathbb{N}\text{ such that }ma\in I^p\}$. Clearly, $\operatorname{Tors}(A/\mathcal{F}^pA)=0$, and so $A/\mathcal{F}^p$ is a free Abelian group. Therefore all factor groups $\mathcal{F}^pA/\mathcal{F}^{p+1}A$ $(p=0,1,\dots)$ are free Abelian. Since $A/I\simeq\mathbb{Z}$, it follows that $\mathcal{F}^1A=I$. Thus $A/\mathcal{F}^1A\simeq\mathbb{Z}$. Since $I$ is a nilpotent ideal of $A$, there exists $k$ such that $I^k=0$. Then $\mathcal{F}^kA=0$. Thus the filtration $A=\mathcal{F}^0A\supset\mathcal{F}^1A\supset\mathcal{F}^2A\supset\dots$ is finite and results of Lemma 3.2 can be applied.

Let $\mathcal{F}^pA_R\overset{\text{def}}{=}R\cap\mathcal{F}^pA$, $\mathcal{F}^p(A\otimes B)\overset{\text{def}}{=}\operatorname{im}(\mathcal{F}^pA\otimes B\to A\otimes B)$, and $\mathcal{F}^pR\overset{\text{def}}{=}R\cap\mathcal{F}^p(A\otimes B)$. If $K$ is one of the rings $A$, $A_R$, $A\otimes B$, or $R$, we set $G^pK\overset{\text{def}}{=}\mathcal{F}^pK/\mathcal{F}^{p+1}K$ and $G^*K\overset{\text{def}}{=}\bigoplus_{p\geq 0}\mathcal{F}^pK/\mathcal{F}^{p+1}K$. Obviously, $\mathcal{F}^pK\cdot\mathcal{F}^qK\subset\mathcal{F}^{p+q}K$ for all $p$ and $q$.

Therefore, $K = \mathcal{F}^0 K \supset \mathcal{F}^1 K \supset \cdots \supset \mathcal{F}^p K \supset \dots$ is a ring filtration. Hence, the adjoint graded group $G^*K$ has a graded ring structure. Since the additive group of $B$ is free, we have a natural ring isomorphism $G^*A \otimes B \simeq G^*(A \otimes B)$.

Since $A_R \otimes 1 \subset R$, we have $G^*A_R \otimes 1 \subset G^*R$. Clearly $G^0(A \otimes B) = (A/I) \otimes B$, and $G^0R$ coincides with the image of the composition $R \to A \otimes B \to (A/I) \otimes B$. By definition of $B_R$, one has $G^0R = 1_{G^*A} \otimes B_R$ (here $1_{G^*A}$ denotes the unit of the ring $G^*A$). Therefore $1_{G^*A} \otimes B_R \subset G^*R$. Since $G^*A_R \otimes 1 \subset G^*R$, $1_{G^*A} \otimes B_R \subset G^*R$, and $G^*R$ is a subring of $G^*A \otimes B$, we have $G^*A_R \otimes B_R \subset G^*R$. Therefore $|G^*(A \otimes B)/G^*R| \le |(G^*A \otimes B)/(G^*A_R \otimes B_R)|$. Applying Lemmas 3.1 and 3.2, we have

$$\left|\frac{A \otimes B}{R}\right| = \left|\frac{G^*(A \otimes B)}{G^*R}\right| \le \left|\frac{G^*A \otimes B}{G^*A_R \otimes B_R}\right| = \left|\frac{G^*A}{G^*A_R}\right|^{r_B} \cdot \left|\frac{B}{B_R}\right|^{r_A} =$$
$$= \left|\frac{A}{A_R}\right|^{r_B} \cdot \left|\frac{B}{B_R}\right|^{r_A} = \left|\frac{A}{A_R}\right|^{r_B} \cdot \left|\frac{A \otimes_{\mathbb{Z}} B}{R + (I \otimes_{\mathbb{Z}} B)}\right|^{r_A}.$$

□

## 4. On the group $\mathrm{CH}^*(X \times Y)$

Let $X$ be a smooth variety. We denote by $\mathcal{F}^p\mathrm{CH}^*(X)$ the group

$$\bigoplus_{i \ge p} \mathrm{CH}^i(X)\,.$$

Let $Y$ be another smooth variety. For a subgroup $A$ of $\mathrm{CH}^*(X)$ and a subgroup $B$ of $\mathrm{CH}^*(Y)$, we denote by $A \boxtimes B$ the image of the composition $A \otimes B \to \mathrm{CH}^*(X) \otimes \mathrm{CH}^*(Y) \to \mathrm{CH}^*(X \times Y)$.

The following assertion is evident (see also [20, §3] or [11]).

Proposition 4.1. *Let $X$ and $Y$ be smooth varieties over $F$. Then*

- *the natural homomorphism $\mathrm{CH}^*(X \times Y) \to \mathrm{CH}^*(Y_{F(X)})$ is surjective,*
- *the kernel of the homomorphism $\mathrm{CH}^*(X \times Y) \to \mathrm{CH}^*(Y_{F(X)})$ contains the group $\mathcal{F}^1\mathrm{CH}^*(X) \boxtimes \mathrm{CH}^*(Y)$.*

□

Corollary 4.2. *If the natural homomorphism $\mathrm{CH}^*(X) \otimes \mathrm{CH}^*(Y) \to \mathrm{CH}^*(X \times Y)$ is bijective and $\mathrm{CH}^*(Y)$ is torsion-free, then the homomorphism $\mathrm{CH}^*(X \times Y) \to \mathrm{CH}^*(Y_{F(X)})$ induces an isomorphism*

$$\frac{\mathrm{CH}^*(X \times Y)}{\mathcal{F}^1\mathrm{CH}^*(X) \boxtimes \mathrm{CH}^*(Y)} \to \mathrm{CH}^*(Y_{F(X)}).$$

*Proof.* Since $\mathrm{CH}^*(X) \otimes \mathrm{CH}^*(Y) \simeq \mathrm{CH}^*(X \times Y)$ and $\mathrm{CH}^*(X)/\mathcal{F}^1\mathrm{CH}^*(X) \simeq \mathrm{CH}^0(X)$, the factor group $\mathrm{CH}^*(X \times Y)/(\mathcal{F}^1\mathrm{CH}^*(X) \boxtimes \mathrm{CH}^*(Y))$ is isomorphic to $\mathrm{CH}^0(X) \otimes_{\mathbb{Z}} \mathrm{CH}^*(Y) \simeq \mathbb{Z} \otimes_{\mathbb{Z}} \mathrm{CH}^*(Y) \simeq \mathrm{CH}^*(Y)$. Thus, it is sufficient to prove that the homomorphism $\mathrm{CH}^*(Y) \to \mathrm{CH}^*(Y_{F(X)})$ is injective. This is obvious since $\mathrm{CH}^*(Y)$ is torsion-free. □

Corollary 4.3. *Let $X$ and $Y$ be smooth varieties and $E/F$ be a field extension such that the natural homomorphism $\mathrm{CH}^*(X_E) \otimes \mathrm{CH}^*(Y_E) \to \mathrm{CH}^*(X_E \times Y_E)$ is bijective and $\mathrm{CH}^*(Y_E)$ is torsion-free. Then there exists an isomorphism*

$$\frac{\mathrm{CH}^*(X_E \times Y_E)}{i_{E/F}(\mathrm{CH}^*(X \times Y)) + \mathcal{F}^1\mathrm{CH}^*(X_E) \boxtimes \mathrm{CH}^*(Y_E)} \simeq \frac{\mathrm{CH}^*(Y_{E(X)})}{i_{E(X)/F(X)}(\mathrm{CH}^*(Y_{F(X)}))}$$

*Proof.* Obvious in view of Corollary 4.2. □

Remark 4.4. It was noticed by the referee that the conditions of Corollary 4.3 (which appear also in Proposition 4.5) hold, if the variety $Y_E$ possess a cellular decomposition (see e.g. [13, Def. 3.2] for the definition of cellular decomposition). In the case of complete varieties $X$ and $Y$, this statement follows e.g. from [19, Th. 6.5]. In the present paper, we shall apply Corollary 4.3 only to the case where $Y_E$ is isomorphic to a projective space.

Proposition 4.5. *Let $X$ and $Y$ be smooth varieties over $F$ and $E/F$ be a field extension such that the following conditions hold*

- *$\mathrm{CH}^*(X_E)$ is a free Abelian group of rank $r_X$,*
- *$\mathrm{CH}^*(Y_E)$ is a free Abelian group of rank $r_Y$,*
- *the canonical homomorphism $\mathrm{CH}^*(X_E) \otimes_{\mathbb{Z}} \mathrm{CH}^*(Y_E) \to \mathrm{CH}^*(X_E \times Y_E)$ is an isomorphism.*

*Then*

$$\left|\frac{\mathrm{CH}^*(X_E \times Y_E)}{i_{E/F}(\mathrm{CH}^*(X \times Y))}\right| \le \left|\frac{\mathrm{CH}^*(X_E)}{i_{E/F}(\mathrm{CH}^*(X))}\right|^{r_Y} \cdot \left|\frac{\mathrm{CH}^*(Y_{E(X)})}{i_{E(X)/F(X)}(\mathrm{CH}^*(Y_{F(X)}))}\right|^{r_X} .$$

*Proof.* Let $A = \mathrm{CH}^*(X_E)$, $A_R = i_{E/F}(\mathrm{CH}^*(X))$ and $I = \bigoplus_{p>0} \mathrm{CH}^p(X_E) = \mathcal{F}^1\mathrm{CH}^*(X_E)$. Let $B = \mathrm{CH}^*(Y_E)$. By our assumption, we have $\mathrm{CH}^*(X_E \times Y_E) \simeq A \otimes_{\mathbb{Z}} B$. We denote by $R$ the image of the composition $\mathrm{CH}^*(X \times Y) \to \mathrm{CH}^*(X_E \otimes Y_E) \simeq A \otimes_{\mathbb{Z}} B$. Clearly, all conditions of Lemma 3.3 hold. Moreover,

$$\left|\frac{\mathrm{CH}^*(X_E \times Y_E)}{i_{E/F}(\mathrm{CH}^*(X \times Y))}\right| = \left|\frac{A \otimes_{\mathbb{Z}} B}{R}\right| \quad \text{and} \quad \left|\frac{\mathrm{CH}^*(X_E)}{i_{E/F}(\mathrm{CH}^*(X))}\right| = \left|\frac{A}{A_R}\right|.$$

By Corollary 4.3 we have

$$\left|\frac{A \otimes_{\mathbb{Z}} B}{R + (I \otimes_{\mathbb{Z}} B)}\right| = \left|\frac{\mathrm{CH}^*(Y_{E(X)})}{i_{E(X)/F(X)}(\mathrm{CH}^*(Y_{F(X)}))}\right|.$$

To complete the prove it suffices to apply Lemma 3.3. □

## 5. The group $\operatorname{Tors}\mathrm{CH}^2(X_\psi \times X_D)$

The aim of this section is Corollary 5.6.

Proposition 5.1 (see [14, §2.1]). *Let $\psi$ be a $(2n+1)$-dimensional quadratic form over a separably closed field. Set $X \overset{\text{def}}{=} X_\psi$ and $d \overset{\text{def}}{=} \dim X = 2n-1$. Then for all $0 \le p \le d$ the group $\mathrm{CH}^p(X)$ is canonically isomorphic to $\mathbb{Z}$ (for other $p$ the group $\mathrm{CH}^p(X)$ is trivial). Moreover,*

- *if $0 \le p < n$, then $\mathrm{CH}^p(X) = \mathbb{Z} \cdot h^p$, where $h \in \mathrm{CH}^1(X)$ denotes the class of a hyperplane section of $X$;*

- *if* $n \le p \le d$, *then* $\mathrm{CH}^p(X) = \mathbb{Z} \cdot l_{d-p}$, *where* $l_{d-p}$ *denotes the class of a linear subspace in* $X$ *of dimension* $d-p$, *besides* $2l_{d-p} = h^p$.

Corollary 5.2. *Let* $\psi$ *be a* $(2n+1)$*-dimensional quadratic form over* $F$ *and let* $X = X_\psi$. *Then*

- $\mathrm{CH}^*(X_{\bar F})$ *is a free Abelian group of rank* $2n$,
- *if* $0 \le p < n$ *then* $|\mathrm{CH}^p(X_{\bar F})/i_{\bar F/F}(\mathrm{CH}^p(X))| = 1$,
- *if* $n \le p \le 2n-1$ *then* $|\mathrm{CH}^p(X_{\bar F})/i_{\bar F/F}(\mathrm{CH}^p(X))| \le 2$,
- $|\mathrm{CH}^*(X_{\bar F})/i_{\bar F/F}(\mathrm{CH}^*(X))| \le 2^n$. □

Proposition 5.3. *Let* $D$ *be a central simple* $F$*-algebra of exponent* $2$ *and of degree* $8$. *Let* $E/L/F$ *be field extensions such that* $\operatorname{ind} D_L = 4$ *and* $\operatorname{ind} D_E = 1$. *Let* $Y = \mathrm{SB}(D)$. *For any* $0 \le p \le \dim Y = 7$, *the group* $\mathrm{CH}^p(Y_E)$ *is canonically isomorphic to* $\mathbb{Z}$. *Moreover, the image of the homomorphism* $i_{E/L} : \mathrm{CH}^p(Y_L) \to \mathrm{CH}^p(Y_E) \simeq \mathbb{Z}$ *contains* $1$ *if* $p = 0, 4$; $2$ *if* $p = 1, 2, 5, 6$; $4$ *if* $p = 3, 7$.

*Proof.* Since $\deg D = 8$ and $\operatorname{ind} D_E = 1$, $Y_E$ is isomorphic to $\mathbb{P}^7_E$. Hence, the group $\mathrm{CH}^p(Y_E) \cong \mathrm{CH}^p(\mathbb{P}^7_E)$ (where $p = 0, \dots, 7$) is generated by the class $h^p$ of a linear subspace ([4]).

The rest part of the proposition is contained in [16, Th.]. For the reader's convenience, we also give a direct construction of the elements required. The class of $Y_L$ itself gives $1 \in i_{E/L}(\mathrm{CH}^0(Y_L))$. Let $\xi$ be the tautological line bundle on the projective space $\mathbb{P}^7_E \simeq Y_E$. Since $\exp D = 2$, the bundle $\xi^{\otimes 2}$ is defined over $F$ and, in particular, over $L$. Its first Chern class gives $2 \in i_{E/L}(\mathrm{CH}^1(Y_L))$. Since $\operatorname{ind} D_L = 4$, the bundle $\xi^{\oplus 4}$ is defined over $L$. Its second Chern class gives $6 \in i_{E/L}(\mathrm{CH}^2(Y_L))$.[1] Thus $2 \in i_{E/L}(\mathrm{CH}^2(Y_L))$. The third Chern class of $\xi^{\oplus 4}$ gives $4 \in i_{E/L}(\mathrm{CH}^3(Y_L))$. The fourth Chern class of $\xi^{\oplus 4}$ gives $1 \in i_{E/L}(\mathrm{CH}^4(Y_L))$. Finally, taking the product of the cycles constructed in codimensions 1, 2, and 3 with the cycle of codimension 4, one gets the cycles of codimensions 5, 6, and 7 required. □

Corollary 5.4. *Under the condition of Proposition 5.3, we have*

$$|\mathrm{CH}^*(Y_E)/i_{E/L}(\mathrm{CH}^*(Y_L))| \le 256 .$$

*Proof.* $\prod_{p=0}^{7} |\mathrm{CH}^p(Y_E)/i_{E/L}(\mathrm{CH}^p(Y_L))| \le 1 \cdot 2 \cdot 2 \cdot 4 \cdot 1 \cdot 2 \cdot 2 \cdot 4 = 256$ . □

Proposition 5.5. *Let* $D$ *be a central division* $F$*-algebra of degree* $8$ *and exponent* $2$. *Let* $\psi$ *be a* $5$*-dimensional quadratic* $F$*-form. Suppose that* $D_{F(\psi)}$ *is not a skewfield. Then* $\operatorname{Tors} G^*K(X_\psi \times X_D) = 0$.

*Proof.* Let $X = X_\psi$ and $Y = X_D$. Corollary 5.2 shows that $\mathrm{CH}^*(X_{\bar F})$ is a free abelian group of rank $r_X = 4$ and $|\mathrm{CH}^*(X_{\bar F})/i_{\bar F/F}(\mathrm{CH}^*(X))| \le 2^2 = 4$.

Since $D$ is a division algebra of degree 8 and $D_{F(\psi)}$ is not division algebra, it follows that $\operatorname{ind} D_{F(X)} = 4$. Applying Corollary 5.4 to the case $L = F(X)$, $E = \bar F(X)$, we have $|\mathrm{CH}^*(Y_{\bar F(X)})/i_{\bar F(X)/F(X)}(\mathrm{CH}^*(Y_{F(X)}))| \le 256$.

[1]In fact, it is enough only to know that the *Grothendieck classes* of the bundles $\xi^{\otimes 2}$ and $\xi^{\oplus 4}$ are in $K(Y_L)$ what can be also seen from the computation of the K-theory.

Since $Y_{\bar F} = \mathrm{SB}(D_{\bar F}) \simeq \mathbb{P}^7_{\bar F}$, the group $\mathrm{CH}^*(Y_{\bar F})$ is a free Abelian of rank $r_Y = 8$ and $\mathrm{CH}^*(X_{\bar F}) \otimes \mathrm{CH}^*(Y_{\bar F}) \simeq \mathrm{CH}^*(X_{\bar F} \times Y_{\bar F})$ (see [3, Prop. 14.6.5]). Thus all conditions of Proposition 4.5 hold for $X$, $Y$, $E = \bar F$ and we have

$$\left| \frac{\mathrm{CH}^*(X_{\bar F} \times Y_{\bar F})}{i_{\bar F/F}(\mathrm{CH}^*(X \times Y))} \right| \leq 4^8 \cdot 256^4 = 2^{48}.$$

Using [29, Th. 4.1 of §8] and [33, Th. 9.1], we get a natural (with respect to extensions of $F$) isomorphism

$$K(X \times Y) \simeq K\big((F^{\times 3} \times C) \otimes_F (F^{\times 4} \times D^{\times 4})\big) \simeq \\ \simeq K\big(F^{\times 12} \times C^{\times 4} \times D^{\times 12} \times (C \otimes_F D)^{\times 4}\big)$$

where $C \stackrel{\mathrm{def}}{=} C_0(\psi)$ is the even Clifford algebra of $\psi$. Note that $C$ is a central simple $F$-algebra of the degree $2^2$. Since $D_{F(\psi)}$ is not a skew field, [25, Th. 1] states that $D \simeq C \otimes_F B$ with some central division $F$-algebra $B$. Therefore, $\operatorname{ind} C = \deg C = 2^2$ and $\operatorname{ind} C \otimes D = \operatorname{ind} B = \deg B = 2$. Hence

$$\left| \frac{K(X_{\bar F} \times Y_{\bar F})}{i_{\bar F/F}(K(X \times Y))} \right| = (\operatorname{ind} C)^4 \cdot (\operatorname{ind} D)^{12} \cdot (\operatorname{ind} C \otimes D)^4 = 2^{2\cdot 4 + 3\cdot 12 + 1\cdot 4} = 2^{48} \ .$$

Applying Proposition 2.4 to the variety $X \times Y$ and $E = \bar F$, we have

$$|\operatorname{Tors} G^* K(X \times Y)| = \frac{|\mathrm{CH}^*(X_{\bar F} \times Y_{\bar F})/i_{\bar F/F}(\mathrm{CH}^*(X \times Y))|}{|K(X_{\bar F} \times Y_{\bar F})/i_{\bar F/F}(K(X \times Y))|} \leq \frac{2^{48}}{2^{48}} = 1 \ .$$

Therefore, $\operatorname{Tors} G^* K(X \times Y) = 0$. □

Applying Lemma 1.9 we get the following

Corollary 5.6. *Under the condition of Proposition 5.5, the group* $\mathrm{CH}^2(X_\psi \times X_D)$ *is torsion-free.* □

## 6. A special case of Theorem 0.1

In this section we prove Theorem 0.1 in the special case where $D$ is a division algebra of degree 8.

Proposition 6.1 ([1, Satz 5.6]). *Let $\psi$ be a quadratic $F$-form of dimension $\geq 5$. The group $H^3(F(\psi)/F)$ is non-trivial iff $\psi$ is a neighbor of an anisotropic 3-Pfister form.*

Proposition 6.2 (see [28, Prop. 4.1 and Rem. 4.1]). *Let $D$ be a central division $F$-algebra of exponent 2. Suppose that $D$ is decomposable (in the tensor product of two proper subalgebras). Then $H^3(F(D)/F) = [D] \cup H^1(F)$.*

Proposition 6.3. *If $D$ and $D'$ are Brauer equivalent central simple $F$-algebras, then the function fields $F(D)$ and $F(D')$ are stably equivalent.*[2]

[2] Two field extensions $E/F$ and $E'/F$ are called *stably equivalent*, if some finitely generated purely transcendental extension of $E$ is isomorphic (over $F$) to some finitely generated purely transcendental extension of $E'$.

*Proof.* Since the algebras $D_{F(D')}$ and $D'_{F(D)}$ are split, the field extensions

$$F(D,D')/F(D') \quad \text{and} \quad F(D,D')/F(D)$$

are purely transcendental. Therefore each of the field extensions $F(D)/F$ and $F(D')/F$ is stably equivalent to the extension $F(D,D')/F$. □

Corollary 6.4. *Fix a quadratic $F$-form $\psi$ and integers $i,j \in \mathbb{Z}$. For any central simple $F$-algebra $D$, the groups $H^i(F(D)/F)$, $H^i(F(D)/F,\mathbb{Q}/\mathbb{Z}(j))$, $H^i(F(\psi,D)/F)$, $H^i(F(\psi,D)/F,\mathbb{Q}/\mathbb{Z}(j))$ only depend on the Brauer class of $D$.* □

Proposition 6.5. *Let $D$ be a central simple $F$-algebra of exponent 2 and let $\psi$ be a quadratic $F$-form. The group $H^3(F(\psi,D)/F,\mathbb{Q}/\mathbb{Z}(2))$ is annihilated by 2.*

*Proof.* Let $\psi_0$ be a 3-dimensional subform of $\psi$. Clearly,

$$H^3(F(\psi,D)/F,\mathbb{Q}/\mathbb{Z}(2)) \subset H^3(F(\psi_0,D)/F,\mathbb{Q}/\mathbb{Z}(2)) .$$

Therefore, it suffices to show that the latter cohomology group is annihilated by 2. Replacing $\psi_0$ by the quaternion algebra $C_0(\psi_0)$, we come to a statement covered by [7, Lemma A.8]. □

Corollary 6.6. *In the conditions of Proposition 6.5, one has*

$$H^3(F(\psi,D)/F,\mathbb{Q}/\mathbb{Z}(2)) = H^3(F(\psi,D)/F) .$$ □

Proposition 6.7. *Let $D$ be a central simple $F$-algebra of exponent 2 and let $\psi$ be a quadratic $F$-form of dimension $\geq 3$. Suppose that $\operatorname{ind} D_{F(\psi)} < \operatorname{ind} D$. Then $\psi$ is not a 3-Pfister neighbor and there is an isomorphism*

$$\frac{H^3(F(\psi,D)/F)}{H^3(F(\psi)/F) + [D]\cup H^1(F)} \simeq \operatorname{Tors}\operatorname{CH}^2(X_\psi \times X_D) .$$

*Proof.* By [9, Prop. 2.2], there is an isomorphism

$$\frac{H^3(F(\psi,D)/F,\mathbb{Q}/\mathbb{Z}(2))}{H^3(F(\psi)/F,\mathbb{Q}/\mathbb{Z}(2)) + H^3(F(D)/F,\mathbb{Q}/\mathbb{Z}(2))} \simeq \\ \simeq \frac{\operatorname{Tors}\operatorname{CH}^2(X_\psi \times X_D)}{pr_\psi^* \operatorname{Tors}\operatorname{CH}^2(X_\psi) + pr_D^* \operatorname{Tors}\operatorname{CH}^2(X_D)} .$$

By Corollary 6.6, we have $H^3(F(\psi,D)/F,\mathbb{Q}/\mathbb{Z}(2)) = H^3(F(\psi,D)/F)$; by [9, Lemma 2.8], we have $H^3(F(\psi)/F,\mathbb{Q}/\mathbb{Z}(2)) = H^3(F(\psi)/F)$; and by [7, Lemma A.8], we have $H^3(F(D)/F,\mathbb{Q}/\mathbb{Z}(2)) = H^3(F(D)/F)$.

Let $D'$ be a division algebra Brauer equivalent to $D$. By Corollary 6.4, we have $H^3(F(D)/F) = H^3(F(D')/F)$; by [18, Prop. 1.1], we have $\operatorname{Tors}\operatorname{CH}^2(X_D) \simeq \operatorname{Tors}\operatorname{CH}^2(X_{D'})$. Since $D'_{F(\psi)}$ is no more a skew field, there is a homomorphism of $F$-algebras $C_0(\psi) \to D'$ ([34, Th. 1], see also [26, Th. 2]). Although the algebra $C_0(\psi)$ is not always central simple, it always contains a non-trivial subalgebra central simple over $F$. Therefore, $D'$ is decomposable, what implies $H^3(F(D')/F) = [D]\cup H^1(F)$ (Proposition 6.2) and $\operatorname{Tors}\operatorname{CH}^2(X_{D'}) = 0$ ([17, Prop. 5.3]). Finally, the existence of a homomorphism $C_0(\psi) \to D'$ implies that $\psi$ is not a 3-Pfister neighbor; therefore $\operatorname{Tors}\operatorname{CH}^2(X_\psi) = 0$ ([14, Th. 6.1]). □

COROLLARY 6.8. *Let $D$ be a central division $F$-algebra of degree 8 and exponent 2. Let $\psi$ be a 5-dimensional quadratic $F$-form. Suppose that $D_{F(\psi)}$ is not a skew field. Then $H^3(F(\psi, D)/F) = [D] \cup H^1(F)$.*

*Proof.* It is a direct consequence of Proposition 6.7, Corollary 5.6, and Proposition 6.1. □

THEOREM 6.9. *Theorem 0.1 is true if $D$ is a division algebra of degree 8.*

*Proof.* Let $\psi_0$ be a 5-dimensional subform of $\psi$. Applying Corollary 6.8, we have $[D] \cup H^1(F) \subset H^3(F(\psi, D)/F) \subset H^3(F(\psi_0, D)/F) = [D] \cup H^1(F)$. Hence $H^3(F(\psi, D)/F) = [D] \cup H^1(F)$.

The assertion on $\operatorname{Tors} \operatorname{CH}^2(X_\psi \times X_D)$ is Corollary 5.6. □

COROLLARY 6.10. *Let $\phi \in I^2(F)$ be a 8-dimensional quadratic form such that $\operatorname{ind} C(\phi) = 8$. Let $D$ be a degree 8 central simple algebra such that $c(\phi) = [D]$. Let $\psi$ be a quadratic form of dimension $\geq 5$ such that $\phi_{F(\psi)}$ is isotropic. Then*

1) *$D$ is a division algebra;*
2) *$D_{F(\psi)}$ is not a division algebra;*
3) *$H^3(F(\psi, D)/F) = [D] \cup H^1(F)$.* □

## 7. PROOF OF COROLLARIES 0.2 AND 0.3

We need several lemmas.

LEMMA 7.1. *Let $\phi \in I^2(F)$ be a 8-dimensional quadratic form and let $D$ be an algebra such that $c(\phi) = [D]$. Then $\phi_{F(D)} \in GP_3(F(D))$.*

*Proof.* We have $c(\phi_{F(D)}) = c(\phi)_{F(D)} = [D_{F(D)}] = 0$. Hence $\phi_{F(D)} \in I^3(F(D))$. Since $\dim \phi = 8$, we are done by the Arason-Pfister Hauptsatz. □

LEMMA 7.2. *Let $\phi, \phi^* \in I^2(F)$ be 8-dimensional quadratic forms such that $c(\phi) = c(\phi^*) = [D]$, where $D$ is a triquaternion division algebra.*[3] *Suppose that there is a quadratic form $\psi$ of dimension $\geq 5$ such that the forms $\phi_{F(\psi,D)}$ and $\phi^*_{F(\psi,D)}$ are isotropic. Then $\phi$ and $\phi^*$ are half-neighbors.*

*Proof.* Lemma 7.1 implies that $\phi_{F(\psi,D)}, \phi^*_{F(\psi,D)} \in GP_3(F(\psi, D))$. By the assumption of the lemma, $\phi_{F(\psi,D)}$ and $\phi^*_{F(\psi,D)}$ are isotropic. Hence $\phi_{F(\psi,D)}$ and $\phi^*_{F(\psi,D)}$ are hyperbolic. Thus $\phi, \phi^* \in W(F(\psi, D)/F)$.

Let $\tau = \phi \perp \phi^*$. Clearly $\tau \in W(F(\psi, D)/F)$. Since $c(\tau) = c(\phi) + c(\phi^*) = [D] + [D] = 0$, we have $\tau \in I^3(F)$. Thus $e^3(\tau) \in H^3(F(\psi, D)/F)$. It follows from Corollary 6.10 that $e^3(\tau) \in [D] \cup H^1(F)$. Hence there exists $s \in F^*$ such that $e^3(\tau) = [D] \cup (s)$. We have $e^3(\tau) = [D] \cup (s) = c(\phi) \cup (s) = e^3(\phi \langle\langle s \rangle\rangle)$. Since $\ker(e^3 : I^3(F) \to H^3(F)) = I^4(F)$, we have $\tau \equiv \phi \langle\langle s \rangle\rangle \pmod{I^4(F)}$. Therefore $\phi + \phi^* = \tau \equiv \phi \langle\langle s \rangle\rangle = \phi - s\phi \pmod{I^4(F)}$. Hence $\phi^* + s\phi \in I^4(F)$. Hence $\phi$ and $\phi^*$ are half-neighbors. □

The following statement was pointed out by Laghribi ([23]) as an easy consequence of the index reduction formula [25].

[3] An $F$-algebra is called *triquaternion*, if it is isomorphic to a tensor product of three quaternion $F$-algebras.

Lemma 7.3. *Let $\psi$ be a quadratic form of dimension $\geq 5$ and $D$ be a division triquaternion algebra. Suppose that $D_{F(\psi)}$ is not a division algebra. Then there exists an 8-dimensional quadratic form $\phi^* \in I^2(F)$ such that $\psi \subset \phi^*$ and $c(\phi^*) = [D]$.* □

*Proof of Corollary 0.2.* Let $D$ be triquaternion algebra such that $c(\phi) = [D]$. Since $\operatorname{ind} C(\phi) = 8$, it follows that $D$ is a division algebra. Since $\phi_{F(\psi)}$ is isotropic, $D_{F(\psi)}$ is not a division algebra. It follows from Lemma 7.3 that there exists an 8-dimensional quadratic form $\phi^* \in I^2(F)$ such that $\psi \subset \phi^*$ and $c(\phi^*) = [D]$. Obviously, all conditions of Lemma 7.2 hold. Hence $\phi$ and $\phi^*$ are half-neighbors. □

Lemma 7.4. *Let $D$ be a division triquaternion algebra over $F$. Then there exist a field extension $E/F$ and an 8-dimensional quadratic form $\phi^* \in I^2(E)$ with the following properties:*

(i) *$D_E$ is a division algebra,*
(ii) *$c(\phi^*) = [D_E]$,*
(iii) *$\phi^*_{E(D)}$ is anisotropic.*

*Proof.* Let $\phi \in I^2(F)$ be an arbitrary $F$-form such that $c(\phi) = [D]$. Let $K = F(X,Y,Z)$ and $\gamma = \phi_K \perp \langle\langle X,Y,Z\rangle\rangle$ be a $K$-form. Let $K = K_0, K_1, \ldots, K_h$; $\gamma_0, \gamma_1, \ldots, \gamma_h$ be a generic splitting tower of $\gamma$.

Since $\gamma \equiv \phi_K \pmod{I^3(K)}$, we have $c(\gamma) = c(\phi_K) = [D_K]$. Since $K/F$ is purely transcendental, $\operatorname{ind} D_K = \operatorname{ind} D = 8$. Hence $\operatorname{ind} C(\gamma) = 8$. It follows from Corollary 1.5 that there exists $s$ such that $\dim \gamma_s = 8$ and $\operatorname{ind} C(\gamma_s) = 8$. We set $E = E_s$, $\phi^* = \gamma_s$.

We claim that the condition (i)–(iii) of the lemma hold. Since $c(\phi^*) = c(\gamma_E) = c(\phi_E) = [D_E]$, condition (ii) holds. Since $[D_E] = c(\phi^*) = c(\gamma_s)$, we have $\operatorname{ind} D_E = \operatorname{ind} C(\gamma_s) = 8$ and thus condition (i) holds.

Now we only need to verify that (iii) holds. Let $M_0/F$ be an arbitrary field extension such that $\phi_{M_0}$ is hyperbolic. Let $M = M_0(X,Y,Z)$. We have $\gamma_M = \phi_M \perp \langle\langle X,Y,Z\rangle\rangle_M$. Clearly $\langle\langle X,Y,Z\rangle\rangle$ is anisotropic over $M$. Since $\phi_M$ is hyperbolic, we have $(\gamma_M)_{an} = \langle\langle X,Y,Z\rangle\rangle_M$ and hence $\dim(\gamma_M)_{an} = 8$. Therefore $\dim(\gamma_M)_{an} = \dim \gamma_s$. By Lemma 1.3, we see that the field extension $ME/M = MK_s/M$ is purely transcendental. Hence $\dim(\gamma_{ME})_{an} = \dim(\gamma_M)_{an} = 8$. Since $(\phi^*_{ME})_{an} = (\gamma_{ME})_{an}$, we see that $\phi^*_{ME}$ is anisotropic. Since $\phi_M$ is hyperbolic, it follows that $[D_M] = c(\phi_M) = 0$. Hence $[D_{ME}] = 0$ and therefore the field extension $ME(D)/ME$ is purely transcendental. Hence $\phi^*_{ME(D)}$ is anisotropic. Therefore $\phi^*_{E(D)}$ is anisotropic. □

Lemma 7.5. *Let $\phi, \phi^* \in I^2(F)$ be 8-dimensional quadratic forms such that $c(\phi) = c(\phi^*) = [D]$, where $D$ is a triquaternion division algebra. Suppose that $\phi^*_{F(D)}$ is anisotropic. Then $\phi_{F(D)}$ is anisotropic.*

*Proof.* Suppose at the moment that $\phi_{F(D)}$ is isotropic. Then letting $\psi \stackrel{\text{def}}{=} \phi^*$, we see that all conditions of Lemma 7.2 hold. Hence $\phi$ and $\phi^*$ are half-neighbors, i.e., there exists $s \in F^*$ such that $\phi^* + s\phi \in I^4(F)$. Therefore $\phi^*_{F(D)} + s\phi_{F(D)} \in I^4(F(D))$. Since $\phi_{F(D)}$ is isotropic, it is hyperbolic and we see that $\phi^*_{F(D)} \in I^4(F(D))$. By the Arason-Pfister Hauptsatz, we see that $\phi^*_{F(D)}$ is hyperbolic. So we get a contradiction to the assumption of the lemma. □

Proposition 7.6. *Let* $\phi \in I^2(F)$ *be an* 8*-dimensional quadratic form such that* $\operatorname{ind} C(\phi) = 8$*. Let* $A$ *be an algebra such that* $c(\phi) = [A]$*. Then* $\phi_{F(A)}$ *is anisotropic.*

*Proof.* Let $D$ be a triquaternion algebra such that $c(\phi) = [D]$. Since $\operatorname{ind} C(\phi) = 8$, $D$ is a division algebra. Let $E/F$ and $\phi^*$ be such that in Lemma 7.4. All conditions of Lemma 7.5 hold for $E$, $\phi_E$, $\phi^*$, and $D_E$. Therefore $\phi_{E(D)}$ is anisotropic. Hence $\phi_{F(D)}$ is anisotropic. Since $[A] = c(\phi) = [D]$, the field extension $F(A)/F$ is stably isomorphic to $F(D)/F$ (Proposition 6.3). Therefore $\phi_{F(A)}$ is anisotropic. □

*Proof of Corollary 0.3.* Suppose at the moment that $\phi_{F(A)} \in I^4(F(A))$. Since $\operatorname{ind} C(\phi) \geq 8$, it follows that $\dim \phi \geq 8$. By Corollary 1.5 there exists a field extension $E/F$ such that $\dim(\phi_E)_{an} = 8$, $\operatorname{ind} C(\phi_E) = 8$. Since $\dim(\phi_E)_{an} = 8$ and $\phi_{E(A)} \in I^4(E(A))$, the Arason-Pfister Hauptsatz shows that $((\phi_E)_{an})_{E(A)}$ is hyperbolic. We get a contradiction to Proposition 7.6. □

## 8. Proof of Theorem 0.1

By Proposition 6.7, there is a surjection

$$\frac{H^3(F(\psi, D)/F)}{[D] \cup H^1(F)} \twoheadrightarrow \operatorname{Tors} \operatorname{CH}^2(X_\psi \times X_D) .$$

Thus, it suffices to prove the second formula of Theorem 0.1.

Proving the second formula, we may assume that $\dim \psi = 5$ (compare to the proof of Theorem 6.9) and $D$ is a division algebra (Corollary 6.4). Under these assumptions, we can write down $D$ as the tensor product $C_0(\psi) \otimes_F B$ (using [25, Th. 1]). In particular, we see that $C_0(\psi)$ is a division algebra, i.e. $\operatorname{ind} C_0(\psi) = \deg C_0(\psi) = 4$.

If $\deg D < 8$, then $D \simeq C_0(\psi)$. In this case, $\psi_{F(D)}$ is a 5-dimensional quadratic form with trivial Clifford algebra; therefore $\psi_{F(D)}$ is isotropic; by this reason, the field extension $F(\psi, D)/F(D)$ is purely transcendental and consequently $H^3(F(\psi, D)/F(D)) = 0$. It follows that

$$H^3(F(\psi, D)/F) = H^3(F(D)/F) = [D] \cup H^1(F) ,$$

where the last equality holds by Proposition 6.2.

If $\deg D > 8$, then $\operatorname{ind} B \geq 4$. Applying the index reduction formula [31, Th. 1.3], we get

$$\operatorname{ind} C_0(\psi)_{F(D)} = \min\{\operatorname{ind} C_0(\psi),\ \operatorname{ind} B\} = 4 .$$

Therefore $\psi_{F(D)}$ is not a 3-Pfister neighbor and by Proposition 6.1 the group $H^3(F(\psi, D)/F(D))$ is trivial. Thus once again

$$H^3(F(\psi, D)/F) = H^3(F(D)/F) = [D] \cup H^1(F) .$$

Finally, if $\deg D = 8$, then we are done by Theorem 6.9 and Proposition 6.7.

## References


[1] Arason, J. Kr. *Cohomologische Invarianten quadratischer Formen.* J. Algebra 36 (1975), 448–491.

[2] Esnault, H., Kahn, B., Levine, M., and Viehweg, E. *The Arason invariant and mod 2 algebraic cycles.* J. Amer. Math. Soc., to appear.

[3] Fulton, W. *Intersection Theory.* Springer-Verlag, 1984.

[4] Hartshorne, R. *Algebraic Geometry.* Springer-Verlag, 1977.

[5] Hoffmann, D. W. *Splitting patterns and invariants of quadratic forms.* Math. Nachr., to appear.
[6] Hurrelbrink, J., Rehmann, U. *Splitting patterns of quadratic forms.* Math. Nachr. 176 (1995), 111–127.
[7] Izhboldin, O. T. *On the nonexcellence of the function fields of Severi-Brauer varieties.* Max-Planck-Institut für Mathematik in Bonn, Preprint MPI 96-159 (1996), 1–28.
[8] Izhboldin, O. T., Karpenko, N. A. *Isotropy of virtual Albert forms over function fields of quadrics.* Prépublications de l'Équipe de Mathématique de Besançon 97/07 (1997), 1–11.
[9] Izhboldin, O. T., Karpenko, N. A. *Isotropy of 6-dimensional quadratic forms over function fields of quadrics.* Prépublications de l'Équipe de Mathématique de Besançon 97/12 (1997), 1–25.
[10] Izhboldin, O. T., Karpenko, N. A. *Isotropy of 8-dimensional quadratic forms over function fields of quadrics.* K-Theory Preprint Archives (http://www.math.uiuc.edu/K-theory/), Preprint N°219, 1997.
[11] Izhboldin, O. T., Karpenko, N. A. *Some new examples in the theory of quadratic forms.* K-Theory Preprint Archives (http://www.math.uiuc.edu/K-theory/), Preprint N°234, 1997.
[12] Kahn, B. *Descente galoisienne et $K_2$ des corps de nombres.* K-Theory 7 (1993), no. 1, 55–100.
[13] Kahn, B. *Motivic cohomology of smooth geometrically cellular varieties.* K-Theory Preprint Archives (http://www.math.uiuc.edu/K-theory/), Preprint N°218, 1997.
[14] Karpenko, N. A. *Algebro-geometric invariants of quadratic forms.* Algebra i Analiz 2 (1991), no. 1, 141–162 (in Russian). Engl. transl.: Leningrad (St. Petersburg) Math. J. 2 (1991), no. 1, 119–138.
[15] Karpenko, N. A. *On topological filtration for Severi-Brauer varieties.* Proc. Symp. Pure Math. 58.2 (1995), 275–277.
[16] Karpenko, N. A. *On topological filtration for Severi-Brauer varieties II.* Transl. Amer. Math. Soc. 174 (1996), no. 2, 45–48.
[17] Karpenko, N. A. *Codimension 2 cycles on Severi-Brauer varieties.* Prépublications de l'Équipe de Mathémathiques de Besançon 96/40 (1996), 1–26. To appear in K-Theory.
[18] Karpenko, N. A. *Cycles de codimension 2 en produits de variétés de Severi-Brauer.* Publications Mathématiques de la Faculté des Sciences de Besançon — Théorie des Nombres, Années 1994/95–1995/96, 1–15.
[19] Karpenko, N. A. *Cohomology of relative cellular spaces and isotropic flag varieties.* Preprint, 1997 (see http://www.uni-muenster.de/math/u/scharlau/publ).
[20] Karpenko, N. A., Merkurjev, A. S. *Chow groups of projective quadrics.* Algebra i Analiz 2 (1990), no. 3, 218–235 (in Russian). Engl. transl.: Leningrad (St. Petersburg) Math. J. 2 (1991), no. 3, 655–671.
[21] Knebusch, M. *Generic splitting of quadratic forms I.* Proc. London Math. Soc. 33 (1976), 65–93.
[22] Knebusch, M. *Generic splitting of quadratic forms II.* Proc. London Math. Soc. 34 (1977), 1–31.

[23] Laghribi, A. *Formes quadratiques en 8 variables dont l'algèbre de Clifford est d'indice 8.* K-Theory, to appear.
[24] Lam, T. Y. *The Algebraic Theory of Quadratic Forms.* Massachusetts: Benjamin 1973 (revised printing: 1980).
[25] Merkurjev, A. S. *Simple algebras and quadratic forms.* Izv. Akad. Nauk SSSR Ser. Mat. 55 (1991), 218–224 (in Russian). English transl.: Math. USSR Izv. 38 (1992), no. 1, 215–221.
[26] Merkurjev, A. S. *K-theory of simple algebras.* Proc. Symp. Pure Math. 58.1 (1995), 65–83.
[27] Panin, I. A. *On the algebraic K-theory of twisted flag varieties.* K-Theory 8 (1994), no. 6, 541–585.
[28] Peyre, E. *Products of Severi-Brauer varieties and Galois cohomology.* Proc. Symp. Pure Math. 58.2 (1995), 369–401.
[29] Quillen, D. *Higher algebraic K-theory I.* Lect. Notes Math. 341 (1973), 85–147.
[30] Scharlau, W. *Quadratic and Hermitian Forms.* Springer, Berlin, Heidelberg, New York, Tokyo (1985).
[31] Schofield, A., van den Bergh, M. *The index of a Brauer class on a Brauer-Severi variety.* Transactions Amer. Math. Soc. 333 (1992), no. 2, 729–739.
[32] Suslin, A. A. *Algebraic K-theory and the norm residue homomorphism.* J. Soviet Math. 30 (1985), 2556–2611.
[33] Swan, R. *K-theory of quadric hypersurfaces.* Ann. Math. 122 (1985), no. 1, 113–154.
[34] Tignol, J.-P. *Réduction de l'indice d'une algèbre simple centrale sur le corps des fonctions d'une quadrique.* Bull. Soc. Math. Belgique 42 (1990), 725–745.

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