# Chern Classes of Fibered Products of Surfaces

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Abstract. In this paper we introduce a formula to compute Chern classes of fibered products of algebraic surfaces. For $f : X \to \mathbb{CP}^2$ a generic projection of an algebraic surface, we define $X_k$ for $k \leq n$ ($n = \deg f$) to be the closure of $k$ products of $X$ over $f$ minus the big diagonal. For $k = n$ (or $n-1$), $X_k$ is called the full Galois cover of $f$ w.r.t. full symmetric group. We give a formula for $c_1^2$ and $c_2$ of $X_k$. For $k = n$ the formulas were already known. We apply the formula in two examples where we manage to obtain a surface with a high slope of $c_1^2/c_2$. We pose conjectures concerning the spin structure of fibered products of Veronese surfaces and their fundamental groups.

Keywords and Phrases: Surfaces, Chern classes, Galois covers, fibered product, generic projection, algebraic surface.

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## 0. Introduction.

When regarding an algebraic surface $X$ as a topological 4-manifold, it has the Chern classes $c_1^2$, $c_2$ as topological invariants. These Chern classes satisfy:

$$c_1^2, \quad c_2 \geq 0$$
$$5c_1^2 \geq c_2 - 36$$
$$\text{Signature} = \tau = \frac{1}{3}(c_1^2 - 2c_2)$$

The famous Bogomolov-Miyaoka-Yau inequality from 1978 (see [Re], [Mi], [Y]) states that the Chern classes of an algebraic surface also satisfy the inequality

$$c_1^2 \leq 3c_2.$$

It is known that this inequality is the best possible since Hirzebruch showed in 1958 that the equality is achieved by complex compact quotients of the unit ball (see [H]).

We want to understand the structure of the moduli space of all surfaces with given $c_1^2$, $c_2$; and, in particular, to populate it with interesting structures of surfaces. As a first step it is necessary to develop techniques to compute Chern classes of different surfaces.

In this paper we compute Chern classes of Galois covers of generic projections of surfaces. This was already computed in [MoTe2] for the case of the full Galois cover, where the product is taken $n$ times ($n$ is the degree of the projection). In this paper we deal with products taken $k$ times, $k < n$, and we manage to give an example of a surface where the slope ($c_1^2/c_2$) is very high (up to 2.73). In subsequent research, using the results of this paper and of our ongoing research on this subject, we plan to further study these constructions, to compute these fundamental groups and to decide when the examples are spin, of positive index, etc. We conjecture that for $X_b$ the Veronese surface of order $b$ greater than 4, $X_k$ is spin if $k$ is even or $b = 2, 3(4)$. We further conjecture that for the Hirzebruch surfaces in general the fundamental groups of $X_k$ are finite.

In [RoTe], we used similar computations to produce a series of examples of surfaces with the same Chern classes and different fundamental groups which are spin manifolds where one fundamental group is trivial and the other one has a finite order which is increasing to infinity. The computations in this paper will lead to more examples of pairs in the $\tau > 0$ area.

We consider in this paper fibered products and Galois covers of generic projections of algebraic surfaces. If $f : X \to \mathbb{CP}^2$ is generic of $\deg n$, we define the $k$-th Galois cover for $k \leq n$ to be $\overline{X \underset{f}{\times} \cdots \underset{f}{\times} X - \Delta}$ where $\Delta$ is the big diagonal and the fibered product is taken $k$ times. There exists a natural projection $g_k : X_k \to \mathbb{CP}^2$, $\deg g_k = n(n-1)\dots(n-k+1)$.

The surface $X_k$ for $k = n$, is called the full Galois cover (i.e., the Galois cover w.r.t. full symmetric group), and is also denoted $X_{\mathrm{Gal}}$ or $\tilde{X}$. Clearly, $\deg(X_{\mathrm{Gal}} \to \mathbb{CP}^2) = n!$. It can be shown that $X_n \simeq X_{n-1}$. The full Galois covers were first treated by Miyaoka in [Mi], who noticed that their signature should be positive. In our papers [MoTe1], [MoTe2], [MoTe3], [MoRoTe], [RoTe], [Te], [FRoTe], we discussed the full Galois covers for $X = f_{|a\ell_1 + b\ell_2|}(\mathbb{CP}^1 \times \mathbb{CP}^1)$, Veronese embeddings and Hirzebruch surfaces. In the papers cited above we computed their fundamental groups (which are finite), the Chern numbers and the divisibility of the canonical divisor (to prove that when considered as 4-manifolds they are spin manifolds). $X_{\mathrm{Gal}}$ are minimal smooth surfaces of general type. Other examples of interest on surfaces in the $\tau > 0$ area can be found in [Ch] and [PPX].

§1. The Main Theorem.
We start with a precise definition.

Definition. *A Galois cover of a generic projection w.r.t. the symmetric group* $\mathbf{S_k}$ (for $\mathbf{k}$ < degree of the generic projection). Let $X \hookrightarrow \mathbb{CP}^N$ be an embedded algebraic surface. Let $f : X \to \mathbb{CP}^2$ be a generic projection, $n = \deg f$. For $1 \le k \le n$, let

$$X \underset{f}{\times} \cdots \underset{f}{\times} X = \{(x_1, \dots x_k) \mid x_i \in X,\ f(x_i) = f(x_j)\ \forall i \forall j\},$$
$$\Delta = \{(x_1, \dots, x_k) \in X \underset{f}{\times} \cdots \underset{f}{\times} X \mid x_i = x_j \ \text{ for some } i \neq j\}$$
$$X_k = \overline{\underbrace{X \underset{f}{\times} \cdots \underset{f}{\times} X}_{k} - \Delta}.$$

$X_k$ is the closure of $X \underset{f}{\times} \cdots \underset{f}{\times} X - \Delta$. $X_k$ is the Galois cover w.r.t. the symmetric group on $k$ elements. We denote $X_0 = \mathbb{CP}^2$.

For every $k \ge 1$ we have the canonical projections $g_k : X_k \to \mathbb{CP}^2$ and a natural projection (on the first $k$ factors) $f_k : X_k \to X_{k-1}$, which satisfy

$$f_1 = g_1 = f$$
$$g_{k-1} f_k = g_k, \quad (k \ge 2).$$

Clearly,

$$\deg g_k = n \cdot (n-1) \dots (n-k+1)$$
$$\deg f_k = n - k + 1,$$
$$X_{n-1} \simeq X_n \ (f_n \text{ is an isomorphism}).$$

.

For $k = n$ (or $n - 1$), we call $X_k$ the Galois cover w.r.t. the full symmetric group or the full Galois cover and denote it also by $X_{\text{Gal}}$.

Remark. $X_k$ is the interesting component in the fibered product $\underbrace{X \underset{f}{\times} \cdots \underset{f}{\times} X}_{k}$

*Notations.*

For the rest of the paper we shall use the following notations:
$n = \deg f$.
$X_k$, the Galois cover of $f : X \to \mathbb{CP}^2$ as above, $k \le n$.
$S =$ the branch curve of $f$ in $\mathbb{CP}^2$ ($S$ is a cuspidal curve)
$m = \deg S$
$\mu = \deg S^*$ ($S^*$ the dual to $S$)
$=$ number of branch points in $S$ w.r.t. a generic projection of $\mathbb{C}^2$ to $\mathbb{C}^1$.
$d =$ number of nodes in $S$
$\rho =$ number of cusps in $S$

THEOREM 1. *The Chern classes of $X_k$ are as follows:*

(a) $$c_1^2(X_1) = 9n + \left(\frac{m}{2} - 6\right) m - \rho - d.$$

*For* $2 \le k \le n-1$

$$\begin{aligned} c_1^2(X_k) &= 9(n-k+1)\dots n \\ &+ \frac{1}{2}[(n-k+1)\dots(n-2)](2n-k-1)k\left(\frac{m}{2}-6\right)m \\ &- [(n-k-1)\dots(n-3)]k\rho \\ &- \frac{1}{2}[(n-k-1)\dots(n-4)](2n-k-5)kd \end{aligned}$$

(b)

$$\begin{aligned} c_2(X_1) &= 3n - 2m + \mu \\ c_2(X_2) &= 3n(n-1) - 2(2n-3)m + (2n-3)\mu + \rho + 2d \\ c_2(X_3) &= 3n(n-1)(n-2) - 3(2n-4)(n-2)m + \frac{3}{2}(2n-4)(n-2)\mu \\ &+ 2(3n-9)d + (3n-8)\rho \end{aligned}$$

*For* $4 \le k \le n-1$

$$\begin{aligned} c_2(X_k) &= 3(n-k+1)\dots n \\ &- (n-k+1)\dots(n-2)(2n-k-1)km \\ &+ \frac{1}{2}(n-k+1)\dots(n-2)(2n-k-1)k\mu \\ &+ (n-k+1)\dots(n-3)(k-1)k\left(\frac{n}{2} - \frac{k+1}{3}\right)\rho \\ &+ [(n-k+1)\dots(n-4)]\frac{k(k+1)}{4}\{(k+6)(k-1) + 4n(n-k-1)\}d \\ &+ [(n-k+1)\dots(n-4)]\{4nk - 2n^2k\}d \end{aligned}$$

REMARKS.

(a) We consider an empty multiplication as 1.

(b) The case $k = n-1 (X_k = X_{\mathrm{Gal}})$, of this Theorem was treated in [MoTe2], Proposition 0.2 (proof there is given by F. Catanese). (See also [MoRoTe]). One can easily see that for $k = n$ the formulas here coincide with the formulas from [MoTe2]. For $c_1^2$ it is enough to use remark (a) about empty multiplication. We get:

$$c_1^2(X_{\mathrm{Gal}}) = c_1^2(X_{n-1}) = \frac{n!}{4}(m-6)^2.$$

Note that $d$ and $\rho$ do not appear in this formula. For $c_2$ we get here (using $(a_1)$)

$$c_2(X_{\mathrm{Gal}}) = c_2(X_{n-1}) = n!\left(3 - m + \frac{1}{4}d + \frac{\mu}{2} + \frac{\rho}{6}\right)$$

which coincide with [MoTe2], using the formula for the degree of the dual curve:

$$\mu = m^2 - m - 2d - 3\rho.$$

*Proof of the Theorem.*

Let $g_k : X_k \to \mathbb{CP}^2$, $f_k : X_k \to X_{k-1}$ be the natural projections. Clearly, $g_1 = f_1 = f$, $g_k = g_{k-1} f_k$, for $k \geq 2$ $\deg f_k = n-k+1$, $\deg g_k = \dfrac{n!}{(n-k)!}$. Let $E_k$ and $K_{X_k}$ be the hyperplane and canonical divisors of $X_k$, respectively ($E_k = g^*(\ell)$ for a line $\ell$ in $\mathbb{CP}^2$).

Let $S_k$ be the branch curve of $f_k$ (in $X_{k-1}$), $m_k$ its degree and $\mu'_k$ the number of branch points that do not come from $S_{k-1}$ ($S_1 = S$). Let $S'_k$ be the ramification locus of $f_k$ (in $X_k$). Let $T'_k$ be the ramification locus of $g_k$ (in $X_k$).

We recall that the branch points in $S$ (or $S_k$) come from two points coming together in the fibre, the cusps from (isolated) occurrences of three points coming together and nodes from 4 points coming together into 2 distinct points. Generically, cusps and nodes are unbranched. We use this observation in the sequel.

To compute $c_1^2(X_k)$ we shall use:

$$c_1^2(X_k) = K_{X_k}^2$$
$$K_{X_k} = -3E_k + T'_k$$

and the following identities.

$$T'_k = \begin{cases} S'_k + f_k^*(T'_{k-1}) & k \geq 2 \\ S'_1 & k = 1 \end{cases}$$
$$T'_k = -\frac{1}{2}S_{k+1} + \frac{1}{2}g_k^*(S)$$

To compute $c_2(X_k)$ we shall assume that all cusps and nodes of $S$ are vertices of a triangulation. Using the standard stratification computations, this implies the following recursive formula:

$$c_2(X_k) = \deg f_k \cdot c_2(X_{k-1}) - 2m_k + \mu'_k.$$

Thus we need to get a formula for $E_k \cdot T'_k$, $S_{k+1} \cdot T'_k$, $m_k$ and $\mu'_k$. We shall use the following 3 claims:

Claim 1.

(i) *Let* $m_k = \deg S_k$. *For* $k \geq 2$, $m_k = (n-k)\dots(n-2)m$, $m_1 = m$.

(ii) *Let* $d_k = \#$ *nodes in* $S_k$. *For* $k \geq 2$, $d_k = (n-k-2)\dots(n-4)d$, $d_1 = d$.

(iii) *Let* $\rho_k = \#$ *cusps in* $S_k$. *For* $k \geq 2$, $\rho_k = (n-k-1)\dots(n-3)\rho$, $\rho_1 = \rho$.

(iv) *Let* $\mu'_k$ *be the number of branch points of* $S_k$ *that do not come from* $S_{k-1}$, $\mu'_k = \mu_k - (n-k+1)\mu_{k-1}$ $(k \geq 2)$ *and* $\mu'_1 = \mu$. *Then for* $k = 2$, $\mu'_2 = (n-2)\mu + \rho + 2d$ *and for* $k \geq 3$ $\mu'_k = (n-k)\dots(n-2)\mu +$ $(n-k)\dots(n-3)(k-1)\rho + [(n-k)\dots(n-4)](k-1)(2n-k-4)d$. *(For* $k = 3$ *the coefficient of* $d$ *is* $2(2n-7)$.*)*

Claim 2.

$$E_k.T'_k = \begin{cases} m & k=1 \\ \frac{1}{2}m[(n-k+1)\dots(n-2)]\{(2n-1)k-k^2\} & k \geq 2 \end{cases}$$

Claim 3.

$$S_{k+1}.T'_k = \begin{cases} 2\rho + 2d & k=1 \\ 2(n-k-1)\dots(n-3)k\rho + (n-k-1)\dots(n-4)(2n-k-5)kd & k \geq 2 \end{cases}$$

*Proof of Claim 1.*

Items (i), (ii) and (iii) are easy to verify from the definition of fibered product. For (iv) we notice that $\{\mu'_k\}$ satisfy the following recursive equations:

$$\mu'_k = (n-k)\mu'_{k-1} + \rho_{k-1} + 2d_{k-1} \quad k \geq 2$$
$$\mu'_1 = \mu.$$

The formula for $\mu'_2, \mu'_3$ follows immediately from the recursive formula. For $k \geq 4$ we substitute the formulas for $\rho_{k-1}$ and $d_{k-1}$ from (ii) and (iii) to get $\mu'_k = (n-k)\mu'_{k-1} + (n-k)\dots(n-3)\rho + 2(n-k-1)\dots(n-4)d$ and we shall proceed by induction. By the induction hypothesis $\mu'_{k-1} = (n-k+1)\dots(n-2)\mu + (n-k+1)\dots(n-3)(k-2)\rho + [(n-k+1)\dots(n-4)](k-2)(2n-k-3)d$.
We substituted the last expression in the previous one to get

$$\begin{aligned} \mu'_k &= (n-k)(n-k+1)\dots(n-2)\mu + (n-k)(n-k+1)\dots(n-3)(k-2)\rho \\ &+ (n-k)(n-k+1)\dots(n-4)(k-2)(2n-k-3)d \\ &+ (n-k)\dots(n-3)\rho + 2(n-k-1)\dots(n-4)d \\ &= (n-k)\dots(n-2)\mu + (n-k)\dots(n-3)(k-1)\rho \\ &+ (n-k)\dots(n-4)\{(k-2)(2n-k-3) + 2(n-k-1)\}d \end{aligned}$$

which coincide with the claim since $(k-2)(2n-k-3) + 2(n-k-1) = (k-1)(2n-k-4)$. □ for Claim 1

*Proof of Claim 2.*

For $k \geq 2$

$$\begin{aligned} E_k.T'_k &= \frac{1}{2}E_k.(g^*_k(S) - S_{k+1}) \\ &= \frac{1}{2}E_k g^*_k(S) - \frac{1}{2}E_k.S_{k+1} \\ &= \frac{1}{2}g^*_k(\ell)g^*_k(S) - \frac{1}{2}E_k.S_{k+1} \\ &= \frac{1}{2}g^*_k(\ell.S) - \frac{1}{2}E_k.S_{k+1} \\ &= \frac{1}{2}(\deg g_k.)m - \frac{1}{2}m_{k+1} \end{aligned}$$

$$= \frac{1}{2}m(n-k+1)\dots n - \frac{1}{2}(n-k-1)(n-k)\dots(n-2)m$$
$$= \frac{1}{2}m[(n-k+1)\dots(n-2)]\{(n-1)n-(n-k-1)(n-k)\}$$
$$= \frac{1}{2}m[(n-k+1)\dots(n-2)]\{2nk-k-k^2\}. \qquad \square \text{ for Claim 2}$$

*Proof of Claim 3.*

Since $T_1' = S_1'$, the formula trivializes for $k=1$. $S_2 \cdot T_1' = S_2 \cdot S_1' = 2\rho + 2d = 2\rho_1 + 2d_1$. For $k \geq 2$

$$\begin{aligned} S_{k+1} \cdot T_k' &= S_{k+1}(f_k(T_{k-1}')' + S_k') \\ &= S_{k+1} \cdot f_k^*(T_{k-1}') + (S_{k+1} \cdot S_k') \\ &= (\deg f_k \big|_{S_{k+1}}) \cdot (S_k \cdot T_{k-1}') + 2\rho_k + 2d_k \\ &= (\deg f_k - 2)(S_k \cdot T_{k-1}') + 2\rho_k + 2d_k \\ &= (n-k-1)(S_k \cdot T_{k-1}') + 2\rho_k + 2d_k. \end{aligned}$$

Denote $a_k = S_{k+1} \cdot T_k'$.

We shall prove the claim by induction using the recursive formula $a_k = (n-k-1)a_{k-1} + 2\rho_k + 2d_k$. For $k=2$ :

$$\begin{aligned} a_2 &= (n-3)a_1 + 2\rho_2 + 2d_2 \\ &= (n-3)(2\rho + 2d) + 2(n-3)\rho + 2(n-4)d \\ &= 4(n-3)\rho + 2d(n-3+n-4) \\ &= 4(n-3)\rho + 2d(2n-7). \end{aligned}$$

Thus the statement is true for $k=2$.

Let $k \geq 3$. Assume the formula is true for $k-1$. We shall prove it for $k$.

$$\begin{aligned} a_k &= (n-k-1)a_{k-1} + 2\rho_k + 2d_k \\ &= (n-k-1)\{2(n-k)\dots(n-3)(k-1)\rho + (k-1)(n-k)\dots(n-4)(2n-k-4)d\} \\ &\quad + 2(n-k-1)\dots(n-3)\rho + 2(n-k-2)\dots(n-4)d \\ &= 2(n-k-1)\dots(n-3)k\rho + (n-k-1)\dots(n-4)\{(2n-k-4)(k-1) + 2(n-k-2)\} \\ &= 2(n-k-1)\dots(n-3)k\rho + (n-k-1)\dots(n-4)\{2nk-k^2-5k\}d. \end{aligned}$$

From the two formulae, we can see that the product $(n-k-1)...(n-4)$ should be 1 for $k \leq 2$. $\square$ for Claim 3

We go back to the proof of the theorem. To prove (a) we write

$$\begin{aligned} c_1^2(X_k) = K_{X_k}^2 &= (-3E_k + T_k')^2 \\ &= 9E_k^2 - 6E_k.T_k' + (T_k')^2 \\ &= 9E_k^2 - 6E_k.T_k' + T_k'\left[-\frac{1}{2}S_{k+1} + \frac{1}{2}g_k^*(S)\right] \\ &= 9E_k^2 - 6E_k \cdot T_k' - \frac{1}{2}T_k' \cdot S_{k+1} + \frac{1}{2}T_k' \cdot g_k^*(S). \end{aligned}$$

Now: $E_k^2 = \deg g_k = (n-k+1)\dots n \left(= \dfrac{n!}{(n-k)!}\right)$. Since $S$ is of deg $m$ $T_k'.g_k^*(S) = mE_k.T_k'$. We substitute the results from Claim 2 and Claim 3 to get (a).

We prove (b) by induction on $k$. For $k = 1$ we take the recursive formula $c_2(X_k) = (n-k+1)c_2(X_{k-1}) - 2m_k + \mu_k'$ and substitute $k = 1$ to get $c_2(X_1) = 3n - 2m + \mu$ which coincides with formula (b) for $k = 1$. We do the same for $k = 2, 3$. To prove $k-1$ implies $k$ we use Claim 1(iv) and (ii) to write

$$\begin{aligned} c_2(X_k) = (n-k+1)c_2(X_{k-1}) &- 2(n-k)\dots(n-2)m + (n-k)\dots(n-2)\mu \\ &+ (k-1)(n-k)\dots(n-3)\rho \\ &+ (n-k)\dots(n-4)(k-1)(2n-k-4)d \end{aligned}$$

When substituting the inductive statement for $c_2(X_{k-1})$ and shifting around terms, we get (b).

□ for the Theorem

§2. A DIFFERENT PRESENTATION OF THE CHERN CLASSES.

PROPOSITION 2. *Let $E$ and $K$ denote the hyperplane and canonical divisors of $X$, respectively. Then the Chern classes of $X_k$ are functions of $c_1^2(X)$, $c_2(X)$,* $\deg(X)$, *$E$, $K$, and $k$.*

*Proof.* (Proof for $X_n$ appeared in [RoTe]) Let $S$ be the branch curve of the generic projection $f : X \to \mathbb{CP}^2$ $(S \subseteq \mathbb{CP}^2)$. By Theorem 1, the Chern classes of $X_k$ depend on $k$, $\deg(S)$, $\deg(X)$ and $\mu, d, \rho$, the number of branch points, nodes and cusps of $S$, respectively.

We shall first show that $\mu, d, \rho$ depends on $c_2(X)$, $\deg X$, $\deg(S)$, $e(E)$ and $g(R)$ where $g$ denotes the genus of an algebraic curve, $e$ denotes the topological Euler characteristic of a space, and $R$ $(\subset X)$ is the ramification locus of $f$ which is, in fact, the non-singular model of $S$.

Recall that $\mu$ also is equal to $\deg(S^*)$, where $S^*$ is the dual curve to $S$. For short we write $n = \deg(X)$, $m = \deg(S)$.

We show this by presenting three linearly independent formulae:

$$\begin{aligned} \mu &= m(m-1) - 2d - 3\rho \\ g(R) &= \frac{(m-1)(m-2)}{2} - d - \rho \\ c_2(X) + n &= 2e(E) + \mu \end{aligned}$$

The first two are well-known formulae for the degree of the dual curve and the genus of a non singular model of a curve. For the third, we may find a Lefschetz pencil of hyperplane sections of $X$ whose union is $X$. The number of singular curves in the pencil is equal to $\mu$. The topological Euler characteristic of the fibration equals $e(X) = e(\mathbb{CP}^1)\cdot e(E) + \mu - n$ ($n$ appears from blowing up $n$ points in the hyperplane sections). The formula follows from $e(\mathbb{CP}^1) = 2$ and $e(X) = c_2(X)$.

We shall conclude by showing that $\deg(S)$, $e(E)$ and $g(R)$ depend on $c_1^2(X)$, $\deg X$ and $E.K$.

This follows from the Riemann-Hurwitz formula, $R = K + 3E$, the adjunction formula $2-2g(C) = -C.(C+K)$, and the fact that $E^2 = \deg X$ and $K^2 = c_1^2(X)$. In fact, we have:

$$\begin{aligned} g(R) &= 1 + \frac{1}{2}R(R+K) = 1 + \frac{1}{2}(K+3E)(2K+3E) \\ e(E) &= 2 - 2g(E) = -E(E+K) \\ \deg(S) &= \deg(R) = E.R = E(K+3E). \end{aligned} \qquad \square$$

From the above proof we can, in fact, get the precise formulae of $c_1^2(X_k)$ and $c_2(X_k)$ in terms of $c_1^2(X)$, $c_2(X)$, $\deg(X)$, $E.K$, and $k$. For certain (computerized) computations, it is easier to work with these formulae rather than those of Theorem 1.

Corollary 2.1. *In the notations of the above proposition:*

$$c_1^2(X_n) = \frac{n!}{4}[(E.K)^2 + 6n(E.K) + 9n^2 - 12(E.K) - 36n + 36]$$

$$c_2(X_n) =$$

$$\frac{n!}{24}[72 - 10c_1^2(X) - 54(E.K) - 114n + 27n^2 + 14c_2(X) + 3(E.K)^2 + 18n(E.K)]$$

Similar formulas can be obtained for $X_k$ for $k < n$.

## §3. Examples.

To use Theorem 1, we need computations of $n$, $m$, $\mu$, $\rho$ and $d$. We compute them for two examples.

Examples 3.1. *For $X = V_b$, a Veronese embedding of order $b$, we have*

$$\begin{aligned} n &= b^2 \\ m &= 3b(b-1) \\ \mu &= 3(b-1)^2 \\ \varphi &= 3(b-1)(4b-5) \\ d &= \frac{3}{2}\ (b-1)(3b^3 - 3b^2 - 14b + 16) \end{aligned}$$

*(see* [MoTe3]*).*

*Proof.* For $n, m, \mu$ and $\rho$, see [MoTe3] and [MoTe4]. Since $\mu = m(m-1) - 2d - 3\rho$, we get the following formula for $d$: $2d = m^2 - m - \mu - 3\varphi$ and thus

$$\begin{aligned} 2d &= 3b(b-1)(3b(b-1)-1) - 3(b-1)^2 - 9(b-1)(4b-5) \\ &= 3(b-1)\{(3b^2-3b-1)b - (b-1) - 3(4b-5)\} \\ &= 3(b-1)\{3b^3 - 3b^2 - 14b + 16\}. \end{aligned}$$

When one substitutes $b = 3$ and $k = 4$, one gets $\frac{c_1^2}{c_2} = 2.73$. By experimental substitutions it seems that for large b, the signature $\tau(X_k)\ \ (= c_1^2 - 2c_2)$, changes from negative to positive at about $\frac{3}{4}n$.

EXAMPLE 3.2. *For $X = X_{t(a,b)} = f_{|a\ell+bC_+|}$ (Hirzebruch surface of order $t$), where $\ell$ is a fiber, $(C_+)^2 = t$, and $a \geq 1$, we have*

$$\begin{aligned} n &= 2ab + tb^2 \\ m &= 6ab - 2a - 2b + t(3b^2 - b) \\ \mu &= 6ab - 4a - 4b + 4 + t(3b^2 - 2b) \\ \varphi &= 24ab - 18a - 18b + 12 + t(12b^2 - 9b) \end{aligned}$$

*Proof.* [MoRoTe], Lemma 7.1.3.

EXAMPLE 3.3. *(in the $\tau < 0$ area)*
*For $X$ a $K3$ surface:*
$K = 0$
$c_1^2(X) = K^2 = 0$
$c_2(X) = 24$
$n = 4$
$m = 12$
$\mu = 36$
$\rho = 24$
$d = 12$
$c_1^2(X_2) = 48$
$c_2(X_2) = 144$
$c_1^2(X_3) = c_1^2(X_{\mathrm{Gal}}) = 216$
$c_2(X_3) = c_2(X_{\mathrm{Gal}}) = 240.$

*Proof.* It is well known that for a $K3$ surfaces $K = 0$, $c_1^2 = 0$, $c_2 = 24$, $S' = 3E$, $n = E^2 = 4$. Using this we can get $m$ and $\mu$:

$$\begin{aligned} m &= S'.E = 3E.E = 3E^2 = 12 \\ \mu &= c_2(X) - 2e(E) + n \quad \text{(see proof of Proposition 2)} \\ &= c_2(X) - 2(2 - 2g(E)) + n \\ &= c_2(X) + 2E.(E + K) + n = 36 \end{aligned}$$

Now from

$$m(m-1) = \mu + 3\rho + 2d$$

and

$$\begin{aligned} m(m-3) &= 2g(S') - 2 + 2\rho + 2d = (K + S'), S' + 2\rho + 2d \\ &= 3E.3E + 2\rho + 2d = 9E^2 + 2\rho + 2d, \end{aligned}$$

we get $2m = \mu - 9E^2 + \rho = 24$ and $\rho = 2m + 9E^2 - \mu = 24$.

Moreover, we get $d = \frac{1}{2}(m(m-1) - \mu - 3\rho) = 12$. We substitute these quantities in the formula from Theorem 1 to get the values of the Chern classes.

REMARK. For $t = 0$, $X_{t(a,b)}$ are actually embeddings of $\mathbb{CP}^1 \times \mathbb{CP}^1$. In [FRoTe], we computed the fundamental group of $X_n = X_{\mathrm{Gal}}$ for $X = X_{t(a,b)}$ which is $\mathbb{Z}_c^{n-2}$ for

$c = g.c.d.(a,b)$. Thus for $(a,b) = 1$ these surfaces are simply connected. All these surfaces are smooth minimal surfaces of general type. For $a \geq 6$, $b \geq 5$, the signature of these surfaces is positive. For five pairs of $(a,b)$, these surfaces have signature 0 (see [MoRoTe]). Four of these surfaces are simply connected and the fifth one for which $a = b = 5$, $\pi_1(X_{\text{Gal}}) = \mathbb{Z}_5^{48}$.

In our ongoing research, we shall apply Theorem 1 and Proposition 2 in order to obtain more examples of non diffeomorphic surfaces or surfaces in different deformation families with the same $c_1^2$ and $c_2$, as well as to compute the slope $\frac{c_1^2}{c_2}$ and to search for higher slopes.

We are also interested in the fundamental groups (in particular, in the finite ones) and the divisibility of the canonical class (in particular, the case where the canonical class is divided by 2, i.e., the spin case), which we will investigate in a subsequent paper. The results in this paper are a basis for producing interesting examples of surfaces with positive index, $(c_1^2 - c_2)$, finite fundamental groups and spin ($K$ even) structure. In particular, we plan to prove the following two conjectures.

Conjecture. *For $X = V_b$, Veronese of order $b$, $b > 4$, we have $X_k$ is a spin manifold $\Leftrightarrow k$ even or $b = 2, 3(4)$.*

Conjecture. *For $X = F_{t(a,b)}$ (the Hirzebruch surface), $\pi_1(X_k)$ is finite.*

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