How Frequent Are Discrete Cyclic Subgroups of Semisimple Lie Groups?

Jörg Winkelmann

Received: February 25, 2001

Communicated by Ulf Rehmann

Abstract. Let G be a non-compact semisimple Lie group. We investigate the asymptotic behaviour of the probability of generating a discrete subgroup.

2000 Mathematics Subject Classification: 22E40
Keywords and Phrases: discrete subgroup, Lie group, generic subgroup, Cartan subgroup

1 Main result

For a locally compact topological group G let us define Δ_G as the set of all $g \in G$ such that the cyclic subgroup $\{g^n : n \in \mathbb{Z}\}$ of G is discrete. If there is no danger of ambiguity, we write simply Δ instead of Δ_G.
Let G be a connected non-compact real semisimple Lie group and μ a Haar measure on G. In a preceding article ([3]) we proved that $\mu(\Delta_G) = \infty$ and that furthermore $\mu(G \setminus \Delta_G) = \infty$ if G contains a compact Cartan subgroup and $\mu(G \setminus \Delta_G) = 0$ otherwise.
During the "Colloquium on Lie Theory and Application" in Vigo in July 2000 K. H. Hofmann suggested to me to investigate the asymptotic behavior of the ratio of volumes of the respective intersections with balls.
This paper is concerned with establishing such an asymptotic description.
The first problem is to make precise what is meant "balls". What is a natural choice of "balls" to be considered here? The first idea would be to consider balls with respect to some Riemannian metric which should be a canonical as possible. However, a non-compact semisimple Lie group does not admit any Riemannian metric invariant under both left and right translations and there is no good reason to discriminate against left or right wingers.
Here we took a different approach. Let K be a maximal compact subgroup of G and consider the double quotient $X = K\backslash G/K$. For a continuous exhaustion

function ρ on X we define "balls" $B_r = \{\rho < r\}$. We demonstrate that with respect to such an exhaustion asymptotically the share of Δ tends to one.
Now let us proceed to a precise statement.
First we recall that an "exhaustion function" ρ on a topological space X is a continuous map $\rho : X \to \mathbb{R}^+ \cup \{0\}$ such that $\rho^{-1}([0,r])$ is compact for all $r \in \mathbb{R}^+$. If H is a subgroup of a topological group G, then an exhaustion function ρ on G is called "H-biinvariant" if $\rho(hgh') = \rho(g)$ for all $g \in G$, $h, h' \in H$.

Theorem. *Let G be a connected, non-compact real semisimple Lie group, Δ the set of all elements $g \in G$ for which the generated subgroup $\{g^n : n \in \mathbb{Z}\}$ is discrete in G, μ a Haar measure on G, K a maximal compact subgroup of G, and $\rho : G \to \mathbb{R}^+ \cup \{0\}$ a K-biinvariant exhaustion function.*
Let $B_r = \{g \in G : \rho(g) < r\}$.
Then

$$\lim_{r\to\infty} \frac{\mu(\Delta \cap B_r)}{\mu(B_r)} = 1.$$

Proof. Let Z denote the center of G. We distinguish three different cases, depending on the cardinality of Z.
Case 1. Here we assume that the center Z is trivial. Then G admits a faithful representation $\lambda : G \to GL(V)$ (for instance, the adjoint representation is faithful.) Note that $\{g^n : n \in \mathbb{Z}\}$ must be discrete if $g \in G$ with $|\operatorname{Tr}(\lambda(g))| > n$. Let K be a maximal compact subgroup of G, $Lie(G) = Lie(K) + \mathbf{p}$ a Cartan decomposition, $\mathbf{a}$ a maximal Abelian subspace of $\mathbf{p}$ and A the corresponding connected Lie subgroup of G. Then (see e.g. [2]) A is a reductive connected and simply-connected Lie group and closed in G. It follows that, in suitably chosen coordinates on V, the image $\lambda(A)$ is a closed subset of the set D^+ of all diagonal matrices with all entries non-negative. This implies in particular that $g \mapsto \operatorname{Tr}(\lambda(g))$ defines an exhaustion function on the closed set A.
Next recall that $G = KAK$ by a result of É. Cartan ([1], see also [2], thm.7.39). We will consider the double coset space $X = K\backslash G/K$ and the natural projection $p : G \to X$. By results due to Cartan (see [2]) $X = K\backslash G/K \simeq A/W$ where $W = N_G(A)/A$ is the (restricted) Weyl group. Since the trace of an endomorphism is invariant under conjugation, $\operatorname{Tr} \circ \lambda|_A$ is W-invariant, and therefore there exists an exhaustion function τ on X such that $\operatorname{Tr} \circ \lambda$ and $\tau \circ p$ coincide on A.
Using the natural projection $p : G \to X \simeq K\backslash G/K$ we define a Borel measure η on X by setting $\eta(U) = \mu(p^{-1}(U))$ for every Borel set $U \subset X$. This is an infinite measure, $\eta(X) = \mu(G) = +\infty$, and for every compact set $C \subset X$ we have $\eta(C) < \infty$, because $p^{-1}(C)$ is compact, too. Let ξ denote the normalized Haar measure on $K \times K$. Then for all $f \in C_c(G)$

$$\int_G f(g) d\mu(g) = \int_X \int_{K\times K} f(kah) d\xi(k,h) d\eta(a).$$

Next we define a function

$$\zeta : \mathrm{End}(\mathbb{R}^n) \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$$

by

$$\zeta(a, R) = \xi\left(S(a, R)\right)$$

where

$$S(a, R) = \{(k, h) \in K : |\operatorname{Tr}(\lambda(k) \cdot a \cdot (\lambda(h)))| < R\}.$$

If $\operatorname{Tr}(a) \neq 0$, then

$$\{(k, h) \in K : \operatorname{Tr}(\lambda(k) \cdot a \cdot (\lambda(h))) = 0\}$$

is a nowhere dense real analytic subset of $K \times K$ and therefore of measure zero. It follows that

$$\lim_{t \to 0} \zeta(a, t) = 0$$

for all $a \in \mathrm{End}(\mathbb{R}^n)$ with $\operatorname{Tr}(a) \neq 0$.
We observe that

$$S(\tilde{a}, R) \subset S(a, R + \epsilon)$$

for $a, \tilde{a} \in \mathrm{End}(\mathbb{R}^n)$ with $\sum_{i=1}^n |\lambda(a)_{ii} - \lambda(\tilde{a})_{ii}| < \epsilon$. Using this, it follows that

$$\lim_{n \to \infty} \zeta(a_n, r_n) = 0$$

for all convergent sequences $(a_n)_n$ in $\mathrm{End}(\mathbb{R}^n)$, $(r_n)_n$ in $\mathbb{R}$ with $\lim r_n = 0$ and $\operatorname{Tr}(\lim a_n) \neq 0$.
This in turns implies, that if we have a compact subset $C \subset \mathrm{End}(\mathbb{R}^n)$ such that $\operatorname{Tr}(c) \neq 0$ for all $c \in C$, then

$$\lim_{t \to 0} \left(\sup_{c \in C} \zeta(c, t) \right) = 0.$$

We now define such a compact set. Let C be the set of all diagonal matrices $diag(d_1, \dots, d_n)$ in $\mathrm{End}(\mathbb{R}^n)$ with $0 \leq d_i \leq 1$ for all i and $\sum_i d_i = 1$.
Now C is a compact set with $\operatorname{Tr}(c) = 1$ for all $c \in C$. By definition of C it is clear that for every $a \in A$ there is an element $c \in C$ such that $c \operatorname{Tr}(\lambda(a)) = \lambda(a)$.
We claim: For every $\epsilon > 0$ there exists a number $R_0 > 0$ such that $\zeta(\lambda(a), n + 1) < \epsilon$ for all $a \in D^+$ with $\operatorname{Tr}(a) \geq R_0$. Indeed, for every ϵ there is a number δ_0 such that

$$\zeta(c, \delta) < \epsilon$$

for all $c \in C$, $\delta \leq \delta_0$.
By the linearity of the trace operator, we have

$$\zeta(c,\delta) = \zeta(xc, x\delta)$$

for all $c \in C$ and $x \in \mathbb{R}^+$. Now let $v \in D^+$. Then $v = xc$ with $c \in C$, $x \in \mathbb{R}^+$ and $\operatorname{Tr}(v) = x$. This implies

$$\zeta(v,\delta) = \zeta(xc,\delta) \leq \zeta(\operatorname{Tr}(v)c,\delta) = \zeta(c, \delta/\operatorname{Tr}(v)).$$

Therefore $\zeta(v,\delta) < \epsilon$ whenever $\delta/\operatorname{Tr}(v) < \delta_0$. Thus

$$\zeta(v, n+1) < \epsilon$$

for all $v \in D^+$ with $\operatorname{Tr}(v) > R_0 = (n+1)/\delta_0$.
Now fix a number $\epsilon > 0$. We will demonstrate that there exists a number $R > 0$ such that

$$\frac{\mu(\Delta \cap B_r)}{\mu(B_r)} > 1 - \epsilon$$

for all $r \geq R$.
We start by choosing R_0 such that $\zeta(a, n+1) < \epsilon/2$ for all $a \in A^+ \subset D^+$ with $\operatorname{Tr}(a) \geq R_0$. Recall that $\eta(X) = +\infty$ and that $\theta : X \to \mathbb{R}_0^+$ is an exhaustion function. Hence we may choose a number $R_1 > R_0$ such that

$$\eta(\{x \in X : \theta(x) \leq R_0\}) < \frac{\epsilon}{2}\eta(\{x \in X : \theta(x) \leq R_1\}).$$

Finally choose R such that $\{\theta \leq R_1\} \subset p(B_R)$.
Now we have for $r > R$:

$$\begin{aligned}
\mu(B_r \setminus \Delta) &\leq \mu(\{g \in B_r : \operatorname{Tr}(\lambda(g)) < n+1\}) \\
&= \int_{x \in p(B_r)} \zeta(x, n+1) d\eta \\
&= \int_{\theta(x) \leq R_0} \zeta(x, n+1) d\eta + \int_{R_0 \leq \theta(x), x \in p(B_r)} \zeta(x, n+1) d\eta \\
&< \eta(\{x \in X : \theta(x) \leq R_0\}) + \int_{R_0 \leq \theta(x), x \in p(B_r)} \frac{\epsilon}{2} d\eta \\
&\leq \frac{\epsilon}{2}\eta(\{x \in X : \theta(x) \leq R_0\}) + \frac{\epsilon}{2}\eta(\{x \in p(B_r) : R_0 \leq \theta(x)\}) \\
&= \epsilon\eta(p(B_r)) = \epsilon\mu(B_r).
\end{aligned}$$

Case 2. Assume that Z is finite, but non-trivial. Let $p : G \to G/Z$ denote the natural projection. Since Z is compact and normal, it is contained in every maximal compact subgroup K. Therefore every K-biinvariant exhaustion functions on G is a pull-back of a $p(K)$-biinvariant exhaustion function on G/Z.

Finiteness of Z furthermore implies that $p^{-1}(\Delta_{G/Z}) = \Delta_G$ and that the Haar measure on G/Z pulls back to a Haar measure on G. Therefore the statement of the theorem for this case follows from the proof for case 1.
Case 3. Assume that Z is infinite. In this case Z is not compact. Since Cartan subgroups are maximally nilpotent and therefore necessarily contain Z, this implies that G admits no compact Cartan subgroups. By the results of [3] it follows that in this case $\mu(G \setminus \Delta) = 0$, implying $\mu(\Delta \cap B_r) = \mu(B_r)$ for all $r \in \mathbb{R}^+$.

□

2 Interpretation from a Lie algebra point of view

One may consider the projection $Lie(G) \to \mathbb{P}(Lie(G))$. In the projective space $\mathbb{P}(Lie(G))$ both the set corresponding to compact Cartan subgroups as well as the set corresponding to non-compact Cartan subgroups contains non-empty open sets, if we assume that G is a non compact semisimple Lie group containing a compact Cartan subgroup. In this sense it seems that one the Lie algebra level the set Δ and its complement look as having the same size. How does this reconcile with our result? The answer may be found in the following reasoning: The correspondence between Lie algebra and Lie group is given by the exponential map. However, the exponential map behaves quite differently for compact and non-compact Cartan subgroups: it is injective on non-compact Cartan subgroups and has infinite kernel for compact Cartan subgroups. Thus, multiplicities are quite different for Lie algebra and Lie groups. Taking these multiplicities into account, it appears only reasonable that on the Lie group Δ dominates if both sides have the same size in $\mathbb{P}(Lie(G))$.

3 Explicit calculations for $SL(2, \mathbb{R})$

In this section, we deduce explicit results for the special case $G = SL_2(\mathbb{R})$.
In this case the KAK-decomposition can be written as the map

$$F : S^1 \times \mathbb{R}^{\geq 1} \times S^1 \to SL_2(\mathbb{R})$$

given by

$$F : (\theta, s, \phi) \mapsto \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} s & \\ & s^{-1} \end{pmatrix} \cdot \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$$

Then

$$F^* d\mu = 2\pi \left(s - s^{-3} \right) \frac{d\theta}{2\pi} \wedge ds \wedge \frac{d\phi}{2\pi}$$

for a Haar measure $d\mu$ on G. We can define a K-biinvariant exhaustion function ρ on $SL_2(\mathbb{R})$ by

$$\rho(g) = \max_{v \in \mathbb{R}^2 \setminus \{(0,0)\}} \frac{||g(v)||}{||v||}.$$

Then $\rho(F(\theta, s, \phi)) = s$.
An element $g \in SL_2(\mathbb{R})$ generates a discrete subgroup if and only if it is diagonalizable or unipotent or a torsion element. It follows that $g \in \Delta$ iff $|\operatorname{Tr}(g)| \geq 2$ or $\frac{1}{2}\operatorname{Tr}(g) = \cos(\frac{q}{2\pi})$ for a rational number $q \in \mathbb{Q}$. Hence $\{g : |\operatorname{Tr}(g)| > 2\} \subset \Delta$ and

$$\mu\left(\Delta \setminus \{g : |\operatorname{Tr}(g)| > 2\}\right) = 0.$$

An easy calculation yields

$$\operatorname{Tr}(F(\theta, s, \phi)) = (s + s^{-1}) \cos(\theta + \phi).$$

It follows that

$$1 - \zeta(s, 2) = \xi\left(\{(\theta, \phi) : |\operatorname{Tr}(F(\theta, s, \phi)| > 2\}\right) = 4 \arccos \frac{2}{s + s^{-1}}.$$

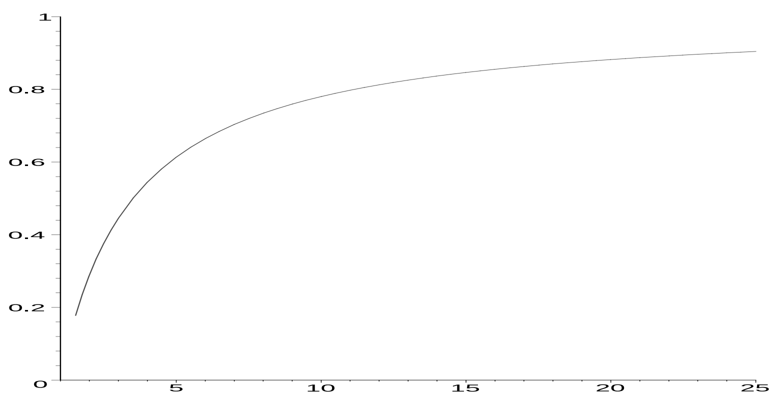
Therefore

$$\mu(B_r) = \int_{s=1}^{r} 2\pi(s - s^{-3}) ds$$

and

$$\mu(B_r \cap \Delta) = \int_{s=1}^{r} 4 \arccos\left(2/(s + s^{-1})\right)\ 2\pi(s - s^{-3}) ds.$$

Using Maple, the graph of the function $f(r) = \mu(B_r \cap \Delta)/\mu(B_r)$ now appears as shown in the graphic below:



References

[1] Cartan, É.: Sur certaines formes riemanniennes remarquables des géométries à groupe fondamental simple. *Annales Sci. École Norm. Sup.* 44, 345–467 (1927)

[2] Knapp, A.W.: Lie Groups Beyond an Introduction. Birkhäuser. 1996.

[3] Winkelmann, J.: Generic subgroups of Lie groups. To appear in *Topology*.

Jörg Winkelmann
Graduate School of Mathematical Sciences
University of Tokyo (Todai)
Komaba, Meguro, Tokyo 153-8914
Japan
jwinkel@member.ams.org