Permanence Properties of the Baum-Connes Conjecture

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Abstract. In this paper we study the stability of the Baum-Connes conjecture with coefficients under various natural operations on the groups. We show that the class of groups satisfying this conjecture is stable under taking subgroups, Cartesian products, and more generally, under certain group extensions. In particular, we show that a group satisfies the conjecture if it has an amenable normal subgroup such that the associated quotient group satisfies the conjecture. We also study a natural induction homomorphism between the topological K-theory of a subgroup H of G and the topological K-theory of G with induced coefficient algebra, and show that this map is always bijective. Using this, we are also able to present new examples of groups which satisfy the conjecture with trivial coefficients.

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0 Introduction.

Given a locally compact group G and a C^*-algebra B equipped with a pointwise continuous action of G by $*$-automorphisms, Baum, Connes and Higson constructed in [2] the topological K-theory $\mathrm{K}_*^{\mathrm{top}}(G;B)$ of G with coefficients in B and an assembly map

$$\mu_{G,B} : \mathrm{K}_*^{\mathrm{top}}(G;B) \to \mathrm{K}_*(B \rtimes_r G).$$

The Baum-Connes conjecture (with coefficients, cf. [2, §9]) asserts that $\mu_{G,B}$ is an isomorphism for all G and for every G-C^*-algebra B. For convenience, we will use the following

Notation. We say that G *satisfies BCC* (Baum-Connes conjecture with coefficients) if $\mu_{G,B}$ is an isomorphism for every G-algebra B. Moreover, we say that G satisfies BCI (resp. BCS) if the assembly map is injective (resp. surjective) for all B. In case we want to specify the coefficient algebra, we simply say that G satisfies BCC (resp. BCI, BCS) for B.

Although BCC has been shown to be true for many groups (for a general overview of recent results we recommend the surveys [24, 28]), it seems now to be clear that there exist examples of groups for which the assembly map is not always surjective (there are counterexamples due to Higson, Lafforgue, Osawa, Skandalis and Yu, which base on a recent announcement by Gromov on the existence of finitely presented groups with certain graph-theoretic properties). However, knowing that the conjecture fails in some cases makes it even more important to be able to describe the class of groups which do satisfy the conjecture. A natural problem in this direction is to investigate how the conjecture behaves under certain standard operations on the group, like passing to (closed) subgroups or taking group extensions.
Partial answers to the extension problem were given in [5], for the case of semi-direct products by a totally disconnected or almost connected group. The argument in [5] is based on the construction of a partial assembly map associated to a semi-direct product, which generalizes and factors the assembly map of the Baum-Connes conjecture.
In [6] we extended the definition of this partial assembly map in order to decompose the assembly map for arbitrary (non-split) group extension: If N is any closed normal subgroup of G and B is a G-algebra, we constructed a natural homomorphism (the partial assembly map)

$$\mu_{N,B}^{G,N} : \mathrm{K}_*^{\mathrm{top}}(G;B) \to \mathrm{K}_*^{\mathrm{top}}(G/N; B \rtimes_r N),$$

which factorizes the assembly map for G in the sense that

$$\mu_{G,B} = \mu_{G/N, B\rtimes_r N} \circ \mu_{N,B}^{G,N},$$

where $\mu_{G/N,B\rtimes_r N}$ denotes the assembly map for G/N with (twisted) coefficient algebra $B \rtimes_r N$. For this construction, we had to use Green's notion of twisted actions which allows to decompose $B\rtimes_r G$ as an iterated twisted crossed product $B\rtimes_r N\rtimes_r G/N$. To make sense of topological K-theory with twisted coefficients, we had to adapt Kasparov's equivariant KK-theory to cover twisted group actions on C^*-algebras. The main results on extensions obtained in [6] are the following: Assume that G has a γ-element (see Definition 1.7 below), and that G/N is either almost connected or totally disconnected. Then G satisfies BCC if G/N and any compact extensions of N in G satisfy BCC.

In this article we want to generalize these results in two directions: Remove the assumption on the topology of the quotient group and lift the requirement that the group has a γ-element. We reached the latter objective in full generality in the case when G/N is totally disconnected (see Theorem 3.3 below), inspired from some ideas exposed in [22], where Oyono-Oyono obtains quite similar results for discrete G. We were also able to reach the other objectives to a very far extend (see the discussion below).

For the investigation of the subgroup problem we study a natural induction homomorphism

$$\operatorname{Ind}_H^G : \mathrm{K}_*^{\mathrm{top}}(H; B) \to \mathrm{K}_*^{\mathrm{top}}(G; \operatorname{Ind}_H^G B),$$

which provides a link between the assembly map for a subgroup H of a group G, with coefficient algebra B and the assembly map for G with coefficients in the induced G-algebra $\operatorname{Ind}_H^G B$ (see Proposition 2.3 below). For discrete G, this map has been studied by Guentner, Higson and Trout in [11] (in the frame of E-theory), where they showed that it is an isomorphism if H is finite. Later, in [21], H. Oyono-Oyono was able to prove the bijectivity of the induction map for arbitrary subgroups of discrete groups. Here we prove that

- The induction homomorphism $\operatorname{Ind}_H^G : \mathrm{K}_*^{\mathrm{top}}(H; B) \to \mathrm{K}_*^{\mathrm{top}}(G; \operatorname{Ind}_H^G B)$ is always bijective (Theorem 2.2).

As a direct consequence we get

- If G satisfies BCC (resp. BCI, BCS), the same is true for every closed subgroup H of G (Theorem 2.5).

Combining this with our previous results on group extensions we are able to make further progress in this direction. In fact we show

- Suppose that N is a closed normal subgroup of G such that N satisfies the Haagerup property (in particular, if N is amenable). Then, if G/N satisfies BCC (resp. BCS), the same is true for G (Corollary 3.14; but see Theorem 3.12 for a more general statement).
- A direct product $G_1 \times G_2$ satisfies BCC if and only if G_1 and G_2 satisfy BCC (Theorem 3.17).

Another application of the bijectivity of the induction homomorphism is given in [7], where it is shown that the generalized Green-Julg theorem (i.e., BCC for *proper* G-algebras) holds for all (second countable) locally compact groups G. Further, in §4 below, we apply the induction isomorphism in a specific example, which hints into the direction of a more general "Mackey-Machine" for the investigation of the Baum-Connes conjecture.

The outline of the paper is as follows: After a short preliminary section (§1), we give a detailed discussion on the induction homomorphism in §2, where we prove all relevant results, except of the bijectivity of this map. In §3 we briefly

discuss the partial assembly map and prove most of our results on group extensions, except of our main technical result on extensions by totally disconnected groups. In §4 we present an example, which illustrates how our results can be used towards a Mackey-Machine approach to the Baum-Connes conjecture. Using some general ideas, we show that for $K = \mathbb{R}$ or $\mathbb{C}$ the Baum-Connes conjecture with trivial coefficients is true for the groups $K^n \rtimes SL_n(K)$, a result which has been known, so far, only for the cases $n \leq 2$.
The most difficult (and technical) results of this paper are the proofs of Theorem 2.2 and Theorem 3.3 on the bijectivity of the induction homomorphism and the bijectivity of the partial assembly map for totally disconnected quotients, respectively. For this reason we decided to devote two extra sections (§5 and §6) to the proofs of these results. There are some substantial similarities in the proofs of these theorems: Both depend deeply on a certain realization of the universal example $\mathcal{E}(G)$ for proper actions of G (which is an important ingredient in the computation of topological K-theory), using the fact that $\mathcal{E}(G)$ can be realized as a simplicial complex if G is totally disconnected. Since the proof of Theorem 3.3 seemed a bit easier (and perhaps more illustrative), we decided to do this result first (§5). Note that the approach via our special realization of $\mathcal{E}(G)$ seems to have a bunch of other important consequences. So, as a further example for the usefulness of this approach, we show in our final section, §7, that the topological K-theory of a group G is continuous in the coefficient algebras, i.e.,

$$\mathrm{K}_*^{\mathrm{top}}(G; \lim_i A_i) = \lim_i \mathrm{K}_*^{\mathrm{top}}(G; A_i)$$

for any inductive limit $\lim_i A_i$ of G-algebras A_i. This result plays an important role in the proof of the generalized Green-Julg theorem given in [7].
In order to avoid unnecessary repetitions, we have chosen to make the following general conventions: ALL C^*-ALGEBRAS (EXCEPT OF MULTIPLIER ALGEBRAS) ARE SUPPOSED TO BE SEPARABLE AND BY GROUP WE MEAN A LOCALLY COMPACT SECOND COUNTABLE HAUSDORFF TOPOLOGICAL GROUP.

1 SOME PRELIMINARIES

Let G be a group. By a *proper G-space* we shall always understand a locally compact space X endowed with an action of G such that the map $G \times X \to X \times X, \quad (g,x) \mapsto (gx, x)$ is continuous and proper (inverse images of compact sets are compact). A *universal example for the proper actions of G, $\mathcal{E}(G)$*, is a proper G-space such that for any other proper G-space Z, there is a continuous and G-equivariant map $F : Z \to \mathcal{E}(G)$ which is unique up to G-equivariant homotopy. Note that $\mathcal{E}(G)$ is uniquely defined up to G-homotopy. The existence of universal proper spaces is shown in [17].
Now let N denote a closed normal subgroup of G. A *twisted action* of (G,N) on a C^*-algebra D (in the sense of Green, [12]) consists of a strongly continuous action by $*$-automorphisms $\alpha : G \to \mathrm{Aut}(D)$ together with a strictly continuous

homomorphism $\tau : N \to UM(D)$ of N into the group of unitaries of the multiplier algebra $M(D)$ of D, such that

$$\alpha_n(d) = \tau_n d \tau_n^* \quad \text{and} \quad \alpha_s(\tau_n) = \tau_{sns^{-1}}, \quad \text{for all } d \in D, s \in G, n \in N.$$

If equipped with such a twisted action, D will be called a (G, N)-algebra. Note that a twisted action of (G, N) should be viewed as a generalization of a G/N-action. In particular, every G/N-algebra can be regarded as a (G, N)-algebra by inflating a given action β of G/N to the twisted action $(\operatorname{Inf} \beta, 1_N)$ of (G, N), and the corresponding twisted crossed products by (G, N) coincide with the ordinary crossed products by G/N. The main advantage of working with twisted actions is, that they allow to decompose crossed products: If B is a G-algebra, then $B \rtimes_r N$ becomes a (G, N)-algebra in a canonical way, so that the iterated (twisted) crossed product $B \rtimes_r N \rtimes_r (G, N)$ is canonically isomorphic to $B \rtimes_r G$. We refer to [6] for more details on these facts and for the construction of the bifunctor $\mathrm{KK}_*^{G,N}(D_1, D_2)$ for pairs (D_1, D_2) of (G, N)-algebras, which extends Kasparov's equivariant $\mathrm{KK}^{G/N}$-theory for G/N-algebras.

Definition 1.1. Let D be a (G, N)-algebra. The *topological* K*-theory of* G/N with coefficient algebra D is

$$\mathrm{K}_*^{\mathrm{top}}(G/N; D) = \lim_{Y} \mathrm{KK}_*^{G,N}\big(C_0(Y), D\big),$$

where the limit is taken over the directed system of all G/N-compact subspaces Y (i.e., $(G/N)\backslash Y$ is compact) of a given universal example $\mathcal{E}(G/N)$ for the proper actions of G/N.

Remark 1.2. In this work we are using a notion of proper G-spaces (resp. G/N-spaces) which differs from the notion used in [2]. This leads to different notions of universal proper G-spaces (resp. G/N-spaces). However, it is shown in [7] that both notions of properness lead to equivalent definitions of the topological K-theory of G (resp. G/N).

If D is a G/N-algebra (viewed as a (G, N)-algebra as explained above), then the above definition of the topological K-theory of G/N with coefficient algebra D coincides with the usual definition of the topological K-theory of G/N with untwisted coefficient algebra D (this follows from [6, Corollary 3.14]). In particular, if $N = \{e\}$ in the above definition, we recover the usual topological K-theory of G with coefficients in the G-algebra D.

For any proper G-space Z, $C_c(Z)$ carries a canonical $C_0(G\backslash Z) - C_c(G \times Z)$ bimodule structure, where we regard $C_c(G \times Z)$ as a dense subalgebra of $C_0(Z) \rtimes G$. This bimodule structure extends to give a $C_0(G\backslash Z) - C_0(Z) \rtimes G$ Hilbert bimodule $\Lambda_{Z,G}$. For reference, the module operations are given on the dense subspaces by

$$
\begin{aligned}
(\varphi \cdot \xi)(z) &= \varphi(Gz)\xi(z)\\
\langle \xi, \eta\rangle(s,z) &= \Delta_G(s)^{-1/2}\overline{\xi(z)}\eta(s^{-1}z)\\
(\xi \cdot f)(z) &= \int_G \xi(s^{-1}z)f(s^{-1}, s^{-1}z)\Delta_G(s)^{-1/2}ds,
\end{aligned}
\tag{1.1}
$$

with $\varphi \in C_0(G\backslash Z)$, ξ, $\eta \in C_c(Z)$, and $f \in C_c(G \times Z)$. Together with the zero operator, we obtain an element (also denoted $\Lambda_{Z,G}$) of the Kasparov group $\mathrm{KK}_0\big(C_0(G\backslash Z), C_0(Z) \rtimes G\big)$ (see [6, §5] for more details). Moreover, if Z is G-compact, i.e., $G\backslash Z$ is compact, then we can pair $\Lambda_{Z,G}$ with the unital homomorphism $\mathbb{C} \to C(G\backslash Z)$ to obtain a canonical element $\lambda_{Z,G} \in \mathrm{K}_0\big(C_0(Z) \rtimes G\big)$. We now recall the definition of the twisted Baum-Connes assembly map as introduced in [6]:

Definition 1.3. Let D be a (G,N)-algebra. The *twisted assembly map* for G/N with coefficients in D, $\mu_{G/N,D} : \mathrm{K}_*^{\mathrm{top}}(G/N; D) \to \mathrm{K}_*\big(D \rtimes_r (G,N)\big)$, is defined inductively by the maps

$$\mu_{G/N,D}[Y] : \mathrm{KK}_*^{G,N}\big(C_0(Y), D\big) \to \mathrm{K}_*\big(D \rtimes_r (G,N)\big),$$

where Y runs through the G/N-compact subspaces of a given realization of $\mathcal{E}(G/N)$ and $\mu_{G/N,D}[Y]$ is defined via the composition of maps

$$
\begin{array}{ccc}
\mathrm{KK}_*^{G,N}\big(C_0(Y), D\big) & \dashrightarrow & \mathrm{K}_*\big(D \rtimes_r (G,N)\big)\\
 & {\scriptstyle j_{N,r}^{G}}\searrow & \uparrow {\scriptstyle \lambda_{Y,G/N}\otimes\cdot}\\
 & & \mathrm{KK}_*\big(C_0(Y) \rtimes (G,N), D \rtimes_r (G,N)\big)
\end{array}
$$

Here $j_{N,r}^G$ denotes the descent in twisted equivariant KK-theory as described in [6, §4].

Remark 1.4. For a G/N-algebra D, viewed as a (G,N)-algebra via inflation, the assembly map of the above definition coincides with the usual Baum-Connes assembly map for G/N with coefficient algebra D. This follows directly from [6, Corollary 3.14]. Of course, if $N = \{e\}$, we get the usual assembly map for G.
It is important to note that by a result of [10], any twisted action of (G,N) is Morita equivalent, and hence $\mathrm{KK}^{G,N}$-equivalent, to an untwisted action of G/N, so that bijectivity, injectivity or surjectivity of the assembly map of Definition 1.3 is equivalent to the corresponding properties of the usual Baum-Connes assembly map for G/N with the corresponding G/N-algebra as coefficient (see [6, 5.6]).

The introduction of twisted coefficients enabled us in [6] to define a partial assembly map for (G,N), which will also play a central role in this paper. We will recall the precise definition of this partial assembly map in §3 below. For its construction we shall need to work with a kind of fundamental class

$$\Lambda_{X,N}^{G,N} \in \mathrm{KK}_0^{G,N}\big(C_0(N\backslash X), C_0(X)\rtimes N\big), \tag{1.2}$$

associated to a proper G-space X, which will play a similar role in the definition of the partial assembly map as the class $\lambda_{Y,G/N}$ in Definition 1.3. We briefly recall its construction: If X is a proper G-space, the given G-action restricts to a proper action of N on X. Thus we can form the $C_0(N\backslash X) - C_0(X)\rtimes N$ bimodule $\Lambda_{X,N}$ as described above. As was shown in [6, §5], there exists a canonical (twisted) action of (G,N) on $\Lambda_{X,N}$, which (again together with the zero operator) provides the element $\Lambda_{X,N}^{G,N}$ of (1.2).

Recall that for two locally compact spaces X and Y, any $*$-homomorphism $\psi : C_0(X)\to C_0(Y)$ is a composition

$$C_0(X) \xrightarrow{\psi_1} C_0(Z) \xrightarrow{\psi_2} C_0(Y),$$

where Z is an open subset of Y, ψ_2 is the canonical inclusion and ψ_1 is induced by a continuous proper map, say $\varphi : Z\to X$, via $\psi_1(f)(z) = f\big(\varphi(z)\big)$, $f\in C_0(X)$. In fact, Z is the open subset of Y corresponding to the ideal $\psi\big(C_0(X)\big)\cdot C_0(Y)\subseteq C_0(Y)$ (note that by an easy application of Cohen's factorization theorem, $\psi\big(C_0(X)\big)\cdot C_0(Y) = \{\psi(f)\cdot g \mid f\in C_0(X), g\in C_0(Y)\}$ is a closed ideal of $C_0(Y)$), and φ is the proper map induced from the non-degenerate $*$-homomorphism $C_0(X)\to \psi\big(C_0(X)\big)\cdot C_0(Y) = C_0(Z); f\mapsto \psi(f)$.

If X and Y are G-spaces and ψ is G-equivariant, then all maps in the above decomposition (and also the map $\varphi : Z\to X$) are G-equivariant. If, moreover, G acts properly on X and Y, and N is a closed normal subgroup of G, then there exist canonical maps

$$C_0(N\backslash X) \xrightarrow{\psi_{1,N}} C_0(N\backslash Z) \xrightarrow{\psi_{2,N}} C_0(N\backslash Y),$$

where the first homomorphism is induced by the proper map

$$\varphi_N : N\backslash Z\to N\backslash X, \quad \varphi_N(Nz) = N\varphi(z),$$

and the second map is induced via the inclusion of the open set $N\backslash Z$ into $N\backslash Y$. Note that the composition $\psi_N := \psi_{2,N}\circ\psi_{1,N}$ satisfies the equation

$$\psi(g\cdot f) = \psi_N(g)\cdot\psi(f), \qquad g\in C_0(N\backslash X), f\in C_0(X).$$

The following lemma will be used frequently throughout this work.

Lemma 1.5 (cf. [6, Lemma 5.13]). *Let $\psi : C_0(X)\to C_0(Y)$ and $\psi_N : C_0(N\backslash X)\to C_0(N\backslash Y)$ be as above. Then*

$$[\Lambda_{X,N}^{G,N}]\otimes j_{\{e\},r}^N([\psi]) = [\psi_N]\otimes[\Lambda_{Y,N}^{G,N}] \quad \textit{in } \mathrm{KK}_0^{G,N}\big(C_0(N\backslash X), C_0(Y)\rtimes N\big),$$

where $j_{\{e\},r}^N : \mathrm{KK}^G\big(C_0(X), C_0(Y)\big)\to \mathrm{KK}^{G,N}\big(C_0(X)\rtimes N, C_0(Y)\rtimes N\big)$ denotes the partial descent of [6, §4]. (Note that by the properness of the N-actions, the maximal crossed products coincide with the reduced crossed products.)

Proof. By the decomposition argument presented above, it is sufficient to prove each of the following special cases:

(1) ψ is induced by a continuous and proper G-map $\varphi : Y \to X$ (as explained above), or

(2) X is an open subset of Y and $\psi : C_0(X) \to C_0(Y)$ is the inclusion.

Since all operators in the Kasparov triples defining the KK-elements of the lemma are the zero operators, it is enough to show that the two $C_0(N\backslash X) - C_0(Y) \rtimes N$ Hilbert bimodules $C_0(N\backslash X) \otimes_{C_0(N\backslash Y)} \Lambda^{G,N}_{Y,N}$ and $\Lambda^{G,N}_{X,N} \otimes_{C_0(X)\rtimes N} \big(C_0(Y) \rtimes N\big)$ are equivariantly isomorphic. Using the formulas for the module operations as given in Equation (1.1) above, we see that $C_0(N\backslash X) \otimes_{C_0(N\backslash Y)} \Lambda^{G,N}_{Y,N}$ is just the closure of $\psi_N\big(C_c(N\backslash X)\big) \cdot C_c(Y) \subseteq \Lambda^{G,N}_{Y,N}$ (pointwise multiplication). Now consider the map

$$\Phi : C_c(X) \odot C_c(N \times Y) \to C_c(Y)$$

defined by

$$\Phi(\xi \otimes f)(y) = \int_N \psi(\xi)(ny) f(n, ny) \Delta_N(n)^{-1/2} dn.$$

A lengthy but straightforward computation shows that Φ is an isometry with respect to the right inner products on $\Lambda^{G,N}_{X,N} \otimes_{C_0(X)\rtimes N} \big(C_0(Y) \rtimes N\big)$ and $\Lambda^{G,N}_{Y,N}$, respectively, and therefore extends to an isometry

$$\Phi : \Lambda^{G,N}_{X,N} \otimes_{C_0(X)\rtimes N} \big(C_0(Y) \rtimes N\big) \to \Lambda^{G,N}_{Y,N}.$$

Factoring $C_c(X)$ as $C_c(N\backslash X) \cdot C_c(X)$, it follows directly from the formula for Φ that it has its image in $\psi_N\big(C_0(N\backslash X)\big) \cdot \Lambda^{G,N}_{Y,N}$. Another short computation shows that Φ respects the module actions and that it is equivariant for the given (G, N)-actions on the modules and algebras (see [6, §5] for the precise formulas of those actions).

Thus the only thing which remains to be checked is the surjectivity of Φ, at least if ψ satisfies either (1) or (2). This is trivial in the case of (2) and we restrict ourselves to (1).

Consider any functions $h \in C_c(N\backslash X)$ and $\eta \in C_c(Y)$. We want to construct $\xi \in C_c(X)$ and $f \in C_c(N \times Y)$ such that $\Phi(\xi \otimes f) = \psi_N(h) \cdot \eta$. For this we choose a function $c : X \to \mathbb{R}$ such that c^2 is a cut-off function for the action of N on X (i.e., the restriction of c^2 to any N-compact subset of X has compact support and $\int_N c^2(nx)dn = 1$ for all $x \in X$). We define $\xi \in C_c(X)$ by $\xi(x) = h(Nx)c(x)$, and we define $f \in C_c(N\times Y)$ by $f(n, ny) = \Delta_N(n)^{1/2} c\big(n\varphi(y)\big)\eta(y)$. Then

$$\begin{aligned}\Phi(\xi \otimes f)(y) &= \int_N \xi\big(n\varphi(y)\big) f(n, ny) \Delta_N(y)^{-1/2} dn \\ &= \int_N h\big(N\varphi(y)\big) c^2\big(n\varphi(y)\big)\eta(y)\, dn \\ &= \big(\psi_N(h) \cdot \eta\big)(y).\end{aligned}$$

This finishes the proof. □

In what follows, we will frequently have to work with the notion of a γ-element, which goes back to the pioneering work of Kasparov [15, 14], and which turned out to be the most important tool for the investigation of the Baum-Connes conjecture. We first have to introduce the notion of proper G-algebras:

DEFINITION 1.6. A G-C^*-algebra $\mathcal{A}$ is called a *proper G-algebra*, if there exists a proper G-space X and a non-degenerate G-equivariant homomorphism $\Phi : C_0(X) \to ZM(\mathcal{A})$, the center of the multiplier algebra of $\mathcal{A}$.

We can now recall the abstract definition of a γ-element:

DEFINITION 1.7 (CF. [25, §5], [14, §3 - 5]). Let G be a group. An element $\gamma_G \in \mathrm{KK}_0^G(\mathbb{C},\mathbb{C})$ is called a *γ-element* for G if

(1) there exists a proper G-algebra $\mathcal{A}$ and (Dirac and dual-Dirac) elements $D \in \mathrm{KK}_0^G(\mathcal{A},\mathbb{C})$, $\eta \in \mathrm{KK}_0^G(\mathbb{C},\mathcal{A})$ such that $\gamma_G = \eta \otimes_{\mathcal{A}} D$;

(2) for any proper G-space Z we have $p^*(\gamma_G) = 1_Z \in \mathcal{R}\mathrm{KK}_0^G\big(Z; C_0(Z), C_0(Z)\big)$, where p maps Z to the one-point set $\{\mathrm{pt}\}$ (see [14, Proposition 2.20]).

REMARK 1.8. If G has a γ-element, then it follows from the work of Kasparov and Tu (see [14, 25]) that G satisfies BCI (i.e., the assembly map $\mu_{G,B}$ is injective for any coefficient algebra B). Moreover, if $\gamma_G = 1 \in \mathrm{KK}_0^G(\mathbb{C},\mathbb{C})$, then G satisfies BCC (we refer to [27] for a concise proof of this result). By a result of Higson and Kasparov ([13], but see also [26]), every group G which satisfies the Haagerup property (in particular every amenable group G) has $1 \in \mathrm{KK}_0^G(\mathbb{C},\mathbb{C})$ as a γ-element, and hence all such groups satisfy BCC for every coefficient algebra B. Moreover, by the work of Kasparov, [15, 14], every group which can be embedded as a closed subgroup of an almost connected group (i.e., a group with compact component group G/G_0) has a γ-element. We refer to [6, §6] for a slightly more detailed account on γ-elements.

2 INDUCTION AND THE BAUM-CONNES CONJECTURE FOR SUBGROUPS

Let H be a closed subgroup of the group G and let B be an H-algebra. In this section we want to discuss the induction homomorphism

$$\mathrm{Ind}_H^G : \mathrm{K}_*^{\mathrm{top}}(H;B) \to \mathrm{K}_*^{\mathrm{top}}(G;\mathrm{Ind}_H^G B)$$

between the topological K-theory of H with coefficients in B and the topological K-theory of G with coefficients in the induced algebra $\mathrm{Ind}_H^G B$. We will then use this homomorphism to show that BCC passes to closed subgroups.
Recall that the induced algebra $\mathrm{Ind}_H^G B$ is defined as

$$\left\{ f \in C_b(G,B) \;\middle|\; \begin{array}{l} h(f(s)) = f(sh^{-1}) \text{ for all } s \in G, h \in H \\ \text{and } sH \mapsto \|f(s)\| \in C_0(G/H) \end{array} \right\}$$

together with the pointwise operations and the supremum norm. If $B = C_0(X)$ is abelian, then $\operatorname{Ind}_H^G C_0(X)$ is canonically isomorphic to $C_0(G \times_H X)$, where $G \times_H X = (G \times X)/H$ (with respect to the diagonal action $h(g,x) = (gh^{-1}, hx)$) denotes the classical induced G-space.
If A and B are two H-algebras, then Kasparov constructed a natural induction homomorphism

$$i_H^G : \mathrm{KK}_*^H(A,B) \to \mathrm{KK}_*^G(\operatorname{Ind}_H^G A, \operatorname{Ind}_H^G B)$$

(see [15, §5] and [14, §3]). Let us briefly recall its construction: Suppose that $x \in \mathrm{KK}_*^H(A,B)$ is represented by a Kasparov triple (E, Φ, T). Similar to the construction of the induced algebras we can form the induced $\operatorname{Ind}_H^G B$-Hilbert module $\operatorname{Ind}_H^G E$ as the set

$$\left\{ \xi \in C_b(G,E) \;\middle|\; \begin{array}{l} h(\xi(s)) = \xi(sh^{-1}) \text{ for all } s \in G, h \in H \\ \text{and } sH \mapsto \|\xi(s)\| \in C_0(G/H) \end{array} \right\},$$

equipped with the pointwise actions and inner products. Pointwise action on the left provides an obvious induced representation $\operatorname{Ind}_H^G \Phi : \operatorname{Ind}_H^G A \to \mathcal{L}(\operatorname{Ind}_H^G E)$. Using a cut-off function $c : G \to [0, \infty[$ for the right translation action of H on G, Kasparov constructs an operator $\tilde{T} \in \mathcal{L}(\operatorname{Ind}_H^G E)$ by the formula:

$$\tilde{T}\xi(g) = \int_H c(gh) h(T(\xi(gh))) dh, \quad \xi \in \operatorname{Ind}_K^G \mathcal{E} \tag{2.1}$$

(see [15, Lemma 2 of §5]), to obtain the Kasparov triple $(\operatorname{Ind}_H^G E, \operatorname{Ind}_H^G \Phi, \tilde{T})$ which represents the element $i_H^G(x) \in \mathrm{KK}_*^G(\operatorname{Ind}_H^G A, \operatorname{Ind}_H^G B)$.
Now suppose that X is an H-compact proper H-space. Then $G \times_H X$ is a G-compact proper G-space and, therefore, there exists a continuous G-map $F : G \times_H X \to \mathcal{E}(G)$ with G-compact image $Y \subseteq \mathcal{E}(G)$. The composition

$$\begin{array}{ccc} \mathrm{KK}_*^H\big(C_0(X), B\big) & \dashrightarrow & \mathrm{KK}_*^G(C_0(Y), \operatorname{Ind}_H^G B) \\ & \searrow^{i_H^G} & \uparrow F^* \\ & & \mathrm{KK}_*^G\big(\operatorname{Ind}_H^G C_0(X), \operatorname{Ind}_H^G B\big) \end{array}$$

provides a well defined homomorphism

$$\operatorname{Ind}_H^G[X] : \mathrm{KK}_*^H(C_0(X), B) \to \mathrm{K}_*^{\mathrm{top}}(G; \operatorname{Ind}_H^G B), \tag{2.2}$$

and it is straightforward to check (using a special case of Lemma 1.5) that the maps $\operatorname{Ind}_H^G[X]$ are compatible with taking inclusions $i : X_1 \to X_2$ (i.e., that $\operatorname{Ind}_H^G[X_2] \circ i^* = \operatorname{Ind}_H^G[X_1]$). Thus, if we let X run through the H-compact subsets of $\mathcal{E}(H)$ we obtain a well defined homomorphism

$$\operatorname{Ind}_H^G : \mathrm{K}_*^{\mathrm{top}}(H; B) \to \mathrm{K}_*^{\mathrm{top}}(G; \operatorname{Ind}_H^G B).$$

Definition 2.1. The homomorphism $\mathrm{Ind}_H^G : \mathrm{K}_*^{\mathrm{top}}(H;B) \to \mathrm{K}_*^{\mathrm{top}}(G;\mathrm{Ind}_H^G B)$ is called the *induction homomorphism* between $\mathrm{K}_*^{\mathrm{top}}(H;B)$ and $\mathrm{K}_*^{\mathrm{top}}(G;\mathrm{Ind}_H^G B)$.

The following theorem is one of the main results of this paper. Since the proof is rather complex and technical, we postpone it to §6 below. For discrete G and finite subgroups H, a similar result (using E-theory) was obtained by Guentner, Higson and Trout in [11], and they asked the question whether the result could be true in more generality. In [21], Oyono-Oyono proves a similar result for arbitrary subgroups of discrete groups.

Theorem 2.2. *Let H be a closed subgroup of G, and let B be an H-algebra. Then the induction map $\mathrm{Ind}_H^G : \mathrm{K}_*^{\mathrm{top}}(H;B) \to \mathrm{K}_*^{\mathrm{top}}(G;\mathrm{Ind}_H^G B)$ is an isomorphism.*

The above theorem has many interesting consequences. It provides a connection between the Baum-Connes conjectures for G and H, as we shall study in more details below. It also allows to prove the fact that the Baum-Connes assembly map

$$\mu_{G,\mathcal{A}} : \mathrm{K}_*^{\mathrm{top}}(G;\mathcal{A}) \to K_*(\mathcal{A} \rtimes_r G)$$

is an isomorphism whenever $\mathcal{A}$ is a proper G-algebra, as is worked out in more detail in [7]. Note that this was an open question for quite some time, and was only known for discrete groups by the work of Guentner, Higson and Trout in [11]. Another use of this isomorphism theorem will be indicated in §4 below.

Proposition 2.3. *Let H be a closed subgroup of the group G, and let B be an H-algebra. Let $x \in \mathrm{KK}_0\big(B \rtimes_r H, (\mathrm{Ind}_H^G B) \rtimes_r G\big)$ be defined by the canonical Morita equivalence between $(\mathrm{Ind}_H^G B) \rtimes_r G$ and $B \rtimes_r H$ (e.g., see [12, Theorem 17]). Then the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{K}_*^{\mathrm{top}}(H;B) & \xrightarrow{\mu_{H,B}} & \mathrm{K}_*(B \rtimes_r H) \\ {\scriptstyle \mathrm{Ind}_H^G}\Big\downarrow & & \cong\Big\downarrow{\scriptstyle \cdot\otimes x} \\ \mathrm{K}_*^{\mathrm{top}}(G;\mathrm{Ind}_H^G B) & \xrightarrow{\mu_{G,\mathrm{Ind}_H^G B}} & \mathrm{K}_*\big((\mathrm{Ind}_H^G B) \rtimes_r G\big). \end{array}$$

For the proof we need

Lemma 2.4. *Let H be a closed subgroup of G. If $\mathcal{E}(G)$ is a universal example for the proper actions of G, then, by restricting the action to H, it is also a universal example for the proper actions of H.*

Proof. Since $\mathcal{E}(G)$ is unique up to G-homotopy (which certainly implies H-homotopy), it is sufficient to show that the result holds for one particular realization of $\mathcal{E}(G)$. By [17], a realization can be constructed as follows: Choose any proper G-space Z and let $\mathcal{E}(G)$ be the set of positive Radon-measures on Z with total mass in the half open interval $]\frac{1}{2},1]$, equipped with the weak$*$-topology and the canonical G-action. Now since the action of G on Z restricts to a proper action of H on Z, the same set of Radon measures provides a realization of $\mathcal{E}(H)$. □

Proof of Proposition 2.3. By the definition of Ind_H^G and Lemma 2.4, it is enough to show that, for a given realization of $\mathcal{E}(G)$, the diagram

$$\begin{array}{ccc}
\mathrm{KK}_*^H\big(C_0(X), B\big) & \xrightarrow{[\Lambda_{X,H}]\otimes j^H_{\{e\},r}(\cdot)} & \mathrm{K}_*(B \rtimes_r H) \\
\big\downarrow{\scriptstyle \mathrm{Ind}_H^G} & & \big\downarrow{\scriptstyle \cdot\otimes x} \\
\mathrm{KK}_*^G\big(C_0(G\times_H X), \mathrm{Ind}_H^G B\big) & \xrightarrow{[\Lambda_{G\times_H X,G}]\otimes j^G_{\{e\},r}(\cdot)} & \mathrm{K}_*\big((\mathrm{Ind}_H^G B)\rtimes_r G\big) \\
\big\downarrow{\scriptstyle F^*} & & \big\downarrow{\scriptstyle =} \\
\mathrm{KK}_*^G\big(C_0(G\cdot X), \mathrm{Ind}_H^G B\big) & \xrightarrow{[\Lambda_{G\cdot X,G}]\otimes j^G_{\{e\},r}(\cdot)} & \mathrm{K}_*\big((\mathrm{Ind}_H^G B)\rtimes_r G\big)
\end{array} \tag{2.3}$$

commutes, where X is any H-compact subset X of $\mathcal{E}(G)$ (also serving as a universal example for proper actions of H). An easy application of Lemma 1.5 implies that the bottom square commutes, so we may restrict our attention to the upper square.

Let y denote the invertible element of $\mathrm{KK}_0\big(C_0(G\times_H X)\rtimes G, C_0(X)\rtimes H\big)$ which is implemented by the canonical Morita equivalence between $C_0(G\times_H X)\rtimes G$ and $C_0(X)\rtimes H$. Then it follows from [14, corollary on p. 176]) that the square

$$\begin{array}{ccc}
\mathrm{KK}_*^H\big(C_0(X), B\big) & \xrightarrow{j^H_{\{e\},r}} & \mathrm{KK}_*\big(C_0(X)\rtimes H, B\rtimes_r H\big) \\
\big\downarrow{\scriptstyle \mathrm{Ind}_H^G} & & \big\downarrow{\scriptstyle y\otimes\cdot\otimes x} \\
\mathrm{KK}_*^G\big(C_0(G\times_H X), \mathrm{Ind}_H^G B\big) & \xrightarrow{j^G_{\{e\},r}} & \mathrm{KK}_*\big(C_0(G\times_H X)\rtimes G, (\mathrm{Ind}_H^G B)\rtimes_r G\big)
\end{array}$$

commutes. So the commutativity of (2.3) will follow if we can show that

$$\begin{array}{ccc}
\mathrm{KK}_*\big(C_0(X)\rtimes H, B\rtimes_r H\big) & \xrightarrow{y\otimes\cdot\otimes x} & \mathrm{KK}_*\big(C_0(G\times_H X)\rtimes G, (\mathrm{Ind}_H^G B)\rtimes_r G\big) \\
\big\downarrow{\scriptstyle [\Lambda_{X,H}]\otimes\cdot} & & \big\downarrow{\scriptstyle [\Lambda_{G\times_H X,G}]\otimes\cdot} \\
\mathrm{KK}_*\big(C_0(H\backslash X), B\rtimes_r H\big) & \xrightarrow[\cdot\otimes x]{} & \mathrm{KK}_*\big(C_0(H\backslash X), (\mathrm{Ind}_H^G B)\rtimes_r G\big)
\end{array}$$

commutes. For this it is enough to prove that

$$[\Lambda_{G\times_H X,G}]\otimes y = [\Lambda_{X,H}] \qquad \text{in } \mathrm{KK}_0\big(C_0(H\backslash X), C_0(X)\rtimes H\big), \tag{2.4}$$

where we identify $G\backslash(G\times_H X)$ with $H\backslash X$ via $G[x,s]\mapsto Hx$. All KK-classes appearing in this equation are given by a Hilbert bimodule together with the zero operator: $\Lambda_{G\times_H X,G}$ (resp. $\Lambda_{X,H}$) is the Hilbert module obtained by taking the completion of $C_c(G\times_H X)$ (resp. $C_c(X)$) with respect to the inner product, the right action of $C_0(G\times_H X)\rtimes G$ (resp. $C_0(X)\rtimes H$) and the left action of $C_0\big(G\backslash(G\times_H X)\big)$ (resp. $C_0(H\backslash X)$) as given in (1.1). The underlying module M for y is obtained by taking the completion of the $C_c\big(G, C_0(G\times_H X)\big)$ – $C_c\big(H, C_0(X)\big)$ bimodule $C_c\big(G, C_0(X)\big)$ with respect to the formulas:

$$\begin{aligned}(\varphi\cdot\eta)(s)(x) &= \int_G \varphi(t)(s,x)\eta(t^{-1})(x)dt\\ \langle\eta_1,\eta_2\rangle(u)(x) &= \int_G \Delta_G(t)^{1/2}\Delta_H(t)^{-1/2}\overline{\eta_1(t)(x)}\eta_2(tu)(u^{-1}x)dt\\ (\eta\cdot f)(s)(x) &= \Delta_G(s)^{1/2}\Delta_H(s)^{-1/2}\int_H \eta(su)(u^{-1}x)f(u^{-1})(u^{-1}x)du,\end{aligned}$$

where $\varphi\in C_c\big(G, C_0(G\times_H X)\big)$, $\eta_1,\eta_2\in C_c\big(G, C_0(X)\big)$ and $f\in C_c\big(H, C_0(X)\big)$. Consider the assignment $\Phi:\ C_c(G\times_H X)\odot C_c\big(G, C_0(X)\big)\to C_c(X)$ defined by

$$\Phi(\xi\otimes\eta)(x)=\int_G \xi(t^{-1},x)\eta(t^{-1})(x)\Delta_G(t)^{-1/2}dt,$$

$\xi\in C_c(G\times_H X)$, $\eta\in C_c\big(G, C_0(X)\big)$. One can check that this map extends to a well defined morphism $\Phi:\ \Lambda_{G\times_H X,G}\otimes_{C_0(G\times_H X)\rtimes G} M\to\Lambda_{X,H}$ which respects the corresponding left and right actions and which is isometric with respect to the $C_0(X)\rtimes H$-valued inner products. To see that it is also surjective let $c: G\cdot X\to[0,\infty[$ be a continuous function such that c^2 is a cut-off function for the proper G-space $G\cdot X$ (cf. proof of Lemma 1.5). Set $\xi(s,x)=c(sx)$. For any $\zeta\in C_c(X)$, set $\eta(s)(x)=c(sx)\zeta(x)\Delta_G(s)^{1/2}$. Then $\zeta=\Phi(\xi\otimes\eta)\in\Lambda_{X,H}$. This proves (2.4). □

As a direct consequence of Theorem 2.2 and Proposition 2.3 we get

Theorem 2.5. *Let H be a closed subgroup of G and let B be an H-algebra. Then the following statements are equivalent:*

(i) H satisfies BCC (resp. BCI, resp. BCS) for B;

(ii) G satisfies BCC (resp. BCI, resp. BCS) for $\mathrm{Ind}_H^G B$.

In particular, if G satisfies BCC (resp. BCI, resp. BCS) for all coefficients, the same is true for H.

We say that a group G satisfies the Baum-Connes conjecture with *abelian coefficients* if the assembly map

$$\mu_{G,A}: \mathrm{K}_*^{\mathrm{top}}(G;A)\to K_*(A\rtimes_r G)$$

is an isomorphism for every commutative C^*-algebra A. Since any commutative H-algebra induces to a commutative G-algebra, we get the following direct corollary of Theorem 2.5:

Corollary 2.6. *Let H be a closed subgroup of G. If G satisfies the Baum-Connes conjecture with abelian coefficients, then the same is true for H.*

3 The Baum-Connes conjecture for group extensions

In this section we want to present our new results on the stability of the Baum-Connes conjecture for group extensions. For this we have to recall from [6] the definition of the partial assembly map

$$\mu_{N,B}^{G,N} : \mathrm{K}_*^{\mathrm{top}}(G;B) \to \mathrm{K}_*^{\mathrm{top}}(G/N; B \rtimes_r N),$$

where $B \rtimes_r N$ is equipped with the decomposition twisted action of (G,N). Let B be any G-algebra and let X be a G-compact proper G-space. Let

$$\Lambda_{X,N}^{G,N} \in \mathrm{KK}_0^{G,N}\left(C_0(N\backslash X), C_0(X) \rtimes N\right)$$

be the fundamental class associated to X as described in §1, (1.2). The composition of maps

$$\begin{array}{ccc} \mathrm{KK}_*^{G}\big(C_0(X), B\big) & \dashrightarrow & \mathrm{KK}_*^{G,N}\big(C_0(N\backslash X), B \rtimes_r N\big) \\ & \searrow^{j_{\{e\},r}^N} & \uparrow{\scriptstyle \Lambda_{X,N}^{G,N}\otimes\cdot} \\ & & \mathrm{KK}_*^{G,N}\big(C_0(X) \rtimes N, B \rtimes_r N\big) \end{array}$$

determines a map

$$\nu[X] : \mathrm{KK}_*^{G}\left(C_0(X), B\right) \to \mathrm{KK}_*^{G,N}\left(C_0(N\backslash X), B \rtimes_r N\right). \qquad (3.1)$$

Now observe that if X is a proper G-space, then $N\backslash X$ is a proper G/N-space, and therefore there exists a homotopically unique continuous G/N-equivariant map $F : N\backslash X \to \mathcal{E}(G/N)$. In particular, there exists a homotopically unique continuous G/N-map $F : N\backslash\mathcal{E}(G) \to \mathcal{E}(G/N)$.

Definition 3.1. Let $F : N\backslash\mathcal{E}(G) \to \mathcal{E}(G/N)$ be as above. For each G-compact subset $X \subseteq \mathcal{E}(G)$ let

$$\mu_{N,B}^{G,N}[X] = F^* \circ \nu[X] : \ \mathrm{KK}_*^{G}\left(C_0(X), B\right) \to \mathrm{KK}_*^{G,N}\left(C_0\big(F(N\backslash X)\big), B \rtimes_r N\right).$$

Then it follows from Lemma 1.5 that the maps $\mu_{N,B}^{G,N}[X]$ are compatible with respect to taking inclusions, and, therefore, they determine a well defined homomorphism

$$\mu_{N,B}^{G,N} : \mathrm{K}_*^{\mathrm{top}}(G;B) \to \mathrm{K}_*^{\mathrm{top}}(G/N; B \rtimes_r N).$$

$\mu_{N,B}^{G,N}$ is called the *partial assembly map* for (G,N) with coefficient algebra B.

The following result was one of the main outcomes of [6], and it is central for the investigations in this section. Recall that if B is a G-algebra and N is a closed normal subgroup of G, then $B \rtimes_r G$ is canonically isomorphic to $B \rtimes_r N \rtimes_r (G,N)$.

Proposition 3.2 (cf. [6, Proposition 5.15]). *The diagram*

$$\begin{array}{ccc} \mathrm{K}_*^{\mathrm{top}}(G;B) & \xrightarrow{\mu_{G,B}} & \mathrm{K}_*(B \rtimes_r G) \\ \mu^{G,N}_{N,B}\big\downarrow & & \big\uparrow\cong \\ \mathrm{K}_*^{\mathrm{top}}(G/N; B\rtimes_r N) & \xrightarrow{\mu_{G/N,B\rtimes_r N}} & \mathrm{K}_*\big((B\rtimes_r N)\rtimes_r (G,N)\big) \end{array}$$

commutes. Thus, if the partial assembly map

$$\mu^{G,N}_{N,B} : \mathrm{K}_*^{\mathrm{top}}(G;B) \to \mathrm{K}_*^{\mathrm{top}}(G/N; B \rtimes_r N)$$

is bijective, then G satisfies BCC (resp. BCI, resp. BCS) for B if and only if G/N satisfies BCC (resp. BCI, resp. BCS) for $B \rtimes_r N$.

Proposition 3.2 gives a strong motivation to study the conditions under which the partial assembly map is an isomorphism. The main technical result in this direction is the following theorem. The proof will be given in §5 below.

Theorem 3.3. *Let $1 \to N \to G \xrightarrow{q} G/N \to 1$ be an extension of groups such that G/N has a compact open subgroup. Let B be any G-algebra and assume that for every compact open subgroup $\dot{K}$ of G/N, the subgroup $q^{-1}(\dot{K})$ of G satisfies BCC with coefficients in B. Then the partial assembly map $\mu^{G,N}_{N,B} : \mathrm{K}_*^{\mathrm{top}}(G;B) \to \mathrm{K}_*^{\mathrm{top}}(G/N; B \rtimes_r N)$ is bijective.*

As a direct corollary of the theorem and of Proposition 3.2 we get:

Corollary 3.4. *Assume that G, N, G/N and B satisfy all assumptions of Theorem 3.3. Then G satisfies BCC (resp. BCI, resp. BCS) for B if and only if G/N satisfies BCC (resp. BCI, resp. BCS) for $B \rtimes_r N$.*

Note that we obtained a similar result in [6, Proposition 7.8] under the additional assumption that G has a γ-element. Although the proof of that special case is easier than the proof of the above result, it can be quite difficult to check the existence of a γ-element in practice. For discrete G, a similar result (without requiring a γ-element) has been obtained by Oyono-Oyono in [22], and the proof of Theorem 3.3, as presented in §5 below, is partly inspired by the ideas of [22].
In [8, Example 3, §5.1] it is shown that if N is a normal subgroup of a group K such that K/N is compact, and if N satisfies the Haagerup property, then K satisfies the Haagerup property. We mentioned earlier (see Remark 1.8) that it follows from the work of Higson, Kasparov and Tu [13, 25] that such groups satisfy BCC. Thus we get

Corollary 3.5. *Let $1 \to N \to G \xrightarrow{q} G/N \to 1$ be an extension of groups such that G/N has a compact open subgroup. Suppose further that N satisfies the Haagerup property (e.g., if N is amenable). Then, if B is a G-algebra, G satisfies BCC (resp. BCI, resp. BCS) for B if and only if G/N satisfies BCC (resp. BCI, resp. BCS) for $B \rtimes_r N$.*

Let G be a group and let G_0 denote the connected component of the identity of G. Then G_0 is a closed normal subgroup of G and G/G_0 is totally disconnected. Thus we may apply the above results to the extension $1 \to G_0 \to G \to G/G_0 \to 1$. In particular, we get

Corollary 3.6. *Assume that G_0 satisfies the Haagerup property and let B be a G-algebra. Then G satisfies BCC (resp. BCI, resp. BCS) for B if and only if G/G_0 satisfies BCC (resp. BCI, resp. BCS) for $B \rtimes_r G_0$.*

In what follows we want to get rid of the condition of G/N having a compact open subgroup. It turns out that, at least if we restrict our attention to property BCC or BCS, we can indeed obtain very far reaching generalizations. A very important tool for this is the use of Theorem 2.2 and its consequences as described in the previous section. The main idea is to reduce to the two cases where the quotient group is either totally disconnected or almost connected (i.e., $(G/N)/(G/N)_0$ is compact). The first case is the one treated above, and the second case was treated in [6] (under the assumption that G has a γ-element). In fact, combining [6, (2) of Proposition 6.7] (note that the injectivity condition in that statement is satisfied by Remark 1.8 if G/N is almost connected) with [6, Proposition 7.6], we get

Theorem 3.7. *Let $1 \to N \to G \overset{q}{\to} G/N \to 1$ be an extension of groups such that G/N is almost connected and such that G has a γ-element. Let B be a G-algebra, and assume that for the maximal compact subgroup $\dot{K}$ of G/N, the group $q^{-1}(\dot{K})$ satisfies BCC for B. Then the partial assembly map $\mu_{N,B}^{G,N} : \mathrm{K}_*^{\mathrm{top}}(G;B) \to \mathrm{K}_*^{\mathrm{top}}(G/N; B \rtimes_r N)$ of Definition 3.1 is bijective. It then follows that G satisfies BCC (resp. BCI, resp. BCS) for B if and only if G/N satisfies BCC (resp. BCI, resp. BCS) for $B \rtimes_r N$.*

As indicated above, we want to combine Theorem 3.7 with Theorem 3.3 in order to cover arbitrary quotients G/N. But before we do this, we want to weaken the assumption on the γ-element in the above theorem. This is done in Lemma 3.9 below, where we show that it is actually enough to assume the existence of a γ-element for the inverse image $K = q^{-1}(\dot{K}) \subseteq G$ of the maximal compact subgroup $\dot{K}$ of G/N. But for the proof of this, we first need another lemma.

For notation: If A is a C^*-algebra and X is a locally compact space, we will write $A(X) := A \otimes C_0(X)$. If A is a G-algebra and X is a G-space, then $A(X)$ carries the diagonal action. Recall that if X is a G-space, K is a closed subgroup of G, and A is a K-algebra, then $\mathrm{Ind}_K^G(A(X)) \cong \big(\mathrm{Ind}_K^G A\big)(X)$ (cf. [14, 3.6]). In fact, both algebras can be viewed as a subalgebra of $C_b(G \times X, A)$: The elements $F \in \mathrm{Ind}_K^G(A(X))$ satisfy the equation $F(gk,x) = k^{-1}\big(F(g,kx)\big)$ and the elements $G \in \big(\mathrm{Ind}_K^G A\big)(X)$ satisfy the equation $G(gk,x) = k^{-1}\big(G(g,x)\big)$. It is then easy to check that

$$\Phi : \mathrm{Ind}_K^G(A(X)) \to \big(\mathrm{Ind}_K^G A\big)(X);\ \Phi(F)(g,x) = F(g, g^{-1}x) \qquad (3.2)$$

is the desired isomorphism.

Lemma 3.8. *Let K be a closed subgroup of G, and let A and B be two K-algebras. Let X be a locally compact G-space. Then the following diagram commutes*

$$\begin{array}{ccc}
\mathrm{KK}_*^K(A,B) & \xrightarrow{\;i_K^G\;} & \mathrm{KK}_*^G(\mathrm{Ind}_K^G A, \mathrm{Ind}_K^G B) \\
\downarrow{\scriptstyle p_X^*} & & \downarrow{\scriptstyle p_X^*} \\
\mathcal{R}\mathrm{KK}_*^K\big(X; A(X), B(X)\big) & & \mathcal{R}\mathrm{KK}_*^G\big(X; (\mathrm{Ind}_K^G A)(X), (\mathrm{Ind}_K^G B)(X)\big) \\
\downarrow{\scriptstyle i^{K,G}} & \nearrow & \\
\mathcal{R}\mathrm{KK}_*^G\big(G \times_K X; \mathrm{Ind}_K^G(A(X)), \mathrm{Ind}_K^G(B(X))\big) & &
\end{array}$$

where p_X^* is induced by the map $p_X : X \to \{pt\}$ (see [14, Proposition 2.20]), i_K^G is the induction morphism defined in [15, Theorem 1 of §5] (see §2 above), $i^{K,G}$ is the induction morphism defined in [14, §3.6], and the bottom slant arrow is obtained by first identifying $G \times_K X$ with $G/K \times X$ via $[g,x] \mapsto (gK, gx)$, then forgetting the action of $C_0(G/K)$, and eventually identifying $\mathrm{Ind}_K^G(A(X))$ with $(\mathrm{Ind}_K^G A)(X)$ (resp. $\mathrm{Ind}_K^G(B(X))$ with $(\mathrm{Ind}_K^G B)(X)$) via the isomorphism of (3.2) above.

Proof. Let $(\mathcal{E}, \Phi, T)$ be any cycle in $\mathbb{E}^K(A,B)$. By construction, the image of the class of this cycle by $p_X^* \circ i_K^G$ is the class in $\mathcal{R}\mathrm{KK}_*^G\big(X; (\mathrm{Ind}_K^G A)(X), (\mathrm{Ind}_K^G B)(X)\big)$ of the cycle

$$P = \big((\mathrm{Ind}_K^G \mathcal{E}) \otimes C_0(X), (\mathrm{Ind}_K^G \Phi) \otimes 1, \tilde{T} \otimes 1\big), \tag{3.3}$$

where the action of $C_0(X)$ is given via the natural inclusions

$$C_0(X) \to M\big((\mathrm{Ind}_K^G A) \otimes C_0(X)\big),\ M\big((\mathrm{Ind}_K^G B) \otimes C_0(X)\big); \quad f \mapsto 1 \otimes f,$$

and where the operator $\tilde{T}$ on $\mathrm{Ind}_K^G \mathcal{E}$ is given by Equation (2.1).
On the other hand, the composition $i^{K,G} \circ p_X^*$ maps the class of $(\mathcal{E}, \Phi, T)$ to the class of the triple

$$Q = \big(\mathrm{Ind}_K^G(\mathcal{E} \otimes C_0(X)), \mathrm{Ind}_K^G(\Phi \otimes 1), \widetilde{T \otimes 1}\big),$$

where the $C_0(G \times_K X)$-actions on the algebras $\mathrm{Ind}_K^G(A \otimes C_0(X))$ (resp. $\mathrm{Ind}_K^G(B \otimes C_0(X))$) are given by

$$(\varphi \cdot F)(g,x) = \varphi([g,x])F(g,x), \text{ for all } \varphi \in C_0(G \times_K X),\ F \in \mathrm{Ind}_K^G(A \otimes C_0(X))$$

(resp. $F \in \mathrm{Ind}_K^G(B \otimes C_0(X))$). We now apply the isomorphism (3.2) to the algebras $\mathrm{Ind}_K^G(A \otimes C_0(X))$ and $\mathrm{Ind}_K^G(B \otimes C_0(X))$. The same formula provides an isomorphism of Hilbert modules $\Psi : \mathrm{Ind}_K^G(\mathcal{E} \otimes C_0(X)) \to (\mathrm{Ind}_K^G \mathcal{E}) \otimes C_0(X)$. If we now identify $G \times_K X$ with $G/K \times X$ as in the lemma, and if we then forget the $C_0(G/K)$ action on the algebras, then a straightforward computation shows that these isomorphisms turn Q into the cycle P of (3.3).

□

Lemma 3.9. *Let $1 \to N \to G \xrightarrow{q} G/N \to 1$ be an extension of groups such that G/N is almost connected. Let $\dot{K}$ be the maximal compact subgroup of G/N and let $K = q^{-1}(\dot{K})$. Assume that K has a γ-element. Then G has a γ-element.*

Proof. Let $\mathcal{A}_K$, (resp. D_K, η_K) be the proper algebra (resp. the Dirac, dual-Dirac element) associated to γ_K, the γ-element of K. By Remark 1.8, G/N also has a γ-element, and we write $\gamma_{G/N}$, $\mathcal{A}_{G/N}$, $D_{G/N}$ and $\eta_{G/N}$ accordingly. Let V be the tangent space of G/K at the point eK and let C_V be the Clifford algebra of V. When viewed as a G-algebra (with trivial N-action), the algebra $\mathcal{A}_{G/N}$ is given by $\mathcal{A}_{G/N} = \mathrm{Ind}_K^G C_V$ (denoted $C_\tau(G/K)$ in [14]).
Define

$$D_G := i_K^G\big(\sigma_{C_V}(D_K)\big) \otimes_{\mathcal{A}_{G/N}} D_{G/N} \in \mathrm{KK}_0^G\left(\mathrm{Ind}_K^G(\mathcal{A}_K \otimes C_V), \mathbb{C}\right) \quad \text{and}$$
$$\eta_G := \eta_{G/N} \otimes_{\mathcal{A}_{G/N}} i_K^G\big(\sigma_{C_V}(\eta_K)\big) \in \mathrm{KK}_0^G\left(\mathbb{C}, \mathrm{Ind}_K^G(\mathcal{A}_K \otimes C_V)\right),$$

where $D_{G/N}$ and $\eta_{G/N}$ are viewed as elements of the respective KK^G-groups by inflating the actions of G/N to G, and $i_K^G : \mathrm{KK}_*^K(A,B) \to \mathrm{KK}_*^G(\mathrm{Ind}_K^G A, \mathrm{Ind}_K^G B)$ denotes Kasparov's induction homomorphism. Since $\mathcal{A}_K$ is K-proper, the algebra $\mathrm{Ind}_K^G(\mathcal{A}_K \otimes C_V)$ is G-proper.
Thus, to see that $\gamma_G = \eta_G \otimes D_G \in \mathrm{KK}_0^G(\mathbb{C},\mathbb{C})$ is a γ-element for G (cf. Definition 1.7), it suffices to check that $p_X^*(\gamma_G) = 1 \in \mathcal{R}\mathrm{KK}_0^G\left(X; C_0(X), C_0(X)\right)$ for every proper G-space X, where p_X denotes the map from X to the one-point set.
Let $z = \sigma_{C_V}(\gamma_K) \in \mathrm{KK}_0^K(C_V, C_V)$. Since X is a proper K-space, it follows that $p_X^*(\gamma_K) = 1$, and, therefore, that $p_X^*(z) = 1 \in \mathcal{R}\mathrm{KK}^K\left(X; C_V(X), C_V(X)\right)$. It then follows that $i^{K,G} \circ p_X^*(z) = 1$, and hence, by Lemma 3.8, that

$$p_X^* \circ i_K^G\big(\sigma_{C_V}(\gamma_K)\big) = p_X^* \circ i_K^G(z) = 1 \in \mathcal{R}\mathrm{KK}^G\big(X; (\mathrm{Ind}_K^G C_V)(X), (\mathrm{Ind}_K^G C_V)(X)\big).$$

On the other hand, the map $p_X : X \to \{\mathrm{pt}\}$ factors G/N-equivariantly through $N\backslash X$, which is a proper G/N-space. Thus, $p_X^*(\gamma_{G/N}) = 1 \in \mathcal{R}\mathrm{KK}_0^G\left(X; C_0(X), C_0(X)\right)$. But this implies

$$\begin{aligned} p_X^*(\gamma_G) &= p_X^*\big(\eta_{G/N} \otimes i_K^G\big(\sigma_{C_V}(\gamma_K)\big) \otimes D_{G/N}\big) \\ &= p_X^*\big(\eta_{G/N}\big) \otimes p_X^*\big(i_K^G(\sigma_{C_V}(\gamma_K))\big) \otimes p_X^*\big(D_{G/N}\big) \\ &= p_X^*\big(\eta_{G/N}\big) \otimes p_X^*\big(D_{G/N}\big) = p_X^*(\gamma_{G/N}) = 1. \end{aligned}$$

□

As a consequence of the above lemma and of Theorem 3.7 we obtain

Corollary 3.10. *Assume that $1 \to N \to G \xrightarrow{q} G/N \to 1$ is a group extension such that G/N is almost connected. Assume further that the inverse image $K = q^{-1}(\dot{K}) \subseteq G$ of the maximal compact subgroup $\dot{K}$ of G/N has a γ-element and satisfies BCC for the given G-algebra B (which is always true if N has the Haagerup property). Then G satisfies BCC (resp. BCI, resp. BCS) for B if and only if G/N satisfies BCC (resp. BCI, resp. BCS) for $B \rtimes_r N$.*

The following proposition is the main step for linking our previous results in order to cover general quotients G/N:

Proposition 3.11. *Let $1 \to N \to G \xrightarrow{q} G/N \to 1$ be an extension of groups and let B be a G-algebra. Denote by $(G/N)_0$ the connected component of the identity in G/N and let $M := q^{-1}\big((G/N)_0\big) \subseteq G$. If $\ddot{H}$ is a compact subgroup of G/M, we denote by $\dot{H} \subset G/N$ the inverse image of $\ddot{H}$ in G/N, and we let H denote the inverse image of $\ddot{H}$ in G. Then the following are true:*

(i) If for every $\ddot{H}$ as above, the group H satisfies BCC for B, then the partial assembly map for (G, M) with coefficients in B is bijective.

(ii) If G/N satisfies BCC (resp. BCS) for every coefficient algebra, then G/M satisfies BCC (resp. BCS) for $B \rtimes_r M$.

Proof. For (i), let us consider the extension $1 \to M \to G \xrightarrow{p} G/M \to 1$. Note that G/M is isomorphic to $(G/N)/(M/N) \cong (G/N)/(G/N)_0$, and hence it is totally disconnected. The condition in (i) is then precisely what we need to apply Theorem 3.3 to this extension.
For (ii), we write G/N as an extension of M/N by G/M:

$$1 \to M/N \to G/N \xrightarrow{r} G/M \to 1. \tag{3.4}$$

Note that G/M is totally disconnected and that the crossed product $B \rtimes_r M$ is isomorphic to $(B \rtimes_r N) \rtimes_r (M, N)$. Thus, applying Proposition 3.2 to extension (3.4), the result will follow if the partial assembly map corresponding to (3.4) with coefficients in $B \rtimes_r N$ is a bijection: If G/N satisfies BCC (resp. BCS) for the coefficient algebra $B \rtimes_r N$, G/M will satisfy BCC (resp. BCS) for the crossed product $B \rtimes_r M \cong (B \rtimes_r N) \rtimes_r (M, N)$.
To see that the partial assembly map for (3.4) with coefficients in $B \rtimes_r N$ is bijective, we apply Theorem 3.3 to this extension. It then follows that it is enough to check that, whenever $\ddot{H}$ is a compact subgroup of G/M, the group $\dot{H} \subseteq G/N$ satisfies BCC for $B \rtimes_r N$. We do this by using the hereditarity result of Theorem 2.5: $\dot{H}$ is a subgroup of G/N which is assumed to satisfy (at least) BCS with arbitrary coefficients. Thus, Theorem 2.5 implies that $\dot{H}$ satisfies (at least) BCS, too. Since $\dot{H}$ is almost connected (as a compact extension of the connected group $M/N \cong (G/N)_0$), it also satisfies BCI for arbitrary coefficient algebras by Remark 1.8. ☐

We now formulate and prove our extension result for arbitrary quotients G/N.

Theorem 3.12. *Let $1 \to N \to G \xrightarrow{q} G/N \to 1$ be an extension of groups and let B be any G-algebra. Assume that for every compact subgroup $\dot{C}$ of G/N, the group $C = q^{-1}(\dot{C})$ has a γ-element and satisfies BCC for B. Then, if G/N satisfies BCC (resp. BCS) for* arbitrary *coefficients, then G satisfies BCC (resp. BCS) for B.*

Proof. We are going to use the reduction argument of Proposition 3.11: Denote by $(G/N)_0$ the connected component of the identity in G/N and let $M = q^{-1}\big((G/N)_0\big) \subseteq G$. Let $\ddot{H}$ be any compact subgroup of G/M, let $\dot{H} \subset G/N$ be the inverse image of $\ddot{H}$ in G/N, and let H denote the inverse image of $\ddot{H}$ in G. Note that $\dot{H}$ is an almost connected group. Let $\dot{K}$ be its maximal compact subgroup. Then $\dot{K}$ is a compact subgroup of G/N, so, by assumption, $K = q^{-1}(\dot{K})$ has a γ-element and satisfies BCC for B. Lemma 3.9 now implies that H itself has a γ-element. Applying Theorem 3.7 to the extension $1 \to N \to H \to \dot{H} \to 1$, it follows that H satisfies BCC for B: Since G/N satisfies BCS for all coefficients, the same is true for $\dot{H} \subseteq G/N$ by Theorem 2.5. Since $\dot{H}$ is almost connected, it also satisfies BCI. Thus Theorem 3.7 applies.
We can now apply Proposition 3.11: By (i), the partial assembly map for (G, M) with coefficients in B in bijective, and, by (ii), G/M satisfies BCC (resp. BCS) for $B \rtimes_r M$. Thus, Proposition 3.2 implies that G satisfies BCC (resp. BCS) for B. □

REMARK 3.13. Note that the statement of Theorem 3.12 is a bit weaker than the statements of Theorems 3.3 and 3.7 above, since it requires that G/N satisfies BCC (resp. BCS) for ALL coefficients, while the previous results only required that G/N satisfies BCC (resp. BCI, BCS) for $B \rtimes_r N$. Also, Theorem 3.12 does not give any information on condition BCI.
If we could show that Theorem 3.7 holds without requiring a γ-element for G, then no reference to γ-elements would be needed in Theorem 3.12 above. However, note that the assumption on the existence of γ-elements for the compact extensions of N in G is much easier to check than the assumption on the existence of a γ-element for G, as we did in [6]. A particularly nice application is given when N satisfies the Haagerup property. As mentioned earlier (see the discussions before Corollary 3.5) the Haagerup property for N implies the Haagerup property for every compact extension of N in G. Thus, all compact extensions of N in G have a γ-element and satisfy BCC (see Remark 1.8). Thus, the following is a direct corollary of Theorem 3.12:

COROLLARY 3.14. *Let N be a closed normal subgroup of G such that N satisfies the Haagerup property (e.g., if N is amenable). Then, if G/N satisfies BCC (resp. BCS), the same is true for G.*

In what follows next, we want to look at the consequences of the above results on the stability of the Baum-Connes conjecture under taking direct products of groups. We need:

LEMMA 3.15. *Let G_1 and G_2 be groups. Suppose that G_1 has a compact open subgroup, or is an almost connected group, and that G_2 has a compact open subgroup. Let B be a $G_1 \times G_2$-algebra, and assume that for every compact subgroup K_2 of G_2, G_1 satisfies BCC for $B \rtimes K_2$. Then the partial assembly map $\mu_{G_1,B}^{G_1 \times G_2, G_1} : \mathrm{K}_*^{\mathrm{top}}(G_1 \times G_2; B) \to \mathrm{K}_*^{\mathrm{top}}(G_2; B \rtimes_r G_1)$ of Definition 3.1 is bijective.*

Proof. Take any compact subgroup K_2 of G_2. If G_1 has a compact open subgroup, we can apply Theorem 3.3 to the extension $K_2 \times G_1$ of K_2 by G_1. Since G_1 satisfies BCC for $B \rtimes K_2$ by assumption, it is enough to check that for every compact subgroup K_1 of G_1, the group $K_2 \times K_1$ satisfies BCC (which is clear). It follows that for every compact subgroup K_2 of G_2, $G_1 \times K_2$ satisfies BCC for B. Thus the result follows from applying Theorem 3.3 to $G_1 \times G_2$.
If G_1 is almost connected, the same is true for $G_1 \times K_2$ (since K_2 is compact), so $G_1 \times K_2$ has a γ-element by Remark 1.8. Replacing Theorem 3.3 by Theorem 3.7 in the above argument gives the result. □

Remark 3.16. In the prove of the theorem below, we shall also need a twisted version of the above lemma, i.e., a version in which the quotient group G/N has a product structure as above. However, this extension follows from the above lemma by the result of [10] that every (G, N)-algebra is Morita equivalent to some G/N-algebra.

Theorem 3.17. *Let G_1 and G_2 be two groups. Then the following statements are true:*

(i) The product group $G = G_1 \times G_2$ satisfies BCC if and only if G_1 and G_2 satisfy BCC.

(ii) Suppose that G_1 satisfies BCC. Then $G = G_1 \times G_2$ satisfies BCS if and only if G_2 satisfies BCS.

(iii) Suppose that G_1 has a compact open subgroup, or is almost connected. Suppose further that G_2 has a compact open subgroup. If G_1 satisfies BCC and G_2 satisfies BCI, then $G = G_1 \times G_2$ satisfies BCI.

Proof. We first prove (i) and (ii). If $G = G_1 \times G_2$ satisfies BCC (resp. BCS), the same is true for G_1 and G_2 by Theorem 2.5. Assume now that G_1 satisfies BCC. Let G_0 (resp. $G_{1,0}$, $G_{2,0}$) denote the connected component of G (resp. G_1, G_2). It is clear that $G_0 = G_{1,0} \times G_{2,0}$. Consider the extension $1 \to G_0 \to G \to G/G_0 \to 1$. The quotient group is totally disconnected. Let B be any G-algebra. By Corollary 3.4, to see that G satisfies BCS (resp. BCC) for B, it is enough to show that

(a) any compact extension of G_0 satisfies BCC, and

(b) G/G_0 satisfies BCS (resp. BCC) for $B \rtimes_r G_0$.

For (a), note that if L is a compact extension of G_0, L is contained in a direct product $L_1 \times L_2$, where L_1 is a compact extension of $G_{1,0}$ and L_2 is a compact extension of $G_{2,0}$. Being a subgroup of G_1 (resp. G_2), L_1 (resp. L_2) satisfies BCS by Theorem 2.5. Both groups being almost connected, they also satisfy BCI, whence BCC. Consider the extension $1 \to L_1 \times \{e\} \to L_1 \times L_2 \to L_2 \to 1$. By Theorem 3.7, to see that $L_1 \times L_2$ satisfies BCC (and hence that L satisfies BCC by Theorem 2.5), it suffices to check

that $L_1 \times K_2$ satisfies BCC whenever K_2 is a compact subgroup of G_2. To see this, we use again Theorem 3.7 to reduce to the group $K_1 \times K_2$, where K_1 is an arbitrary compact subgroup of G_1. But compact groups satisfy BCC.
In order to check (b), first observe that we just saw in particular, that compact extensions of $G_{1,0}$ in G_1 (resp. of $G_{2,0}$ in G_2) satisfy BCC. Hence it follows from Theorem 3.3 and Proposition 3.2 that $G_1/G_{1,0}$ satisfies BCC with twisted coefficients in $A \rtimes_r G_{1,0}$, where A_1 is any G_1-algebra. A similar result holds for $G_2/G_{2,0}$.
In particular, it follows that $G_1/G_{1,0}$ satisfies BCC with (twisted) coefficients in

$$B \rtimes_r G_0 \rtimes_r \dot{K}_2 = B \rtimes_r G_{2,0} \rtimes_r \dot{K}_2 \rtimes_r G_{1,0} = B \rtimes_r K_2 \rtimes_r G_{1,0},$$

where $\dot{K}_2$ is any compact subgroup of $G_2/G_{2,0}$ and K_2 denotes its inverse image in G_2. Note that in the above formula we took the freedom to write the twisted crossed products by the pairs $(K_2 \times G_{1,0}, G_0)$ (in the first crossed product) and $(K_2, G_{2,0})$ (in the second crossed product) simply as crossed products by the common quotient $\dot{K}_2$. Using the definition of the twisted crossed products (see [6]), it is fairly straightforward (but tedious) to check that all three crossed products in the above formula do coincide.
A similar argument shows that $G_2/G_{2,0}$ satisfies BCS (resp. BCC) with coefficients in the algebra

$$B \rtimes_r G_0 \rtimes_r G_1/G_{1,0} = B \rtimes_r G_1 \rtimes_r G_{2,0}.$$

Using the twisted version of Lemma 3.15 (replacing G_1 by $G_1/G_{1,0}$, G_2 by $G_2/G_{2,0}$ and B by $B \rtimes_r G_0 = B \rtimes_r G_{1,0} \rtimes_r G_{2,0}$), we see that the partial assembly map for the extension of $G_1/G_{1,0}$ by $G_2/G_{2,0}$ and with coefficients in $B \rtimes_r G_0$ is bijective. Composing this with the assembly map for $G_2/G_{2,0}$ with coefficient algebra $B \rtimes_r G_0 \rtimes_r G_1/G_{1,0} = B \rtimes_r G_1 \rtimes_r G_{2,0}$, we get (b).
We have now completed the proofs of (i) and (ii). (iii) is a direct consequence of Lemma 3.15 and Proposition 3.2.

□

4 An example for the Baum-Connes conjecture with trivial coefficients

In this section we want to show how the results of the previous sections may be combined in order to produce new examples for the validity of the Baum-Connes conjecture without coefficients. The methods we use here give a hint into a direction of a more general "Mackey-machine" for computing the topological K-theory of group extensions via an induction process.
The basic idea is to use our partial assembly map to write $\mathrm{K}_*^{\mathrm{top}}(G; \mathbb{C})$ as $\mathrm{K}_*^{\mathrm{top}}(G/N; C_r^*(N))$, where N is a closed normal subgroup of G (see the main results of §3 above). In good cases, we might be able to decompose $C_r^*(N)$ into finitely many pieces (i.e., G-invariant subquotients) which are induced from

smaller groups, which satisfy the conjecture for the respective coefficient algebras. The bijectivity of the induction homomorphism (see Theorem 2.2) then gives the conjecture for the original pieces, and, using excision, we end up with the desired result for G. Below, we will give some explicit examples for this procedure.

But before we present the examples, we need to mention some results on the functorial properties of the topological K-theory of a fixed group G, viewed as a functor on the category of G-C^*-algebras.

By a result of Kasparov and Skandalis (see [17, Appendix]), it is known that for any proper G-algebra D the functor $A \mapsto \mathrm{KK}^G(D, A)$ is half exact. Replacing D by $C_0(X)$, for X a G-compact subspace of $\mathcal{E}(G)$, and taking the limit over X, implies that the topological K-theory functor $A \mapsto \mathrm{K}_*^{\mathrm{top}}(G; A)$ is half exact, too. Since this functor is also homotopy invariant and satisfies Bott-periodicity, it follows from some general arguments (which, for instance, are outlined in [4, Chapter IX]) that it satisfies excision in the sense that every short exact sequence

$$0 \to I \to A \to A/I \to 0$$

of G-algebras induces a natural six-term exact sequence

$$\begin{array}{ccccc} \mathrm{K}_0^{\mathrm{top}}(G; I) & \longrightarrow & \mathrm{K}_0^{\mathrm{top}}(G; A) & \longrightarrow & \mathrm{K}_0^{\mathrm{top}}(G; A/I) \\ {\scriptstyle\partial}\uparrow & & & & \downarrow{\scriptstyle\partial} \\ \mathrm{K}_1^{\mathrm{top}}(G; A/I) & \longleftarrow & \mathrm{K}_1^{\mathrm{top}}(G; A) & \longleftarrow & \mathrm{K}_1^{\mathrm{top}}(G; I). \end{array}$$

If G satisfies BCC, then it follows from the half exactness of $\mathrm{K}_*^{\mathrm{top}}(G, \cdot)$ and the naturality of the assembly map that the functor

$$A \mapsto \mathrm{K}_*(A \rtimes_r G)$$

has to be half exact, too. Thus we see that BCC can only hold for G if G is K-*exact* in the sense that for every short exact sequence $0 \to I \to A \to A/I \to 0$ of G-algebras, the natural sequence

$$\mathrm{K}_*(I \rtimes_r G) \to \mathrm{K}_*(A \rtimes_r G) \to \mathrm{K}_*(A/I \rtimes_r G)$$

is exact in the middle term. By the same general arguments as used above, K-exactness of G implies that every short exact sequence of G-algebras induces a natural six-term exact sequence for the K-theories of the reduced crossed products.

Proposition 4.1. *Assume that G is* K-*exact and that $0 \to I \to A \to A/I \to 0$ is a short exact sequence of G-algebras. Let $\partial : \mathrm{K}_{*+1}^{\mathrm{top}}(G; A/I) \to \mathrm{K}_*^{\mathrm{top}}(G; I)$ and $\delta : \mathrm{K}_{*+1}(A/I \rtimes_r G) \to \mathrm{K}_*(I \rtimes_r G)$ denote the boundary maps in the respective* *six-term exact sequences. Then the diagram*

$$\begin{array}{ccc} \mathrm{K}^{\mathrm{top}}_{*+1}(G;A/I) & \xrightarrow{\partial} & \mathrm{K}^{\mathrm{top}}_{*}(G;I) \\ {\scriptstyle \mu_{G,A/I}}\downarrow & & \downarrow{\scriptstyle \mu_{G,I}} \\ \mathrm{K}_{*+1}(A/I\rtimes_r G) & \xrightarrow[\delta]{} & \mathrm{K}_{*}(I\rtimes_r G) \end{array}$$

commutes. In fact, the assembly map commutes with all maps in the respective six-term exact sequences.

Proof. This result follows easily from the naturality of the assembly map and the general construction of the boundary maps: By [4, Theorem 21.4.3], it follows that the boundary maps can be factorized via K-theory maps coming from $*$-homomorphisms $\Phi : C_0(\mathbb{R}) \otimes A/I \to C_q$ and $e : I \to C_q$, where C_q denotes the mapping cone for the quotient map $q : A \to A/I$. More precisely, using suspension, the above diagram splits into the diagram

$$\begin{array}{ccccc} \mathrm{K}^{\mathrm{top}}_{*}\big(G;C_0(\mathbb{R})\otimes A/I\big) & \xrightarrow{\Phi_*} & \mathrm{K}^{\mathrm{top}}_{*}(G;C_q) & \xleftarrow[\cong]{e_*} & \mathrm{K}^{\mathrm{top}}_{*}(G;I) \\ {\scriptstyle \mu_{G,C_0(\mathbb{R})\otimes A/I}}\downarrow & & {\scriptstyle \mu_{G,C_q}}\downarrow & & \downarrow{\scriptstyle \mu_{G,I}} \\ \mathrm{K}_{*}\big((C_0(\mathbb{R})\otimes A/I)\rtimes_r G\big) & \xrightarrow[\Phi_*]{} & \mathrm{K}_{*}(C_q\rtimes_r G) & \xleftarrow[e_*]{\cong} & \mathrm{K}_{*}(I\rtimes_r G), \end{array}$$

which commutes by the naturality of the assembly map. The result then follows from the fact that the assembly map commutes with Bott periodicity. □

Using the above observations, an easy application of the Five Lemma gives the following general principle:

Proposition 4.2. *Suppose that G is* K*-exact and let $0 \to I \to A \to A/I \to 0$ be a short exact sequence of G-algebras. If G satisfies BCC for two of the algebras I, A and A/I, then it satisfies BCC for all three algebras.*

We are now ready to present our example.

Example 4.3. Let $K = \mathbb{R}$ or $\mathbb{C}$. The semi-direct product groups $K^n \rtimes SL_n(K)$, $n \in \mathbb{N}^*$, (where the action of $SL_n(K)$ on K^n is by matrix multiplication) satisfy the Baum-Connes conjecture *without* coefficients (i.e., with coefficient algebra $\mathbb{C}$). We believe that this was known before only for the cases $n \leq 2$.

The proof is by induction on n. For short, let us write $H_n = SL_n(K)$ and $G_n = K^n \rtimes H_n$. For $n = 1$, the conclusion holds. By induction, take $n > 1$ and let us assume that the conclusion holds for $n - 1$.
Since K^n is abelian, hence amenable, it follows from Theorem 3.7 that G_n satisfies the Baum-Connes conjecture for $\mathbb{C}$ if and only H_n satisfies BCC for

$C_r^*(K^n) \cong C_0(K^n)$. Moreover, the Gelfand transform carries the decomposition action of H_n on $C^*(K^n)$ to the action of H_n on $C_0(K^n)$ given by the formula

$$(x \cdot f)(\eta) = f(x^* \cdot \eta), \quad x \in SL_n(K), \eta \in K^n,$$

where x^* denotes the adjoint of the matrix x.
There are two orbits under this action of H_n on K^n: $\{0\}$ and $K^n \smallsetminus \{0\}$. Let $\eta = (1, 0, \ldots, 0)^t \in K^n$. Then a short computation shows that the stabilizer of η under the above action is isomorphic to $K^{n-1} \rtimes SL_{n-1}(K) = G_{n-1}$, and therefore we get

$$C_0(K^n \setminus \{0\}) \cong C_0(H_n/G_{n-1}) \cong \operatorname{Ind}_{G_{n-1}}^{H_n} \mathbb{C}.$$

By [19], we know that the semi-simple group $SL_n(K)$ satisfies BCC for $\mathbb{C}$ and by the induction assumption we know that G_{n-1} also satisfies BCC for $\mathbb{C}$. Using Theorem 2.5 it follows that H_n satisfies BCC for $C_0(K^n \setminus \{0\})$. Thus, applying Proposition 4.2 to the short exact sequence $0 \to C_0(K^n \setminus \{0\}) \to C_0(K^n) \to \mathbb{C} \to 0$ gives the result.

Remark 4.4. A similar argument can be used to show that $\mathbb{Q}_p^n \rtimes GL_n(\mathbb{Q}_p)$ satisfies BCC with coefficients in $\mathbb{C}$. Since $GL_n(\mathbb{Q}_p)$ can be written as the product $\mathbb{Q}_p 1 \cdot GL_n(\mathcal{O})$, where $GL_n(\mathcal{O})$ is the compact group of invertible matrices with p-adic integer entries, it is a (non-direct) product of an amenable group and a compact group, and therefore exact by a standard argument (e.g., see [18]). Moreover, by results of Baum, Higson and Plymen [3] and Lafforgue [19], it is known that $GL_n(\mathbb{Q}_p)$ satisfies BCC for $\mathbb{C}$. Now the same procedure as used in the above example, using Theorem 3.3 instead of Theorem 3.7, gives the result.

5 Proof of Theorem 3.3

In this section we give the proof of Theorem 3.3. As indicated in the introduction, some of the main ideas (and intermediate results) used in this proof will also be applied in the proofs of the bijectivity of the induction homomorphism as given in §6, and the continuity of topological K-theory with respect to the coefficients as presented in §7. First we recall the statement of the theorem.

Theorem 3.3. *Let $1 \to N \to G \xrightarrow{q} G/N \to 1$ be an extension of groups such that G/N has a compact open subgroup. Let B be a G-algebra and assume that for every compact open subgroup $\dot{K}$ of G/N, the subgroup $q^{-1}(\dot{K})$ of G satisfies BCC for B. Then the partial assembly map $\mu_{N,B}^{G,N} : \mathrm{K}_*^{\mathrm{top}}(G; B) \to \mathrm{K}_*^{\mathrm{top}}(G/N; B \rtimes_r N)$ of Definition 3.1 is bijective.*

As mentioned in the introduction, the proof relies on a special realization of a universal example for the proper actions of G. In order to obtain this special realization, we start with the following easy observation:

LEMMA 5.1. *Let G_1 and G_2 be groups. If $\mathcal{E}(G_1)$ (resp. $\mathcal{E}(G_2)$) is a realization of the universal example for the proper actions of G_1 (resp. G_2), then $\mathcal{E}(G_1) \times \mathcal{E}(G_2)$, equipped with the product action of $G_1 \times G_2$, is a realization of the universal example for the proper actions of $G_1 \times G_2$.*

Proof. The action of $G_1 \times G_2$ on $\mathcal{E}(G_1) \times \mathcal{E}(G_2)$ is clearly proper. To show the universal property, take any space X endowed with a proper action of $G_1 \times G_2$. This action restricts to proper actions of G_1 and G_2 on X, so there exist continuous G_i-maps $F_i : X \to \mathcal{E}(G_i)$, $i = 1, 2$. Then $F : X \to \mathcal{E}(G_1) \times \mathcal{E}(G_2); F(x) = (F_1(x), F_2(x))$ is a continuous $G_1 \times G_2$-equivariant map. Conversely, every continuous $G_1 \times G_2$-map is of this form. Thus the uniqueness (up to homotopy), follows from the uniqueness of $\mathcal{E}(G_1)$ and $\mathcal{E}(G_2)$. □

If a group G has at least one compact open subgroup, the constructions in [17] provide a realization of $\mathcal{E}(G)$ as a simplicial complex (the constructions in [17] are given for discrete G; the adaptations for the more general case of groups with compact open subgroup are given in the discussion following [6, Lemma 7.10]). Summing up the results of [17] and [6, §7], we obtain

PROPOSITION 5.2. *Let G be a group having a compact open subgroup K. Then there exists a realization $\mathcal{E}(G)$ of the universal example for proper actions of G, such that*

(i) $\mathcal{E}(G)$ is the geometric realization of a locally finite simplicial complex on which G acts properly and simplicially,

(ii) If S is any simplex of $\mathcal{E}(G)$, $\overset{\circ}{S}$ its interior, and $g \in G$, then either g acts as the identity on S or $g\overset{\circ}{S} \cap \overset{\circ}{S} = \emptyset$.

In this section we will from now on assume that N is a closed normal subgroup of G such that G/N has a compact open subgroup. Thus, in what follows next, we can always assume that $\mathcal{E}(G/N)$ has the structure of a simplicial complex as described in Proposition 5.2 above. The following lemma shows how this provides a special realization for the universal example for the proper actions of G:

LEMMA 5.3. *Let $\mathcal{E}(G)$ be a universal example for the proper actions of G. Then the Cartesian product $\mathcal{E}(G) \times \mathcal{E}(G/N)$, endowed with the diagonal action $g(x, y) = (gx, \dot{g}y)$, is also a universal example for the proper actions of G.*

Proof. $\mathcal{E}(G) \times \mathcal{E}(G/N)$ is a universal example for $G \times G/N$ by Lemma 5.1. Because G can be seen as a closed subgroup of $G \times G/N$ via the map $g \mapsto (g, \dot{g})$, the result follows from Lemma 2.4. □

The main advantage of taking $\mathcal{E}(G) \times \mathcal{E}(G/N)$ as a universal example for the proper actions of G comes from the fact that the simplicial structure of $\mathcal{E}(G/N)$ allows us to use induction arguments on the dimension of simplices and to "compress" to smaller subgroups.

Remark 5.4. In what follows we will denote by π_2 the projection of $\mathcal{E}(G) \times \mathcal{E}(G/N)$ onto the second factor $\mathcal{E}(G/N)$. Moreover, if we restrict the diagonal G-action on $\mathcal{E}(G) \times \mathcal{E}(G/N)$ to N, the quotient space $N\backslash(\mathcal{E}(G) \times \mathcal{E}(G/N))$ can be identified with $(N\backslash\mathcal{E}(G)) \times \mathcal{E}(G/N)$, and we will always denote by p_2 the second projection of $(N\backslash\mathcal{E}(G)) \times \mathcal{E}(G/N)$ onto $\mathcal{E}(G/N)$. Note that by the universal property of $\mathcal{E}(G/N)$, $p_2 : N\backslash(\mathcal{E}(G) \times \mathcal{E}(G/N)) \to \mathcal{E}(G/N)$ is *the* unique (up to G/N-equivariant homotopy) continuous G/N-equivariant map.

We are now using $\mathcal{E}(G) \times \mathcal{E}(G/N)$ to compute the topological K-theory of G. We start with

Definition 5.5. Let Y be a G/N-compact subset of $\mathcal{E}(G/N)$. Then we define $\mathrm{K}_*^{\mathrm{top}}\langle Y\rangle(G;B)$ to be the inductive limit

$$\mathrm{K}_*^{\mathrm{top}}\langle Y\rangle(G;B) := \lim_{X} \mathrm{KK}_*^G\big(C_0(X), B\big),$$

over all G-compact subspaces X of $\mathcal{E}(G) \times \mathcal{E}(G/N)$ which satisfy $\pi_2(X) = Y$.

Lemma 5.6. *Assume that G/N has a compact open subgroup and that $\mathcal{E}(G/N)$ has the simplicial structure described in Proposition 5.2. Let $\mathcal{F}$ denote the family of subsets Y of $\mathcal{E}(G/N)$ such that Y is the G/N-saturation of a finite union of simplices of $\mathcal{E}(G/N)$. Then the topological* K*-theory of G with coefficients in the G-algebra B can be computed by the formula*

$$\mathrm{K}_*^{\mathrm{top}}(G;B) = \lim_{Y\in\mathcal{F}} \mathrm{K}_*^{\mathrm{top}}\langle Y\rangle(G;B).$$

Proof. Using $\mathcal{E}(G) \times \mathcal{E}(G/N)$ as a realization of the universal example for the proper actions of G, $\mathrm{K}_*^{\mathrm{top}}(G;B)$ can be computed as $\mathrm{K}_*^{\mathrm{top}}(G;B) = \lim_Z \mathrm{KK}_*^G\big(C_0(Z), B\big)$, where Z runs through the family of G-compact subsets of $\mathcal{E}(G) \times \mathcal{E}(G/N)$. Clearly, any such Z is contained in a G-compact subset X of $\mathcal{E}(G) \times \mathcal{E}(G/N)$ satisfying $\pi_2(X) = Y$ for some $Y \in \mathcal{F}$: Choose any $Y \in \mathcal{F}$ such that $\pi_2(Z) \subseteq Y$. Y can be written as $Y = G/N \cdot K$, where K is a compact subset of $\mathcal{E}(G/N)$. Take any point x in $\mathcal{E}(G)$ and put $X = Z \cup G \cdot \big(\{x\} \times K\big) \subseteq \mathcal{E}(G) \times \mathcal{E}(G/N)$. □

To each piece of the above decomposition of $\mathrm{K}_*^{\mathrm{top}}(G;B)$ via the elements $Y \in \mathcal{F}$ corresponds a piece of the partial assembly map for (G,N): If Y is a G-compact subset of $\mathcal{E}(G/N)$ and X is a G-compact subset of $\mathcal{E}(G)\times\mathcal{E}(G/N)$ (viewed as a universal example for the proper actions of G) such that $\pi_2(X) = Y$, we obtain from Definition 3.1 a well defined morphism

$$\mu_{N,B}^{G,N}[X] : \mathrm{KK}_*^G(C_0(X), B) \to \mathrm{KK}_*^{G,N}(C_0(Y), B \rtimes_r N)$$

(note that the map $F : N\backslash\mathcal{E}(G) \to \mathcal{E}(G/N)$ of Definition 3.1 is given by the projection $p_2 : N\backslash\big(\mathcal{E}(G) \times \mathcal{E}(G/N)\big) \to \mathcal{E}(G/N)$, and that $p_2(X) = Y$). It follows from Lemma 1.5 that the maps $\mu_{N,B}^{G,N}[X]$ commute with the maps induced by taking inclusions. Thus we may define

DEFINITION 5.7. For any G/N-compact subset Y of $\mathcal{E}(G/N)$ we define

$$\mu_{N,B}^{G,N}\langle Y\rangle : \mathrm{K}_*^{\mathrm{top}}\langle Y\rangle(G;B) \to \mathrm{KK}_*^{G,N}\big(C_0(Y), B\rtimes_r N\big)$$

as the map which is obtained inductively from the morphisms

$$\mathrm{KK}_*^{G}\big(C_0(X), B\big) \xrightarrow{\mu_{N,B}^{G,N}[X]} \mathrm{KK}_*^{G,N}\big(C_0(Y), B\rtimes_r N\big),$$

where X runs through all G-compact subspaces of $\mathcal{E}(G)\times\mathcal{E}(G/N)$ which satisfy $\pi_2(X)=Y$.

We observe that not only $\mathrm{K}_*^{\mathrm{top}}(G;B)$ but also the partial assembly map associated to (G,N) can be recovered from the decomposition described above. This follows directly from the definitions.

LEMMA 5.8. *Let N, G, $\mathcal{E}(G/N)$, and $\mathcal{F}$ be as in Lemma 5.6. Then the partial assembly map $\mu_{N,B}^{G,N} : \mathrm{K}_*^{\mathrm{top}}(G;B) \to \mathrm{K}_*^{\mathrm{top}}(G;B\rtimes_r N)$ of Definition 3.1 can be computed inductively from the maps*

$$\mu_{N,B}^{G,N}\langle Y\rangle : \mathrm{K}_*^{\mathrm{top}}\langle Y\rangle(G;B) \to \mathrm{KK}_*^{G,N}\big(C_0(Y), B\rtimes_r N\big), \quad Y\in\mathcal{F}.$$

In view of Lemma 5.8, the proof of Theorem 3.3 reduces to the proof of:

PROPOSITION 5.9. *Let N, G, $\mathcal{E}(G/N)$ and $\mathcal{F}$ be as in Lemma 5.6. Let $q: G\to G/N$ be the quotient map and let B be any G-algebra. Assume further that for any compact open subgroup $\dot{K}$ of G/N, the subgroup $q^{-1}(\dot{K})$ of G satisfies BCC for B. Then, for any $Y\in\mathcal{F}$, the map*

$$\mu_{N,B}^{G,N}\langle Y\rangle : \mathrm{K}_*^{\mathrm{top}}\langle Y\rangle(G;B) \to \mathrm{KK}_*^{G,N}\big(C_0(Y), B\rtimes_r N\big)$$

is bijective.

To prove Proposition 5.9, we use two ingredients (which goes back to some ideas used in [11, Chapter 12] and [21, §5]). The first is to make an induction on the maximal dimension of the simplices generating Y, using a Mayer-Vietoris argument. For this we need a relative version of Definition 5.7:

DEFINITION 5.10. Let Y be a G/N-compact subset of $\mathcal{E}(G/N)$ and let Y_0 be an open (in the relative topology) G/N-equivariant subset of Y. For any G-compact set $X\subseteq\mathcal{E}(G)\times\mathcal{E}(G/N)$ satisfying $\pi_2(X)=Y$ we put $X_0 = X\cap\pi_2^{-1}(Y_0)$. Consider the composition of maps

$$\begin{array}{ccc} \mathrm{KK}_*^{G}\big(C_0(X_0), B\big) & \dashrightarrow & \mathrm{KK}_*^{G,N}\big(C_0(Y_0), B\rtimes_r N\big) \\ & {\searrow}^{\nu[X_0]} & \uparrow p_2^* \\ & & \mathrm{KK}_*^{G,N}\big(C_0(N\backslash X_0), B\rtimes_r N\big) \end{array}$$

where $p_2 : N\backslash(\mathcal{E}(G) \times \mathcal{E}(G/N)) \to \mathcal{E}(G/N)$ denotes the projection onto the second factor and $\nu[X_0]$ is as in Equation (3.1). Using Lemma 1.5 we see that these maps induce a well defined map

$$\mu_{N,B}^{G,N}\langle Y_0\rangle : \lim_{X} \mathrm{KK}_*^G\big(C_0(X_0), B\big) \to \mathrm{KK}_*^{G,N}\big(C_0(Y_0), B \rtimes_r N\big),$$

on the inductive limit, where X runs through all G-compact subsets of $\mathcal{E}(G) \times \mathcal{E}(G/N)$ which project onto Y.

Remark 5.11. It is important to note that both, the limit $\lim_X \mathrm{KK}_*^G\big(C_0(X_0), B\big)$, and the map $\mu_{N,B}^{G,N}\langle Y_0\rangle$, only depend on the space Y_0 and not on the particular choice of the G-compact set $Y \supseteq Y_0$. To see this, it is enough to observe that, if $Y_0 \subseteq Y \subseteq Y_1$ such that Y and Y_1 are G/N-compact and Y_0 is open in Y_1, then for any G-compact set $X \subseteq \mathcal{E}(G) \times \mathcal{E}(G/N)$ with $\pi_2(X) = Y$, there exists a G-compact set $X_1 \subseteq \mathcal{E}(G) \times \mathcal{E}(G/N)$ with $\pi_2(X_1) = Y_1$ and $X \subseteq X_1$ (then $X_0 \subseteq X_{1,0}$, and, conversely, if X_1 is given, and if we put $X = X_1 \cap \pi_2^{-1}(Y)$, we get $X_{1,0} \subseteq X_0$).

We will make use of Definition 5.10 in our Mayer-Vietoris argument. The second ingredient for the proof of Proposition 5.9 is a reduction argument based on an isomorphism in KK-theory, which makes it possible to simplify the group $\mathrm{KK}_*^{G,N}\big(C_0(Y_0), B \rtimes_r N\big)$, if Y_0, as a G-space, is induced from an open subgroup of G. The following characterization of induced spaces is taken from [9]:

Proposition 5.12 (cf. [9, Corollary 2]). *Let Y be a locally compact G-space and let C be a closed subgroup of G. Then Y is G-homeomorphic to an induced space $G \times_C T$, for some C-space T, if and only if there exists a continuous G-equivariant map $p : Y \to G/C$. In that case, the C-space T can be chosen as $T = p^{-1}(\{eC\}) \subseteq Y$, and a G-homeomorphism is given by the mapping $G \times_C T \to Y; [(s, x)] \mapsto sx$.*

Remark 5.13. Assume that G/N has a compact open subgroup and that $\mathcal{E}(G/N)$ has the simplicial structure of Proposition 5.2. By part (ii) of that proposition, the above characterization of induced spaces shows immediately that any subspace Z of $\mathcal{E}(G) \times \mathcal{E}(G/N)$ which projects onto the G-saturation of the interior $\mathring{S}$ of a simplex S of $\mathcal{E}(G/N)$ is an induced space $Z = G \times_C T$, where C is the stabilizer of S under the action of G and $T = \pi_2^{-1}(\mathring{S}) \cap Z$. Since the action of G/N on $\mathcal{E}(G/N)$ is continuous, simplicial, and proper, C is an open subgroup of G and C/N is a compact subgroup of G/N.

One of the basic ideas of the proof of Theorem 3.3 is to "compress" to the open subgroup $C \subseteq G$ of the above remark. For this we want to use a more general compression isomorphism in twisted equivariant KK-theory, which we are now going to describe.

So assume that C is an open subgroup of G containing the closed normal subgroup N of G, and let A be a (C, N)-algebra. Let $\mathrm{Ind}_C^G A$ denote the induced algebra of A. Then $\operatorname{Ind}_C^G A$ is a (G,N)-algebra in a canonical way: If $\tau : N \to UM(A)$ is the twist for the original C-action on A, then the twist $\operatorname{Ind}\tau : N \to UM(\operatorname{Ind}_C^G A)$ for the induced G-action on $\operatorname{Ind}_C^G A$ is given by the formula

$$\big(\operatorname{Ind}\tau_n \cdot F\big)(g) = \tau_{g^{-1}ng} \cdot F(g), \quad F \in \operatorname{Ind}_C^G A.$$

Since C is open in G, there exists a canonical (C,N)-equivariant embedding $i_A : A \to \operatorname{Ind}_C^G A$, given by

$$\big(i_A(a)\big)(g) = \begin{cases} g^{-1}(a) & \text{if } g \in C; \\ 0 & \text{if } g \notin C. \end{cases}$$

For any (G,N)-algebra D, the compression homomorphism

$$\operatorname{comp}_C^G : \mathrm{KK}_*^{G,N}\left(\operatorname{Ind}_C^G A, D\right) \to \mathrm{KK}_*^{C,N}(A,D),$$

is then defined as the composition

$$\mathrm{KK}_*^{G,N}\left(\operatorname{Ind}_C^G A, D\right) \stackrel{\operatorname{res}_C^G}{\to} \mathrm{KK}_*^{C,N}\left(\operatorname{Ind}_C^G A, D\right) \stackrel{i_A^*}{\to} \mathrm{KK}_*^{C,N}(A,D).$$

It is shown in [5] (extending earlier results of [11] and [21]) that the compression map is an isomorphism if $N = \{e\}$ and C is a compact open subgroup of G. But for our purposes it is necessary to get rid of these assumptions. Thanks to a recent result of Ralf Meyer, this is indeed possible:

Proposition 5.14. *The map*

$$\operatorname{comp}_C^G : \mathrm{KK}_*^{G,N}\left(\operatorname{Ind}_C^G A, D\right) \to \mathrm{KK}_*^{C,N}(A,D)$$

is an isomorphism.

Proof. We first note that the result is invariant under passing to Morita equivalent twisted actions in both variables: First, if we replace D by a Morita equivalent (G,N)-algebra D', say, and if $y \in \mathrm{KK}_0^{G,N}(D,D')$ is the corresponding invertible element, then the statement follows from the commutativity of the diagram

$$\begin{array}{ccc} \mathrm{KK}_*^{G,N}\left(\operatorname{Ind}_C^G A, D\right) & \xrightarrow{\operatorname{comp}_C^G} & \mathrm{KK}_*^{C,N}(A,D) \\ {\scriptstyle \cdot\otimes y}\downarrow\cong & & \cong\downarrow{\scriptstyle \cdot\otimes \operatorname{res}_C^G(y)} \\ \mathrm{KK}_*^{G,N}\left(\operatorname{Ind}_C^G A, D'\right) & \xrightarrow{\operatorname{comp}_C^G} & \mathrm{KK}_*^{C,N}(A,D'). \end{array}$$

Secondly, if we replace A by a Morita equivalent (C,N)-algebra A', and if $x \in \mathrm{KK}_0^{C,N}(A',A)$ denotes the corresponding invertible element, the statement follows from the commutativity of

$$\begin{array}{ccc}
\mathrm{KK}_*^{G,N}\big(\mathrm{Ind}_C^G A, D\big) & \xrightarrow{\mathrm{comp}_C^G} & \mathrm{KK}_*^{C,N}(A,D) \\
{\scriptstyle i_C^G(x)\otimes\cdot}\big\downarrow\cong & & \cong\big\downarrow{\scriptstyle x\otimes\cdot} \\
\mathrm{KK}_*^{G,N}\big(\mathrm{Ind}_C^G A', D\big) & \xrightarrow{\mathrm{comp}_C^G} & \mathrm{KK}_*^{C,N}(A',D),
\end{array}$$

which follows from the equation $[i'_A] \otimes \mathrm{res}_C^G\big(i_C^G(x)\big) = x \otimes [i_A]$ in $\mathrm{KK}_0^{C,N}(A', \mathrm{Ind}_C^G A)$.

Since every twisted action of (G,N) (resp. (C,N)) is Morita equivalent to an ordinary action of G/N (resp. C/N) by [10, Theorem 1], it follows from the results of [6, §3] that we may assume without loss of generality that $N = \{e\}$ and all actions are untwisted. Moreover, since an action $\alpha : C \to \mathrm{Aut}(A)$ is Morita equivalent to the stabilized action $\alpha \otimes \mathrm{Ad}\,\lambda : C \to \mathrm{Aut}\big(A \otimes \mathcal{K}(L^2(C))\big)$, where λ denotes the left regular representation of C (a Morita equivalence is given by $(A \otimes L^2(C), \alpha \otimes \lambda)$), we can use [20, Proposition 3.2] in order to assume without loss of generality that every element $\alpha \in \mathrm{KK}^C(A,D)$ can be represented by a Kasparov triple $(\mathcal{E}, \Phi, T)$, such that $\Phi(A)\mathcal{E} = \mathcal{E}$ and such that T is a C-equivariant operator on $\mathcal{E}$. Moreover, by [20, Proposition 3.4], we can also assume that the homotopies between equivalent triples have the same properties.

Using these reductions, we can now follow the constructions of [5, Lemma 4.11] (see also [21]) to build an inverse

$$\mathrm{inf}_C^G : \mathrm{KK}_*^C(A,D) \to \mathrm{KK}_*^G(\mathrm{Ind}_C^G A, D)$$

for the compression homomorphism comp_C^G: Let $\alpha \in \mathrm{KK}_*^C(A,D)$ be represented by a Kasparov triple $(\mathcal{E}, \Phi, T)$ with the properties as described above. Consider the complex vector space E consisting of all continuous functions $\xi : G \to \mathcal{E}$ such that

- $\xi(gc) = c^{-1}(\xi(g))$ for all $g \in G$, $c \in C$;
- the map $gC \mapsto \|\xi(g)\|$ has finite support in G/C.

Then E becomes a G-equivariant pre-Hilbert D-module by defining the D-valued inner product, the right D-action on E, and the action of G on E by

$$\langle \xi, \eta\rangle_D = \sum_{\dot g \in G/C} g(\langle \xi(g), \eta(g)\rangle_D), \quad (\xi \cdot d)(g) = \xi(g) \cdot g^{-1}(d), \quad \text{and}$$

$$(g \cdot \xi)(g') = \xi(g^{-1}g'),$$

for all $g, g' \in G$, $\xi, \eta \in E$ and $d \in D$. Let $\tilde{\mathcal{E}}$ denote the completion of E and define $\tilde\Phi : \mathrm{Ind}_C^G A \to \mathcal{L}(\tilde{\mathcal{E}})$ and an operator $\tilde T \in \mathcal{L}(\tilde{\mathcal{E}})$ by

$$\big(\tilde\Phi(F) \cdot \xi\big)(g) = \Phi\big(F(g)\big) \cdot \big(\xi(g)\big) \quad \text{and} \quad \big(\tilde T \xi\big)(g) = T\big(\xi(g)\big),$$

for $F \in \operatorname{Ind}_C^G A$ and $\xi \in E$. We want to define

$$\inf_C^G\big([(\mathcal{E},\Phi,T)]\big) = [(\tilde{\mathcal{E}},\tilde{\Phi},\tilde{T})] \in \mathrm{KK}^G\big(\operatorname{Ind}_C^G A, D\big). \tag{5.1}$$

For this we first have to show that $(\tilde{\mathcal{E}},\tilde{\Phi},\tilde{T})$ is a Kasparov triple in $\mathbb{E}^G(\operatorname{Ind}_C^G A, D)$. Since $\tilde{T}$ is clearly G-equivariant, it is enough to check that $[\tilde{T},\tilde{\Phi}(F)]$, $(\tilde{T}^2-1)\tilde{\Phi}(F)$, and $(\tilde{T}^*-\tilde{T})\tilde{\Phi}(F)$ are compact operators on $\tilde{\mathcal{E}}$ for all $F \in \operatorname{Ind}_C^G A$. Since $\tilde{T}$ is G-equivariant, we may replace F by any translate of it, and since the finite sums of the translates of the elements of the form $i_A(a)$, $a \in A$, are dense in $\operatorname{Ind}_C^G A$, we may even assume that $F = i_A(a)$ for some $a \in A$. Now observe that $\mathcal{E}$ embeds (C-equivariantly) as a direct summand of $\tilde{\mathcal{E}}$ via

$$\big(i_{\mathcal{E}}(w)\big)(g) = \begin{cases} g^{-1}(w) & \text{if } g \in C,\\ 0 & \text{if } g \notin C.\end{cases}$$

This induces a corresponding embedding $i_{\mathcal{K}(\mathcal{E})} : \mathcal{K}(\mathcal{E}) \to \mathcal{K}(\tilde{\mathcal{E}})$, and it follows directly from the formulas that $[\tilde{T},\tilde{\Phi}(i_A(a))] = i_{\mathcal{K}(\mathcal{E})}\big([T,\Phi(a)]\big)$, $(\tilde{T}^2-1)\tilde{\Phi}(i_A(a)) = i_{\mathcal{K}(\mathcal{E})}\big((T^2-1)\Phi(a)\big)$, and $(\tilde{T}^*-\tilde{T})\tilde{\Phi}(i_A(a)) = i_{\mathcal{K}(\mathcal{E})}\big((T^*-T)\Phi(a)\big)$, and hence all three elements are in $\mathcal{K}(\tilde{\mathcal{E}})$. Since the assignment $(\mathcal{E},\Phi,T) \mapsto (\tilde{\mathcal{E}},\tilde{\Phi},\tilde{T})$ preserves homotopy (we just apply the same construction to a homotopy), it is now clear that (5.1) determines a well defined map in KK-theory.
It is easy to check that $\mathrm{comp}_C^G \circ \inf_C^G$ is the identity on $\mathrm{KK}^C(A,D)$: Write $\tilde{\mathcal{E}} = \mathcal{E} \bigoplus \mathcal{F}$, with respect to the C-equivariant embedding $i_{\mathcal{E}} : \mathcal{E} \to \tilde{\mathcal{E}}$ considered above. Then show that $\tilde{\Phi} \circ i_A$ decomposes as $\Phi \bigoplus 0$ under the above decomposition of $\tilde{\mathcal{E}}$, from which it follows that $\mathrm{comp}_C^G \circ \inf_C^G\big([\mathcal{E},\Phi,T])\big) = [(\mathcal{E},\Phi,T)] \bigoplus [(\mathcal{F},0,\tilde{T})] = [(\mathcal{E},\Phi,T)] \in \mathrm{KK}^C(A,D)$.
Conversely, to see that $\inf_C^G \circ \mathrm{comp}_C^G = \mathrm{id}_{\mathrm{KK}^G(\operatorname{Ind}_C^G A,D)}$, we start with a Kasparov triple $(\mathcal{F},\Psi,S)$ representing a class in $\mathrm{KK}^G(\operatorname{Ind}_C^G A,D)$. Passing to the stabilization $A \otimes \mathcal{K}(L^2(G))$, if necessary (equipped with action $\alpha \otimes \operatorname{Ad}\lambda_G$), we can use the equation $\operatorname{Ind}_C^H\big(A \otimes \mathcal{K}(L^2(G))\big) = \big(\operatorname{Ind}_H^G A\big) \otimes \mathcal{K}(L^2(G))$ in order to apply Meyer's result [20, Proposition 3.2] to the induced algebra $\operatorname{Ind}_C^G A$. Thus we may assume without loss of generality that

(1) $\Psi(\operatorname{Ind}_C^G A)\mathcal{F} = \mathcal{F}$, and

(2) the operator $S \in \mathcal{L}(\mathcal{F})$ is G-equivariant.

We can use (1) to define a family $\{p_{\dot g} \mid \dot g \in G/C\}$ of projections on $\mathcal{F}$ by $p_{\dot g}\big(\Psi(F)\xi\big) = \Psi(F|_{gC})\xi$, for $F \in \operatorname{Ind}_C^G A$. We may then assume additionally that

(3) $p_{\dot g}S = Sp_{\dot g}$ for all $\dot g \in G/C$.

In fact, if S does not satisfy this condition, then we pass to the compact perturbation $S' = \sum_{\dot g \in G/C} p_{\dot g} S p_{\dot g}$ of S, which then satisfies (1)–(3) (to see that S' is a compact perturbation of S, i.e., that $(S-S')\Psi(F) \in \mathcal{K}(\mathcal{F})$ for all $F \in \operatorname{Ind}_C^G A$, one first observes that $(S-S')\Psi(F) = \sum_{G/C}(S - p_{\dot g}S)\Psi(F|_{gC})$, where the sum converges in the norm topology, and then one uses the compactness of $[S, \Psi(F|_{gC})]$ to see that each summand is a compact operator). Using these properties, we easily check that $(\mathcal{E}, \Phi, T) := (p_{\dot{e}}\mathcal{F}, p_{\dot{e}}\Psi p_{\dot{e}}, p_{\dot{e}} S p_{\dot{e}})$ is a representative for $\mathrm{comp}_C^G([\mathcal{F}, \Psi, S])$. A straightforward computation then shows that

$$\Theta : \tilde{\mathcal{E}} \to \mathcal{F}; \quad \Theta(\xi) = \sum_{\dot{g} \in G/C} g(\xi(g))$$

is an isomorphism which intertwines $\tilde{\Phi}$ with Ψ and $\tilde{T}$ with S. □

The main reduction argument for the proof of Proposition 5.9 is contained in the following lemma. We resume the situation of Lemma 5.6, i.e., we assume that G/N has a compact open subgroup, and $\mathcal{E}(G/N)$ has the structure of a simplicial complex as in Proposition 5.2. Moreover, for a simplex S of $\mathcal{E}(G/N)$ we let $\overset{\circ}{S}$ denote its interior, n its dimension and C the open subgroup of G which stabilizes S (with respect to the inflated action of G on $\mathcal{E}(G/N)$).

Lemma 5.15. *For any simplex S of $\mathcal{E}(G/N)$, let $Y \in \mathcal{F}$ be the G-saturation of S in $\mathcal{E}(G/N)$, and let Y_0 be the open subset of Y generated by $\overset{\circ}{S}$ under the action of G. Let X be a G-compact subspace of $\mathcal{E}(G) \times \mathcal{E}(G/N)$ such that $\pi_2(X) = Y$, and let X_0 be the open subset of X defined by $X_0 = X \cap \pi^{-1}(Y_0)$. Then:*

(i) After enlarging X, if necessary, we may assume that there exists a C-compact subset T of $\mathcal{E}(G)$ such that X_0 is G-homeomorphic to the induced space $G \times_C (T \times \overset{\circ}{S})$.

(ii) For every G-algebra B, the diagram

$$\begin{array}{ccc}
\mathrm{KK}_i^G\big(C_0(X_0), B\big) & \xrightarrow{\mu_{N,B}^{G,N}[X_0]} & \mathrm{KK}_i^{G,N}\big(C_0(Y_0), B \rtimes_r N\big) \\
\mathrm{comp}_C^G \downarrow \cong & & \cong \downarrow \mathrm{comp}_C^G \\
\mathrm{KK}_i^C\big(C_0(T \times \overset{\circ}{S}), B\big) & \xrightarrow{\mu_{N,B}^{C,N}[T \times \overset{\circ}{S}]} & \mathrm{KK}_i^{C,N}\big(C_0(\overset{\circ}{S}), B \rtimes_r N\big) \\
\beta\otimes . \downarrow \cong & & \cong \downarrow \beta\otimes . \\
\mathrm{KK}_{i+n}^C\big(C_0(T), B\big) & \xrightarrow{\mu_{N,B}^{C,N}[T]} & \mathrm{KK}_{i+n}^{C,N}(\mathbb{C}, B \rtimes_r N)
\end{array}$$

commutes, where $\beta \in \mathrm{KK}_n\big(\mathbb{C}, C_0(\mathbb{R}^n)\big)$ denotes the Bott element.

Proof. Since S generates Y as a G-space, we can choose a compact subset $L \subseteq \mathcal{E}(G) \times S$ such that L generates X as a G-space and such that $\pi_2(L) = S$. Let $T = C \cdot \pi_1(L)$, where $\pi_1 : \mathcal{E}(G) \times \mathcal{E}(G/N) \to \mathcal{E}(G)$ denotes the projection on the first factor, and let $X' = G \cdot (T \times S)$. Then $X \subseteq X'$, X' is G-compact, and $X'_0 \cong G \times_C (T \times \overset{\circ}{S})$ by Remark 5.13. Thus, replacing X by X' gives (i).

For the proof of (ii), first note that β can be seen as an element of $\mathrm{KK}_n^{C,N}\big(\mathbb{C}, C_0(\overset{\circ}{S})\big)$ because the action of C on $\overset{\circ}{S}$ is trivial and $\overset{\circ}{S}$ is homeomorphic to $\mathbb{R}^n$. To see the commutativity of the upper square of the diagram, we first have to introduce some notation:

- $i_1 : C_0(T\times \overset{\circ}{S}) \to C_0(X_0)$ and $i_2 : C_0(\overset{\circ}{S}) \to C_0(Y_0)$ denote the canonical inclusions (recall that because C is open in G, $\overset{\circ}{S}$ and $T\times \overset{\circ}{S}$ are open subsets of Y_0 and X_0 respectively).

- $q_1 : C_0(Y_0) \to C_0(N\backslash X_0)$ and $q_2 : C_0(\overset{\circ}{S}) \to C_0\big(N\backslash(T\times \overset{\circ}{S})\big)$ are the homomorphisms induced by the second projection $p_2 : N\backslash\big(\mathcal{E}(G)\times \mathcal{E}(G/N)\big) \to \mathcal{E}(G/N)$ (note that the restrictions of p_2 to $N\backslash X_0 \to Y_0$ and to $N\backslash(T\times \overset{\circ}{S}) \to \overset{\circ}{S}$ are proper maps).

The corresponding elements in the various equivariant KK-groups are denoted by the same letters. Using the definitions of $\mu_{N,B}^{C,N}[X_0]$, $\mu_{N,B}^{C,N}[T\times \overset{\circ}{S}]$ and comp_C^G (see Definition 5.10 and Proposition 5.14 above), we get for all $\alpha \in \mathrm{KK}_*^G\big(C_0(X_0), B\big)$:

$$\mu_{N,B}^{C,N}[T\times \overset{\circ}{S}] \circ \mathrm{comp}_C^G(\alpha) = q_2 \otimes \Lambda^{C,N}_{T\times\overset{\circ}{S},N} \otimes j^N_{\{e\},r}(i_1 \otimes \mathrm{res}_C^G(\alpha)).$$

On the other hand we have

$$\mathrm{comp}_C^G \circ \mu_{N,B}^{G,N}[X_0](\alpha) = i_2 \otimes \mathrm{res}_C^G\big(q_1 \otimes \Lambda^{G,N}_{X_0,N} \otimes j^N_{\{e\},r}(\alpha)\big).$$

But it is clear from Equation (1.1) that $\mathrm{res}_C^G(\Lambda^{G,N}_{X_0,N})$ is nothing but $\Lambda^{C,N}_{X_0,N}$. Using the fact that $\mathrm{res}_C^G(j^N_{\{e\},r}(\alpha)) = j^N_{\{e\},r}(\mathrm{res}_C^G(\alpha))$ (cf. [6, (2) Remark 4.6]), we note that the commutativity of the upper square of the diagram reduces to the equality:

$$i_2 \otimes \mathrm{res}_C^G(q_1) \otimes \Lambda^{C,N}_{X_0,N} = q_2 \otimes \Lambda^{C,N}_{T\times\overset{\circ}{S},N} \otimes j^N_{\{e\},r}(i_1),$$

which follows from Lemma 1.5.

To see the commutativity of the lower square of the diagram, we first observe that, since N (as a subgroup of C) acts trivially on $\overset{\circ}{S}$, we have

$$p_2^*(\Lambda^{C,N}_{T\times\overset{\circ}{S},N}) = \sigma_{C_0(\overset{\circ}{S})}\big(p_2^*(\Lambda^{C,N}_{T,N})\big) \quad \text{and} \quad j^N_{\{e\},r}\big(\sigma_{C(T)}(\beta)\big) = \sigma_{C(T)\rtimes N}(\beta),$$

where for any (C,N)-algebra D, $\sigma_D : \mathrm{KK}^{C,N}(A,B) \to \mathrm{KK}^{C,N}(A\otimes D, B\otimes D)$ denotes the external tensor product operator. Using this and the commutativity of the Kasparov product over $\mathbb{C}$, we compute for $\alpha \in \mathrm{KK}_i^{C,N}(C_0(T\times \overset{\circ}{S}), B)$:

$$\begin{aligned}
\mu_{N,B}^{C,N}[T](\beta\otimes\alpha) &= p_2^*(\Lambda_{T,N}^{C,N}) \otimes_{C(T)\rtimes N} j_{\{e\},r}^N\Big(\sigma_{C_0(T)}(\beta)\otimes_{C_0(T\times\overset{\circ}{S})}\alpha\Big)\\
&= p_2^*(\Lambda_{T,N}^{C,N}) \otimes_{C(T)\rtimes N} \big(\sigma_{C_0(T)\rtimes N}(\beta)\big) \otimes_{C_0(T\times\overset{\circ}{S})\rtimes N} \big(j_{\{e\},r}^N(\alpha)\big)\\
&= \Big(p_2^*(\Lambda_{T,N}^{C,N}) \otimes_{\mathbb{C}} \beta\Big) \otimes_{C_0(T\times\overset{\circ}{S})\rtimes N} \big(j_{\{e\},r}^N(\alpha)\big)\\
&= \Big(\beta \otimes_{\mathbb{C}} p_2^*(\Lambda_{T,N}^{C,N})\Big) \otimes_{C_0(T\times\overset{\circ}{S})\rtimes N} \big(j_{\{e\},r}^N(\alpha)\big)\\
&= \Big(\beta \otimes_{C_0(\overset{\circ}{S})} \sigma_{C_0(\overset{\circ}{S})}\big(p_2^*(\Lambda_{T,N}^{C,N})\big)\Big) \otimes_{C_0(T\times\overset{\circ}{S})\rtimes N} \big(j_{\{e\},r}^N(\alpha)\big)\\
&= \beta \otimes_{C_0(\overset{\circ}{S})} \Big(p_2^*(\Lambda_{T\times\overset{\circ}{S},N}^{C,N}) \otimes_{C_0(T\times\overset{\circ}{S})\rtimes N} \big(j_{\{e\},r}^N(\alpha)\big)\Big)\\
&= \beta \otimes \mu_{N,B}^{C,N}[T\times \overset{\circ}{S}](\alpha).
\end{aligned}$$

□

In what follows next, we still assume that G/N has a compact open subgroup, and that $\mathcal{E}(G/N)$ has the structure of a simplicial complex. As before, $\mathcal{F}$ denotes the family of G-saturations of a finite union of simplices in $\mathcal{E}(G/N)$.

Proposition 5.16. *Let B be a G-algebra such that for every compact subgroup $\dot{K}$ of G/N the group $K = q^{-1}(\dot{K}) \subseteq G$ satisfies BCC for B. Let $Y \in \mathcal{F}$ and let W be a finite set of simplices whose union generates Y as a G/N-space. Define* $\dim(Y)$ *to be the highest dimension of simplices in W and let $Y_0 \subseteq \mathcal{E}(G/N)$ be the G/N-saturation of the interiors of the simplices of dimension* $\dim(Y)$ *in W.*

(i) Assume that $\dim(Y) = n > 0$. *Then the partial assembly map*

$$\mu_{N,B}^{G,N}\langle Y_0\rangle : \lim_X \mathrm{KK}_*^G\big(C_0(X_0), B\big) \to \mathrm{KK}_*^{G,N}\big(C_0(Y_0), B\rtimes_r N\big)$$

of Definition 5.10 is bijective (recall that the limit is taken over the G-compact subsets X of $\mathcal{E}(G)\times\mathcal{E}(G/N)$ which satisfy $\pi_2(X) = Y$ and $X_0 = X\cap \pi_2^{-1}(Y_0)$).

(ii) Assume $\dim(Y) = 0$. *Then the partial assembly map*

$$\mu_{N,B}^{G,N}\langle Y\rangle : \lim_X \mathrm{KK}_*^G\big(C_0(X), B\big) \to \mathrm{KK}_*^{G,N}\big(C_0(Y), B\rtimes_r N\big)$$

of Definition 5.7 is bijective.

Proof. To show that (i) holds, note first that Y_0 is a finite union of disjoint spaces, each being the G-saturation of the interior $\overset{\circ}{S}$ of only one simplex S in $\mathcal{E}(G/N)$. We can therefore assume that Y is the G-saturation of S. It is not hard to check that the diagram of Lemma 5.15 is compatible with taking the inductive limit over the G-compact subsets X of $\mathcal{E}(G)\times\mathcal{E}(G/N)$ which satisfy

$\pi_2(X) = Y$. Moreover, by part (i) of Lemma 5.15, it follows that taking the limits over all $X_0 = X \cap \pi_2^{-1}(Y_0)$ such that X projects onto Y is the same as taking the limit over the sets $G \times_C (T \times \overset{\circ}{S})$, where T runs through the C-compact subsets of $\mathcal{E}(G)$. Thus, taking the limit over T of the diagram in part (ii) of Lemma 5.15 gives:

$$\begin{array}{ccc}
\lim_X \mathrm{KK}_i^G\big(C_0(X_0), B\big) & \xrightarrow{\mu_{N,B}^{G,N}\langle Y\rangle} & \mathrm{KK}_i^{G,N}\big(C_0(G/C), B \rtimes_r N\big) \\
\cong\downarrow & & \downarrow\cong \\
\lim_T \mathrm{KK}_{i+n}^C(C_0(T), B) & & \mathrm{KK}_{i+n}^{C,N}(\mathbb{C}, B \rtimes_r N) \\
=\downarrow & & \downarrow= \\
\mathrm{K}_{i+n}^{\mathrm{top}}(C; B) & \xrightarrow[\mu_{N,B}^{C,N}]{} & \mathrm{K}_{i+n}^{\mathrm{top}}(C/N; B \rtimes_r N),
\end{array}$$

where the first upper vertical arrows are given by the compositions $\mathrm{Bott} \circ \mathrm{comp}_C^G$. We now use the assumption that C satisfies BCC for B: Since C/N is compact (and thus satisfies BCC), Proposition 3.2 implies that the partial assembly map $\mu_{N,B}^{C,N}$ is a bijection. The above diagram then completes the proof of part (i) of the proposition.
For (ii), the same argument applies, starting from the fact that Y is a finite union of disjoint "induced spaces" $G/N \cdot x \cong G/C$, where x denotes a vertex of $\mathcal{E}(G/N)$ and C its stabilizer under the action of G. No Bott map (and thus no dimension shift) is required to get the analogue of the above diagram in this case. □

As we have already suggested, we are going to use Proposition 5.16 for an induction argument on the maximal dimension of the simplices involved. To do this, we need to be able to put the above maps into a six-term exact sequence in KK-theory, namely the Mayer-Vietoris sequence associated to the inclusion $Y_0 \to Y$.

Lemma 5.17. *Let Y, $n = \dim(Y)$, and Y_0 be as in Proposition 5.16. Assume further that $n > 0$. Then Y_0 is a nonempty open subset of Y and $Y_1 = Y \setminus Y_0$ is an element of $\mathcal{F}$ and satisfies $\dim(Y_1) = n - 1$. Furthermore, for any G-compact subset X of $\mathcal{E}(G) \times \mathcal{E}(G/N)$ such that $\pi_2(X) = Y$, we write $X_0 = X \cap \pi_2^{-1}(Y_0)$ and $X_1 = X \cap \pi_2^{-1}(Y_1)$. Then we get two equivariant exact sequences of commutative C^*-algebras:*

$$\begin{aligned}
\delta: &\quad 0 \to C_0(X_0) \to C_0(X) \to C_0(X_1) \to 0 \quad \textit{and} \\
d: &\quad 0 \to C_0(Y_0) \to C_0(Y) \to C_0(Y_1) \to 0
\end{aligned}$$

which determine elements $[\delta] \in \mathrm{KK}_1^G\big(C_0(X_1), C_0(X_0)\big)$ and $[d] \in \mathrm{KK}_1^G\big(C_0(Y_1), C_0(Y_0)\big)$ such that

$$[p_2^*]\otimes[\Lambda_{X_1,N}^{G,N}]\otimes j_{\{e\},r}^N([\delta]) = [d]\otimes[p_2^*]\otimes[\Lambda_{X_0,N}^{G,N}] \in \mathrm{KK}_1^{G,N}\big(C_0(Y_1), C_0(X_0)\rtimes N\big).$$

Proof. The existence of $[\delta] \in \mathrm{KK}_1^G\big(C_0(X_1), C_0(X_0)\big)$ and $[d] \in \mathrm{KK}_1^G\big(C_0(Y_1), C_0(Y_0)\big)$ is a particular case of [16, Corollary of Proposition 6.2]. For the equation, we first consider the extensions

$$\begin{array}{cccccccccc} d: & 0 & \longrightarrow & C_0(Y_0) & \longrightarrow & C_0(Y) & \longrightarrow & C_0(Y_1) & \longrightarrow & 0, \\ & & & \big\downarrow{\scriptstyle p_2^*} & & \big\downarrow{\scriptstyle p_2^*} & & \big\downarrow{\scriptstyle p_2^*} & & \\ \delta_N: & 0 & \longrightarrow & C_0(N\backslash X_0) & \longrightarrow & C_0(N\backslash X) & \longrightarrow & C_0(N\backslash X_1) & \longrightarrow & 0. \end{array}$$

Applying [23, Lemma 1.5] to this diagram implies that $[p_2^*]\otimes[\delta_N] = [d]\otimes[p_2^*] \in \mathrm{KK}_1^{G,N}\big(C_0(Y_1), C_0(N\backslash X_0)\big)$. Thus, it is enough to check that

$$[\Lambda^{G,N}_{X_1,N}] \otimes j^N_{\{e\},r}([\delta]) = [\delta_N] \otimes [\Lambda^{G,N}_{X_0,N}] \in \mathrm{KK}_1^{G,N}\big(C_0(N\backslash X_1), C_0(X_0)\rtimes N\big). \tag{5.2}$$

According to [1, Remarque 7.5, (2)], $[\delta_N]$ and $[\delta]$ are obtained from the Bott element $\beta \in \mathrm{KK}_1\big(\mathbb{C}, C_0(]0,1[)\big)$. To be more precise, recall from [4, §19.5] that if

$$c: 0 \to J \to A \to A/J \to 0$$

is a semi-split short exact sequence of C^*-algebras (i.e., there exists a completely positive section $q: A/J \to A$), then the canonical embedding

$$e: J \to C_q := C_0\big([0,1[, A\big)/C_0\big(]0,1[, J\big)$$

determines a KK-equivalence $[e] \in \mathrm{KK}_0(J, C_q)$. The same computations show that if the above short exact sequence is equivariant with respect to an action of a group G and if q can be chosen to be equivariant as well, then $e: J \to C_q := C_0\big([0,1[, A\big)/C_0\big(]0,1[, J\big)$ determines a KK-equivalence $[e] \in \mathrm{KK}_0^G(J, C_q)$ (where G acts trivially on $[0,1]$). Moreover, if we also consider the canonical inclusion

$$i: C_0\big(]0,1[, A/J\big) \to C_q,$$

then it follows from [1, Remarque 7.5, (2)] that the element $[c] \in \mathrm{KK}_1^G(A/J, J)$ coming from any equivariantly semi-split short exact sequence as above satisfies the equation

$$[e]\otimes[c] = \sigma_{A/J}(\beta)\otimes[i]. \tag{5.3}$$

We want to apply this to the short exact sequences δ and δ_N. For this define Z by $Z = \big(X\times[0,1[\big) \smallsetminus \big(X_0\times]0,1[\big)$. Then $C_0(Z)$ is the algebra C_q corresponding to the extension δ, and $C_0(N\backslash Z)$ becomes the substitute for C_q with respect to the extension δ_N. Let $e: C_0(X_0) \to C_0(Z)$ and $e_N: C_0(N\backslash X_0) \to C_0(N\backslash Z)$ denote the canonical inclusions (which, by the above discussion, are KK-equivalences) and let i and i_N denote the canonical inclusions of $C_0\big(X_1\times]0,1[\big)$

and $C_0\big(N\setminus(X_1\times]0,1[)\big)$ into $C_0(Z)$ and $C_0(N\setminus Z)$, respectively. Using this notation we now compute

$$\begin{aligned}
& \Lambda^{G,N}_{X_1,N}\otimes j^N_{\{e\},r}([\delta]) = [\delta_N]\otimes\Lambda^{G,N}_{X_0,N}\\
\Leftrightarrow\quad & \Lambda^{G,N}_{X_1,N}\otimes j^N_{\{e\},r}([\delta]\otimes[e]) = [\delta_N]\otimes\Lambda^{G,N}_{X_0,N}\otimes j^N_{\{e\},r}([e])\\
\Leftrightarrow\quad & \Lambda^{G,N}_{X_1,N}\otimes j^N_{\{e\},r}([\delta]\otimes[e]) = [\delta_N]\otimes[e_N]\otimes\Lambda^{G,N}_{Z,N}\ ,\qquad \text{by Lemma 1.5}\\
\Leftrightarrow\quad & \Lambda^{G,N}_{X_1,N}\otimes j^N_{\{e\},r}\big(\sigma_{C_0(X_1)}(\beta)\otimes[i]\big) = \sigma_{C_0(N\setminus X)}(\beta)\otimes[i_N]\otimes\Lambda^{G,N}_{Z,N}\ ,\ \text{by (5.3)}\\
\Leftrightarrow\quad & \Lambda^{G,N}_{X_1,N}\otimes j^N_{\{e\},r}\big(\sigma_{C_0(X_1)}(\beta)\otimes[i]\big) = \sigma_{C_0(N\setminus X)}(\beta)\otimes\Lambda^{G,N}_{X_1\times]0,1[,N}\otimes j^N_{\{e\},r}([i])
\end{aligned}$$

where the last line uses Lemma 1.5. Since G acts trivially on $]0,1[$, it follows that

$$\Lambda^{G,N}_{X_1\times]0,1[,N} = \sigma_{C_0\left(]0,1[\right)}(\Lambda^{G,N}_{X_1,N}). \tag{5.4}$$

On the other hand, since $\beta\in\mathrm{KK}_1\left(\mathbb{C},C_0\big(]0,1[\big)\right)$ (inflated to the various equivariant KK-groups), it follows that

$$j^N_{\{e\},r}\big(\sigma_{C_0(X_1)}(\beta)\big) = \sigma_{C_0(X_1)\rtimes N}(\beta). \tag{5.5}$$

Using (5.4) and (5.5), the above computation shows that it is enough to prove that $\big(\Lambda^{G,N}_{X_1,N}\otimes\sigma_{C_0(X_1)\rtimes N}(\beta)\big)\otimes j^N_{\{e\},r}([i])$ is equal to $\big(\sigma_{C_0(N\setminus X)}(\beta)\otimes\sigma_{C_0(]0,1[)}(\Lambda^{G,N}_{X_1,N})\big)\otimes j^N_{\{e\},r}([i])$ to get Equation (5.2). Using Kasparov's notations, this becomes

$$\big(\Lambda^{G,N}_{X_1,N}\otimes_{\mathbb{C}}\beta\big)j^N_{\{e\},r}([i]) = \big(\beta\otimes_{\mathbb{C}}\Lambda^{G,N}_{X_1,N}\big)j^N_{\{e\},r}([i]),$$

which is a consequence of the commutativity of the Kasparov product over $\mathbb{C}$ (see [14, Theorem 2.14]). This finishes the proof. □

We are now able to complete the proof of Proposition 5.9. This will also complete the proof of Theorem 3.3 since, as noted earlier, the theorem is a consequence of Proposition 5.9 and Lemma 5.8.

Proof of Proposition 5.9. We are going to make an induction on the dimension of $Y\in\mathcal{F}$. Let $Y\in\mathcal{F}$ such that $\dim(Y)=0$. Then $\mu^{G,N}_{N,B}\langle Y\rangle$ is bijective by (ii) of Proposition 5.16.
Let n be an arbitrary non-negative integer, and assume that $\mu^{G,N}_{N,B}\langle Z\rangle$ is bijective for all $Z\in\mathcal{F}$ such that $\dim(Z)\le n$.
Take $Y\in\mathcal{F}$ such that $\dim(Y)=n+1$, and let W be a finite set of simplices in $\mathcal{E}(G/N)$ which generate Y under the action of G. Define Y_0 to be the G-saturation of the union of the interiors of the simplices of dimension $n+1$ in W. Then Y_0 is open in Y and $Y_1 = Y\setminus Y_0$ is an element of $\mathcal{F}$ which has dimension less or equal to n.
Consider any G-compact subset X of $\mathcal{E}(G)\times\mathcal{E}(G/N)$ which satisfies $\pi_2(X)=Y$ and put $X_0 = X\cap\pi_2^{-1}(Y_0)$ and $X_1 = X\cap\pi_2^{-1}(Y_1)$. Using Lemma 5.17, we

obtain two long exact sequences in equivariant KK-theory, where the boundary maps are given by Kasparov product with the elements $[\delta]$ and $[d]$, respectively. Using [23, Lemma 1.5] we see that the sequence for X is compatible with taking inclusions of G-compact sets. Thus we can form the inductive limit over the G-compact subsets X of $\mathcal{E}(G) \times \mathcal{E}(G/N)$ which satisfy $\pi_2(X) = Y$, to obtain a diagram

$$\begin{array}{ccc}
\uparrow & & \uparrow \\
\lim_X \mathrm{KK}^G_{i+1}\big(C_0(X_1), B\big) & \xrightarrow{\mu^{G,N}_{N,B}\langle Y_1\rangle} & \mathrm{KK}^{G,N}_{i+1}\big(C_0(Y_1), B \rtimes_r N\big) \\
\uparrow & & \uparrow \\
\lim_X \mathrm{KK}^G_{i}\big(C_0(X_0), B\big) & \xrightarrow{\mu^{G,N}_{N,B}\langle Y_0\rangle} & \mathrm{KK}^{G,N}_{i}\big(C_0(Y_0), B \rtimes_r N\big) \\
\uparrow & & \uparrow \\
\lim_X \mathrm{KK}^G_{i}\big(C_0(X), B\big) & \xrightarrow{\mu^{G,N}_{N,B}\langle Y\rangle} & \mathrm{KK}^{G,N}_{i}\big(C_0(Y), B \rtimes_r N\big) \\
\uparrow & & \uparrow \\
\lim_X \mathrm{KK}^G_{i}\big(C_0(X_1), B\big) & \xrightarrow{\mu^{G,N}_{N,B}\langle Y_1\rangle} & \mathrm{KK}^{G,N}_{i}\big(C_0(Y_1), B \rtimes_r N\big) \\
\uparrow & & \uparrow \\
\lim_X \mathrm{KK}^G_{i+1}\big(C_0(X_0), B\big) & \xrightarrow{\mu^{G,N}_{N,B}\langle Y_0\rangle} & \mathrm{KK}^{G,N}_{i+1}\big(C_0(Y_0), B \rtimes_r N\big) \\
\uparrow & & \uparrow
\end{array}$$

in which the vertical sequences are exact. Using Lemma 5.17, it follows from the definition of the horizontal maps that the diagram commutes. By the induction hypothesis, the two horizontal arrows corresponding to Y_1 are bijective, and part (i) of Proposition 5.16 ensures that those corresponding to Y_0 are also bijective. Thus, it follows from the Five Lemma that $\mu^{G,N}_{N,B}\langle Y\rangle$ is bijective, too. □

6 Proof of the induction isomorphism

In this section we give the proof of the bijectivity of the induction homomorphism as stated in Theorem 2.2. For convenience, let's restate the theorem:

Theorem 2.2. *Let H be a closed subgroup of a group G, and let B be an H-algebra. Then the map $\mathrm{Ind}_H^G : \mathrm{K}^{\mathrm{top}}_*(H; B) \to \mathrm{K}^{\mathrm{top}}_*(G; \mathrm{Ind}_H^G B)$ is an isomorphism.*

As in the proof of Theorem 3.3 (see §5), we will use a special realization of the universal proper space for G to obtain a certain simplicial structure which allows an induction argument based on excision. In fact, if G_0 denotes the connected component of the identity in G, then G/G_0 is a totally disconnected group, and therefore has a compact open subgroup. Thus, by Proposition

5.2, there exists a realization of $\mathcal{E}(G/G_0)$ as a simplicial complex. If $\mathcal{E}(G)$ is any realization of the universal proper G-space, then, by Lemma 5.3, $\mathcal{E}(G) \times \mathcal{E}(G/G_0)$ equipped with the diagonal G-action is also a universal proper G-space, which we will use throughout this section to compute the topological K-theories of G and H.
The strategy used in the previous section will allow us to reduce the problem of the bijectivity of the induction homomorphism to the case of almost connected groups. But by Kasparov's work, every almost connected group has a γ-element (see Definition 1.7 and Remark 1.8). So, as a first step, we start with giving the proof under the extra condition that G has a γ-element. For this we have to use the following general lemma about the image of the assembly map in the presence of a γ-element for G. Note that this lemma is well known to the experts (it is implicitly contained in the work of Kasparov and Tu [15, 14, 25]). However, it seems that there exist no direct references. Thus, for the reader's convenience, we present a short argument building on [25, Proposition 5.23].

Lemma 6.1. *Assume that G has a γ-element $\gamma = \eta \otimes_{\mathcal{A}} D \in \mathrm{KK}^G(\mathbb{C},\mathbb{C})$ (see Definition 1.7). Then, for every G-algebra B, the assembly map induces an isomorphism between $\mathrm{K}_*^{\mathrm{top}}(G;B)$ and the γ-part*

$$\gamma\big(\mathrm{K}_*(B \rtimes_r G)\big) := \mathrm{K}_*(B \rtimes_r G) \otimes j_r^G(\sigma_B(\gamma)) \subseteq \mathrm{K}_*(B \rtimes_r G)$$

of $\mathrm{K}_*(B \rtimes_r G)$.

Proof. We will use the facts that the assembly map $\mu_{G,B}$ is injective, whenever G has a γ-element ([25, Proposition 5.23]) and that $\mu_{G,D}$ is surjective if D is a proper G-algebra (which follows from the descent isomorphism of [17]). It follows from part (2) of Definition 1.7 (see [6, Remark 6.4]) that the right Kasparov product with $\sigma_B(\gamma)$ determines the identity map on $\mathrm{KK}_*^G(C_0(X), B)$ for every proper G-space X. Thus, γ acts as the identity on $\mathrm{K}_*^{\mathrm{top}}(G;B)$ via right Kasparov product. This easily implies that the image of the assembly map $\mu_{G,B}$ lies in the γ-part of $\mathrm{K}_*(B \rtimes_r G)$, and we get a commutative diagram

$$\begin{array}{ccc}
\mathrm{K}_*^{\mathrm{top}}(G;B) & \xrightarrow{\mu_{G,B}(\cdot)} & \gamma_G\big(\mathrm{K}_*(B \rtimes_r G)\big) \\
{\scriptstyle \cdot\otimes\eta}\big\downarrow & & \big\downarrow{\scriptstyle \cdot\otimes j_r^G(\sigma_B(\eta))} \\
\mathrm{K}_*^{\mathrm{top}}(G;B\otimes\mathcal{A}) & \xrightarrow{\mu_{G,B\otimes\mathcal{A}}(\cdot)} & \gamma_G\Big(\mathrm{K}_*\big((B\otimes\mathcal{A}) \rtimes_r G\big)\Big) \\
{\scriptstyle \cdot\otimes D}\big\downarrow & & \big\downarrow{\scriptstyle \cdot\otimes j_r^G(\sigma_B(D))} \\
\mathrm{K}_*^{\mathrm{top}}(G;B) & \xrightarrow{\mu_{G,B}(\cdot)} & \gamma_G\big(\mathrm{K}_*(B \rtimes_r G)\big).
\end{array}$$

Since $B\otimes\mathcal{A}$ is a proper G-algebra, the middle horizontal row is a bijection, and, by the above discussion, the composition of the left-hand side vertical rows is the identity on $\mathrm{K}_*^{\mathrm{top}}(G;B)$. Finally, since γ is an idempotent in $\mathrm{KK}_0^G(\mathbb{C},\mathbb{C})$ by [25, Proposition 5.20], the composition of the right-hand side vertical arrows is the identity of $\gamma\big(\mathrm{K}_*(B \rtimes_r G)\big)$. Now a straightforward diagram chase gives the result. □

Lemma 6.2. *Let H be a closed subgroup of G and assume that G has a γ-element. Then, for every G-algebra B, the induction homomorphism Ind_H^G : $\mathrm{K}_*^{\mathrm{top}}(H;B) \to \mathrm{K}_*^{\mathrm{top}}(G;\mathrm{Ind}_H^G B)$ is bijective.*

Proof. Let γ_G be the γ-element of G. Then $\gamma_H = \mathrm{res}_H^G(\gamma_G)$ is the γ-element of H by [6, Remark 6.4]. Let $x \in \mathrm{KK}_*\left((\mathrm{Ind}_H^G B) \rtimes_r G, B \rtimes_r H\right)$ denote the invertible element implementing the canonical Morita equivalence between $(\mathrm{Ind}_H^G B) \rtimes_r G$ and $B \rtimes_r H$. As was already noted for the proof of [6, Proposition 6.9], the corollary on page 176 of [14] and item (2) of [15, Theorem 1 of §5] imply:

$$j_r^G\big(\sigma_{\mathrm{Ind}_H^G B}(\gamma_G)\big) = x \otimes_{B\rtimes_r H} j_r^H\big(\sigma_B(\gamma_H)\big) \otimes_{B\rtimes_r H} x^{-1}. \tag{6.1}$$

Together with Proposition 2.3, this implies that the two squares of the following diagram are commutative:

$$\begin{array}{ccc}
\mathrm{K}_*^{\mathrm{top}}(H;B) & \xrightarrow{\mathrm{Ind}_H^G} & \mathrm{K}_*^{\mathrm{top}}(G;\mathrm{Ind}_H^G B) \\
{\scriptstyle \mu_{H,B}}\big\downarrow & & \big\downarrow{\scriptstyle \mu_{G,\mathrm{Ind}_H^G B}} \\
\mathrm{K}_*(B \rtimes_r H) & \xrightarrow{\cdot\otimes x^{-1}} & \mathrm{K}_*\left((\mathrm{Ind}_H^G B) \rtimes_r G\right) \\
{\scriptstyle \cdot\otimes j_r^H\left(\sigma_B(\gamma_H)\right)}\big\downarrow & & \big\downarrow{\scriptstyle \cdot\otimes j_r^G\left(\sigma_{\mathrm{Ind}_H^G B}(\gamma_G)\right)} \\
\gamma_H\big(\mathrm{K}_*(B \rtimes_r H)\big) & \xrightarrow[\cdot\otimes x^{-1}]{} & \gamma_G\Big(\mathrm{K}_*\left((\mathrm{Ind}_H^G B) \rtimes_r G\right)\Big),
\end{array}$$

where $\gamma_H\big(\mathrm{K}_*(B \rtimes_r H)\big)$ (resp. $\gamma_G\Big(\mathrm{K}_*\left((\mathrm{Ind}_H^G B) \rtimes_r G\right)\Big)$) denotes the γ-part of $\mathrm{K}_*(B \rtimes_r H)$ (resp. $\mathrm{K}_*\left((\mathrm{Ind}_H^G B) \rtimes_r G\right)$). But Lemma 6.1 implies that the compositions of the vertical arrows are isomorphisms. Further, since the middle row of the above diagram is an isomorphism, Equation 6.1 also implies that the bottom arrow is an isomorphism. But then the top arrow has to be an isomorphism, too. □

As noted above, our aim is to reduce the proof of the general result of Theorem 2.2 to the special case where G is almost connected, in which case the result follows from Lemma 6.2. We start the reduction argument with some preliminaries:

Lemma 6.3. *Let H be a closed subgroup of G, let C be an open subgroup of G, and let B be an H-algebra. For each $\ddot{g}$ in the double coset space $H\backslash G/C$ (which is a discrete countable space) we put $C_H^g = C \cap g^{-1}Hg \subseteq C$, and we view B as a C_H^g-algebra by putting $g^{-1}hg \cdot b := h \cdot b$, $h \in H$, $b \in B$. Then the induced algebra $\mathrm{Ind}_H^G B$ is C-equivariantly isomorphic to $\bigoplus_{\ddot{g}\in H\backslash G/C} \mathrm{Ind}_{C_H^g}^C B$.*

Similarly, if $\mathcal{E}$ is an H-equivariant B-Hilbert module, there is a C-equivariant isomorphism between $\operatorname{Ind}_H^G \mathcal{E}$, viewed as a C-equivariant $\operatorname{Ind}_H^G B$-Hilbert module, and the $\bigoplus \operatorname{Ind}_{C_H^g}^C B$-Hilbert module $\bigoplus_{\ddot{g}\in H\backslash G/C} \operatorname{Ind}_{C_H^g}^C \mathcal{E}$.

Proof. Chose a set of representatives $\Gamma = \{g_0,\ g_1,\ ...,g_n,\ ...\}$ for $H\backslash G/C$ in G. By definition, $\operatorname{Ind}_H^G B$ is the subalgebra of $C_b(G,B)$ (the C^*-algebra of continuous bounded functions on G with values in B) consisting of all functions f which satisfy the conditions:

(i) $f(st) = t^{-1}\cdot f(s)$ for any $s\in G$ and $t\in H$,

(ii) $sH \mapsto \|f(s)\|$ is an element of $C_0(G/H)$.

The G action is given by $(s\cdot f)(t) = f(s^{-1}t)$, for $s,t\in G$.
For f in $\operatorname{Ind}_H^G B$ and $g\in\Gamma$, we define $\phi_{\ddot{g}} : \operatorname{Ind}_H^G B \to \operatorname{Ind}_{C_H^g}^C B$ by

$$\big(\phi_{\ddot{g}}(f)\big)(s) = g^{-1}\big(f(sg^{-1})\big), \quad \text{for all } s\in C.$$

It is straightforward to check that $\phi_{\ddot{g}}$ is a well defined C-equivariant $*$-homomorphism. Note that for any f in $\operatorname{Ind}_H^G B$ and any $\epsilon > 0$, there is a compact set $K\subseteq G/H$ such that, for all s in G, s belongs to K if $||f(s)|| \geq \epsilon/2$. Thus, if we denote by $\ddot{K}$ the image of K^{-1} in $H\backslash G/C$, we see that $||\phi_{\ddot{g}}(f)|| < \epsilon$ whenever $\ddot{g}$ does not belong to the compact set $\ddot{K}$. Hence the sequence $\Phi(f) = \big(\phi_{\ddot{g}}(f)\big)_{g\in\Gamma}$ belongs to $\bigoplus_{\ddot{g}\in H\backslash G/C} \operatorname{Ind}_{C_H^g}^C B$. It is readily seen that this defines an isomorphism Φ between $\operatorname{Ind}_H^G B$ and $\bigoplus_{\ddot{g}\in H\backslash G/C} \operatorname{Ind}_{C_H^g}^C B$: If $\lambda = (\lambda_g)_{g\in\Gamma} \in \bigoplus_{\ddot{g}\in H\backslash G/C} \operatorname{Ind}_{C_H^g}^C B$, we define $\Psi(\lambda)\in C_b(G,B)$ by $\Psi(\lambda)(s) = h^{-1}g_i^{-1}\lambda_{g_i}(c)$, whenever s is equal to cg_ih with $g_i\in\Gamma$. Then Ψ takes values in $\operatorname{Ind}_H^G B$ and Ψ is inverse to Φ.
A similar computation implies the decomposition $\operatorname{Ind}_H^G \mathcal{E} \cong \bigoplus \operatorname{Ind}_{C_H^g}^C \mathcal{E}$. □

Lemma 6.4. *Let H be a closed subgroup of G, let C be an open subgroup of G, and let A and B be two H-algebras. For $g\in G$ let*

$$\lambda_g : \operatorname{Ind}_{C_H^g}^C A \to \bigoplus_{\ddot{g}'\in H\backslash G/C} \operatorname{Ind}_{C_H^{g'}}^C A \quad \textit{and} \quad \rho_g : \bigoplus_{\ddot{g}'\in H\backslash G/C} \operatorname{Ind}_{C_H^{g'}}^C B \to \operatorname{Ind}_{C_H^g}^C B$$

denote the canonical C-equivariant inclusions and projections, respectively. Then, using the direct sum decomposition provided by Lemma 6.3, the diagram

$$\begin{array}{ccc}
\mathrm{KK}_*^H(A,B) & \xrightarrow{i_H^G} & \mathrm{KK}_*^G(\operatorname{Ind}_H^G A, \operatorname{Ind}_H^G B) \\
\Big\downarrow \mathrm{res}^H_{C_H^g} & & \Big\downarrow \mathrm{res}^G_C \\
\mathrm{KK}_*^{C_H^g}(A,B) & & \mathrm{KK}_*^C(\bigoplus \operatorname{Ind}_{C_H^{g'}}^C A, \bigoplus \operatorname{Ind}_{C_H^{g''}}^C B) \\
\Big\downarrow i^C_{C_H^g} & & \Big\downarrow \rho_g^* \\
\mathrm{KK}_*^C(\operatorname{Ind}_{C_H^g}^C A, \operatorname{Ind}_{C_H^g}^C B) & \xleftarrow[\lambda_{g,*}]{} & \mathrm{KK}_*^C(\bigoplus \operatorname{Ind}_{C_H^{g'}}^C A, \operatorname{Ind}_{C_H^g}^C B)
\end{array}$$

commutes (here the restriction $\operatorname{res}^H_{C^g_H} : \mathrm{KK}^H_*(A,B) \to \mathrm{KK}^{C^g_H}_*(A,B)$ *is defined by first restricting to* $H \cap gCg^{-1} = gC^g_H g^{-1}$ *and then* identifying *this group with* C^g_H *via conjugation). Moreover, if* $Hg'C \neq HgC$*, then* $[\lambda_g] \otimes \operatorname{res}^G_C \circ i^G_H(\alpha) \otimes [\rho_{g'}] = 0$ *for all* $\alpha \in \mathrm{KK}^H_*(A,B)$*.*

Proof. Note that for the proof of the commutativity of the above diagram we may replace A by any H-algebra A' which is H-equivariantly Morita equivalent to A (a similar statement holds for B, but we only need this for A). This follows easily from the fact that any H-equivariant Morita equivalence X between two H-algebras A and A' induces to a G-equivariant Morita equivalence $\operatorname{Ind}_H^G X$ between $\operatorname{Ind}_H^G A$ and $\operatorname{Ind}_H^G A'$, which by Lemma 6.3 also has a C-equivariant direct sum decomposition. Using this one checks that each map in the above diagram commutes with the respective Morita equivalences.
Replacing A by the Morita equivalent H-algebra $A \otimes \mathcal{K}(L^2(H))$, if necessary (with action given by $\alpha \otimes \operatorname{Ad}\lambda$, where $\alpha : H \to \operatorname{Aut}(A)$ denotes the given action on A and λ denotes the right regular representation of H), we can now use Meyer's result [20, Proposition 3.2] in order to assume that every $\alpha \in \mathrm{KK}^H_*(A,B)$ can be represented by a Kasparov triple $(\mathcal{E},\Phi,T) \in \mathbb{E}^H(A,B)$ such that $\Phi(A)\mathcal{E} = \mathcal{E}$ and such that T is H-equivariant. Using the formulas for the definitions of i^G_H and $i^C_{C^g_H}$, respectively (see §2), it follows from the decomposition of $\operatorname{Ind}_H^G \mathcal{E}$ as given in Lemma 6.3 and the H-equivariance of T, that

$$\operatorname{res}^G_C\left(\operatorname{Ind}_H^G \mathcal{E}, \operatorname{Ind}_H^G \Phi, \tilde{T}\right) \cong \Big(\bigoplus_{H\backslash G/C} \operatorname{Ind}^C_{C^g_H} \mathcal{E}_g, \bigoplus_{H\backslash G/C} \operatorname{Ind}^C_{C^g_H} \Phi_g, \bigoplus_{H\backslash G/C} \tilde{T}_g\Big)$$

in $\mathbb{E}^C(\operatorname{Ind}_H^G A, \operatorname{Ind}_H^G B)$, where $(\mathcal{E}_g, \Phi_g, T_g)$ denotes the cycle in $\mathbb{E}^{C^g_H}_*(A,B)$ obtained by first restricting the H action to $gCg^{-1} \cap H$, and then identifying $gCg^{-1} \cap H$ with C^g_H via the isomorphism given by conjugation with g. The result now follows immediately from this decomposition. □

The next result will be extended to arbitrary groups in §7 below (see Proposition 7.1). We only need here the weaker version where we assume that G is an almost connected group. As for the induction homomorphism, the proof of the general case will be done by a reduction to this case where G is almost connected.

Lemma 6.5. *Let* G *be an almost connected group and let* $B = \lim_i B_i$ *be an inductive limit of* G*-algebras* B_i*,* $i \in I$ *(with* G*-equivariant structure maps). Then*

$$\mathrm{K}^{\mathrm{top}}_*(G;B) \cong \lim_i \mathrm{K}^{\mathrm{top}}_*(G;B_i),$$

where the isomorphism is obtained from the morphisms $f_{i,*} :\ \mathrm{K}^{\mathrm{top}}_*(G;B_i) \to \mathrm{K}^{\mathrm{top}}_*(G;B)$*, which are induced by the canonical maps* $f_i :\ B_i \to B$*.*

Proof. Because G is almost connected, the functor which associates to a G-algebra the corresponding reduced crossed-product algebra is continuous with respect to taking inductive limits (in fact this holds whenever G is an exact group in the sense of [18]). We also know that G has a γ-element. So the lemma is an immediate consequence of the isomorphism $\mathrm{K}_*^{\mathrm{top}}(G;B)\cong\gamma\big(\mathrm{K}_*(B\rtimes_r G)\big)$ of Lemma 6.1 and of the continuity of K-theory ([4]). $\square$

We now come back to the proof of Theorem 2.2. As noted before, we use $\mathcal{E}(G)\times\mathcal{E}(G/G_0)$ as a universal example for the proper actions of G and H (see Lemmas 5.3 and 2.4 above), and assume that $\mathcal{E}(G/G_0)$ has the simplicial structure described in Proposition 5.2. As mentioned before, we can do this because G/G_0 is totally disconnected and, therefore, it has a compact open subgroup. In the following, we denote by $\mathcal{F}_G$ the family of all subsets of $\mathcal{E}(G/G_0)$ which are G-saturation of finite unions of simplices of $\mathcal{E}(G/G_0)$ (cf. Lemma 5.6, the subscript here is to prevent confusion between G- and H-actions).
As shown in Lemma 5.6, we can use $\mathcal{F}_G$ to compute $\mathrm{K}_*^{\mathrm{top}}(G;\mathrm{Ind}_H^G B)$ in the following way:

$$\mathrm{K}_*^{\mathrm{top}}(G;\mathrm{Ind}_H^G B)=\varinjlim_{Z\in\mathcal{F}_G}\ \varinjlim_{\substack{Y\subset\mathcal{E}(G)\times Z\\ Y\ G\text{-compact}}}\mathrm{KK}_*^G\big(C_0(Y),\mathrm{Ind}_H^G B\big). \tag{6.2}$$

But since $\mathcal{E}(G)\times\mathcal{E}(G/G_0)$ (with action restricted to H) serves also as a realization of the universal example for proper H-actions, we can use $\mathcal{F}_G$ also for the computation of $\mathrm{K}_*^{\mathrm{top}}(H;B)$:

$$\mathrm{K}_*^{\mathrm{top}}(H;B)=\varinjlim_{Z\in\mathcal{F}_G}\ \varinjlim_{\substack{Y\subset\mathcal{E}(G)\times Z\\ Y\ G\text{-compact}}}\ \varinjlim_{\substack{X\subset Y\\ X\ H\text{-compact}\\ G\cdot X=Y}}\mathrm{KK}_*^H\big(C_0(X),B\big). \tag{6.3}$$

The above formulas correspond to those of Lemma 5.6. Although these formulas look a bit complicated, they offer the advantage of breaking down the computation of the two topological K-theories into pieces which correspond to each other via the induction morphism and on which we can do an induction argument on the dimension $\dim(Z)$ of the elements $Z\in\mathcal{F}_G$.
In effect, using the above notations, note that $C_0(Y)=F^*\circ\mathrm{Ind}_H^G\big(C_0(X)\big)$, where $F:G\times_H X\to\mathcal{E}(G)\times\mathcal{E}(G/G_0)$ is defined by $F([s,x])=s\cdot x$. Thus, it follows from the definition of the induction homomorphism that, on $\mathrm{KK}_*^H(C_0(X),B)$, it factorizes via the diagram

$$\begin{array}{ccc}
\mathrm{KK}_*^H(C_0(X),B) & \xrightarrow{F^*\circ i_H^G} & \mathrm{KK}_*^G(C_0(Y),\mathrm{Ind}_H^G B)\\
\downarrow & & \downarrow\\
\mathrm{K}_*^{\mathrm{top}}(H;B) & \xrightarrow[\mathrm{Ind}_H^G]{} & \mathrm{K}_*^{\mathrm{top}}(G;\mathrm{Ind}_H^G B).
\end{array}$$

In what follows next we will simply write Ind_H^G for the map $F^* \circ i_H^G$ in the above diagram. Hence, in view of formula (6.2) and (6.3), the conclusion of Theorem 2.2 will follows from:

Proposition 6.6. *For each $Z \in \mathcal{F}_G$, the induction map Ind_H^G induces an isomorphism*

$$\varinjlim_{\substack{Y \subset \mathcal{E}(G)\times Z \\ Y\ G\text{-compact}}} \varinjlim_{\substack{X \subset Y \\ X\ H\text{-compact} \\ G\cdot X = Y}} \mathrm{KK}_*^H\big(C_0(X), B\big) \xrightarrow{\mathrm{Ind}_H^G} \varinjlim_{\substack{Y \subset \mathcal{E}(G)\times Z \\ Y\ G\text{-compact}}} \mathrm{KK}_*^G\big(C_0(Y), \mathrm{Ind}_H^G B\big).$$

As mentioned earlier, we want to use induction on $n = \dim(Z)$. As in the proof of Proposition 5.9, we introduce the following notations: Z_0 is the G/G_0-saturation of the interiors of the simplices of dimension n generating Z, $Z_1 = Z \setminus Z_0$; and we put

$$Y_0 = Y \cap (\mathcal{E}(G) \times Z_0); \qquad X_0 = X \cap (\mathcal{E}(G) \times Z_0);$$
$$Y_1 = Y \cap (\mathcal{E}(G) \times Z_1); \qquad X_1 = X \cap (\mathcal{E}(G) \times Z_1).$$

Note that

$$C_0(Y) = F^* \circ \mathrm{Ind}_H^G\big(C_0(X)\big), \qquad C_0(Y_0) = F^* \circ \mathrm{Ind}_H^G\big(C_0(X_0)\big),$$

$$\text{and} \quad C_0(Y_1) = F^* \circ \mathrm{Ind}_H^G\big(C_0(X_1)\big),$$

and that we have the exact sequences:

$$\delta: \quad 0 \longrightarrow C_0(X_0) \longrightarrow C_0(X) \longrightarrow C_0(X_1) \longrightarrow 0, \quad \text{and}$$
$$d: \quad 0 \longrightarrow C_0(Y_0) \longrightarrow C_0(Y) \longrightarrow C_0(Y_1) \longrightarrow 0.$$

Each of these two short exact sequences gives rise to a long exact sequence in equivariant KK-theory, which are linked by the induction homomorphisms Ind_H^G:

$$\begin{array}{ccc}
\uparrow & & \uparrow \\
\mathrm{KK}^H_{i+1}\big(C_0(X_1),B\big) & \xrightarrow{\mathrm{Ind}_H^G} & \mathrm{KK}^{G,N}_{i+1}\big(C_0(Y_1),\mathrm{Ind}_H^G B\big) \\
\uparrow & & \uparrow \\
\mathrm{KK}^H_{i}\big(C_0(X_0),B\big) & \xrightarrow{\mathrm{Ind}_H^G} & \mathrm{KK}^{G,N}_{i}\big(C_0(Y_0),\mathrm{Ind}_H^G B\big) \\
\uparrow & & \uparrow \\
\mathrm{KK}^H_{i}\big(C_0(X),B\big) & \xrightarrow{\mathrm{Ind}_H^G} & \mathrm{KK}^{G,N}_{i}\big(C_0(Y),\mathrm{Ind}_H^G B\big) \\
\uparrow & & \uparrow \\
\mathrm{KK}^H_{i}\big(C_0(X_1),B\big) & \xrightarrow{\mathrm{Ind}_H^G} & \lim_Y \mathrm{KK}^{G,N}_{i}\big(C_0(Y_1),\mathrm{Ind}_H^G B\big) \\
\uparrow & & \uparrow \\
\mathrm{KK}^H_{i-1}\big(C_0(X_0),B\big) & \xrightarrow{\mathrm{Ind}_H^G} & \mathrm{KK}^{G,N}_{i-1}\big(C_0(Y_0),\mathrm{Ind}_H^G B\big) \\
\uparrow & & \uparrow
\end{array} \tag{6.4}$$

We need

Lemma 6.7. *The above diagram commutes.*

Proof. The only slight difficulty arises at the square

$$\begin{array}{ccc}
\mathrm{KK}^H_{i+1}(C_0(X_1),B) & \xrightarrow{\mathrm{Ind}_H^G} & \mathrm{KK}^G_{i+1}(C_0(Y_1),\mathrm{Ind}_H^G B) \\
{\scriptstyle [\delta]\otimes}\uparrow & & \uparrow{\scriptstyle [d]\otimes} \\
\mathrm{KK}^H_{i}(C_0(X_0),B) & \xrightarrow{\mathrm{Ind}_H^G} & \mathrm{KK}^G_{i}(C_0(Y_0),\mathrm{Ind}_H^G B).
\end{array} \tag{6.5}$$

By the naturality of the boundary maps, we may assume without loss of generality that $Y = G \times_H X$ (and then $Y_i = G \times_H X_i$, $i = 0,1$), and that Ind_H^G coincides with Kasparov's induction i_H^G. We then follow the constructions in the proof of Lemma 5.17: Define the spaces

$$T = \big(X \times [0,1[\big) \smallsetminus \big(X_0 \times]0,1[\big) \text{ and } W = \big((G \times_H X) \times [0,1[\big) \smallsetminus \big((G \times_H X_0) \times]0,1[\big).$$

Let $e_X : C_0(X_0) \to C_0(T)$ and $e_{G\times_H X} : C_0(G \times_H X_0) \to C_0(W)$ denote the canonical inclusions, and let i_X and $i_{G\times_H X}$ denote the canonical inclusions of $C_0\big(X_1 \times]0,1[\big)$ and $C_0\big((G \times_H X_1) \times]0,1[\big)$ into $C_0(T)$ and $C_0(W)$, respectively. A short computation shows that $[e_{G\times_H X}] = i_H^G([e_X])$ and $[i_{G\times_H X}] = i_H^G([i_X])$ (where H and G act trivially on $[0,1]$). Moreover, we know from the discussion in the proof of Lemma 5.17 that $[e_X]$ and $[e_{G\times_H X}]$ are KK-equivalences and that

$$[e_X]\otimes[\delta] = \sigma_{C_0(X_1)}(\beta)\otimes[i_X] \quad \text{and} \quad [e_{G\times_H X}]\otimes[d] = \sigma_{C_0(G\times_H X_1)}(\beta)\otimes[i_{G\times_H X}],$$

where $\beta \in \mathrm{KK}_1\big(\mathbb{C}, C_0(]0,1[)\big)$ denotes the Bott-element, viewed as an element of the equivariant KK-groups with respect to the trivial group actions. Using the fact that i_H^G preserves Kasparov products, we now get

$$\begin{aligned}
[e_{G\times_H X}] \otimes [d] &= \sigma_{C_0(G\times_H X_1)}(\beta) \otimes [i_{G\times_H X}] \\
&= i_H^G\big(\sigma_{C_0(X_1)}(\beta) \otimes [i_X]\big) \\
&= i_H^G\big([e_X] \otimes \delta\big) \\
&= [e_{G\times_H X}] \otimes i_H^G([\delta]).
\end{aligned}$$

Since $[e_{G\times_H X}]$ is a KK-equivalence, it follows that $[d] = i_H^G([\delta])$, which easily implies the commutativity of (6.5). □

We are now taking limits of Diagram (6.4): First we are taking the inductive limit over the H-compact sets X such that $X \subset Y$ and $G \cdot X = Y$, and then we take the limit over the G-compact subsets Y of $\mathcal{E}(G) \times Z$. As a result, we obtain the commutative diagram

$$\begin{array}{ccc}
\uparrow & & \uparrow \\
\lim_Y \lim_X \mathrm{KK}_{i+1}^G\big(C_0(X_1), B\big) & \xrightarrow{\mathrm{Ind}_H^G} & \lim_Y \mathrm{KK}_{i+1}^{G,N}\big(C_0(Y_1), \mathrm{Ind}_H^G B\big) \\
\uparrow & & \uparrow \\
\lim_Y \lim_X \mathrm{KK}_{i}^G\big(C_0(X_0), B\big) & \xrightarrow{\mathrm{Ind}_H^G} & \lim_Y \mathrm{KK}_{i}^{G,N}\big(C_0(Y_0), \mathrm{Ind}_H^G B\big) \\
\uparrow & & \uparrow \\
\lim_Y \lim_X \mathrm{KK}_{i}^G\big(C_0(X), B\big) & \xrightarrow{\mathrm{Ind}_H^G} & \lim_Y \mathrm{KK}_{i}^{G,N}\big(C_0(Y), \mathrm{Ind}_H^G B\big) \\
\uparrow & & \uparrow \\
\lim_Y \lim_X \mathrm{KK}_{i}^G\big(C_0(X_1), B\big) & \xrightarrow{\mathrm{Ind}_H^G} & \lim_Y \mathrm{KK}_{i}^{G,N}\big(C_0(Y_1), \mathrm{Ind}_H^G B\big) \\
\uparrow & & \uparrow \\
\lim_Y \lim_X \mathrm{KK}_{i-1}^G\big(C_0(X_0), B\big) & \xrightarrow{\mathrm{Ind}_H^G} & \lim_Y \mathrm{KK}_{i-1}^{G,N}\big(C_0(Y_0), \mathrm{Ind}_H^G B\big) \\
\uparrow & & \uparrow
\end{array}$$

Using the same induction argument as in the proof of Proposition 5.16 of the previous section (based on the Five Lemma), the demonstration of Proposition 6.6, and hence the proof of Theorem 2.2 reduces to show

Lemma 6.8. *Let Z be an element of the family $\mathcal{F}_G$.*

(i) If $\dim(Z) > 0$, *then the map*

$$\lim_{\substack{\longrightarrow \\ Y\subset\mathcal{E}(G)\times Z \\ Y\ G\text{-compact}}} \lim_{\substack{\longrightarrow \\ X\subset Y \\ X\ H\text{-compact} \\ G\cdot X = Y}} \mathrm{KK}_i^H\big(C_0(X_0), B\big) \xrightarrow{\mathrm{Ind}_H^G} \lim_{\substack{\longrightarrow \\ Y\subset\mathcal{E}(G)\times Z \\ Y\ G\text{-compact}}} \mathrm{KK}_i^G\big(C_0(Y_0), \mathrm{Ind}_H^G B\big)$$

is a bijection.

(ii) If $\dim(Z) = 0$*, then the map*

$$\lim_{\substack{\longrightarrow \\ Y \subset \mathcal{E}(G)\times Z \\ Y\ G\text{-compact}}} \lim_{\substack{\longrightarrow \\ X \subset Y \\ X\ H\text{-compact} \\ G\cdot X = Y}} \mathrm{KK}_i^H\big(C_0(X), B\big) \stackrel{\mathrm{Ind}_H^G}{\longrightarrow} \lim_{\substack{\longrightarrow \\ Y \subset \mathcal{E}(G)\times Z \\ Y\ G\text{-compact}}} \mathrm{KK}_i^G\big(C_0(Y), \mathrm{Ind}_H^G B\big)$$

is a bijection.

Proof. We will only show part (i), since part (ii) follows by almost the same (but somewhat easier) arguments. So assume that $\dim(Z) > 0$. By the definition of $\mathcal{F}_G$, the space Z_0 is a disjoint union of finitely many spaces Z_0^i, $i = 1, 2, ..., k$, each of the form $Z_0^i = G/G_0 \cdot \overset{o}{S}_i$, where the S_i are simplices of dimensions $\dim(Z)$ of $\mathcal{E}(G/G_0)$. Setting $Z^i = G/G_0 \cdot S_i$, $Y_0^i = Y \cap (\mathcal{E}(G/G_0) \times Z_0^i)$ and $X_0^i = X \cap Y_0^i$, we obtain finite partitions of Y_0 and of X_0:

$$C_0(Y) = \bigoplus_{i=1}^{k} C_0(Y_0^i) \quad \text{and} \quad C_0(X) = \bigoplus_{i=1}^{k} C_0(X_0^i).$$

Note that these decompositions are compatible with the morphism Ind_H^G, so it is enough to give a proof of Lemma 6.8 in the case where Z is the G/G_0-saturation of a single simplex S of $\mathcal{E}(G/G_0)$. Further, the inductive limits over Y are taken over the G-compact subspaces of $\mathcal{E}(G) \times Z$. But any such space can be embedded in a G-compact set of the special form $Y = G \cdot (K \times S)$, where K is a compact subset of $\mathcal{E}(G)$. Hence, we can assume that every set Y which appears in the formula of the inductive limit is of this special kind.
Denote by $\dot{C}$ the stabilizer of S under the action of G/G_0, and let $C := q^{-1}(\dot{C}) \subseteq G$. Then

$$Y = G \cdot (K \times S) = G \cdot \big((C \cdot K) \times S\big) \quad \text{and} \quad Y_0 = G \times_C \big((C \cdot K) \times \overset{o}{S}\,\big).$$

For every double coset $\ddot{g} \in H\backslash G/C$, we consider the space

$$Y^{\ddot{g}} = Hg \cdot (C \cdot K \times S).$$

It is a closed H-invariant subspace of Y, and any H-compact subspace X of Y can be written as

$$X = \cup_{\ddot{g} \in F_X} X^{\ddot{g}}, \quad \text{with } X^{\ddot{g}} := X \cap Y^{\ddot{g}},$$

where F_X is a *finite* subset of $H\backslash G/C$. Put $X_0^{\ddot{g}} = X^{\ddot{g}} \cap Y_0$. We record the fact that each $X^{\ddot{g}}$ is an H-compact subspace of $Y^{\ddot{g}}$, and that X_0 is the disjoint union of the $X_0^{\ddot{g}}$, $\ddot{g} \in F_X$. As a consequence, we get

$$\mathrm{KK}_*^H\big(C_0(X_0), B\big) = \bigoplus_{\ddot{g} \in F_X} \mathrm{KK}_*^H\big(C_0(X_0^{\ddot{g}}), B\big).$$

Moreover, any H-compact subset of $Y^{\ddot g} = Hg\cdot(C\cdot K\times S)$ can be realized as a subset of an H-compact set of the form $Hg\cdot(L\times S)$, for L a compact subset of $C\cdot K$ satisfying $C\cdot L = C\cdot K$. Thus, when taking the inductive limit over the H-compact sets X which satisfy $G\cdot X = Y$, we can always enlarge X in order to assume that for every $\ddot g\in F_X$, $X^{\ddot g} = Hg\cdot(L\times S)$, for some compact subset $L\subseteq C\cdot K$ such that $C\cdot L = C\cdot K$. Moreover, it follows from this that $X_0^{\ddot g} = Hg\cdot(L\times \overset{\circ}{S})$. Thus, we obtain:

$$\varinjlim_{\substack{X\subset Y=G\cdot(K\times S)\\ X\ H\text{–compact}\\ GX=Y}} \mathrm{KK}_i^H\big(C_0(X_0),B\big) = \bigoplus_{\ddot g\in H\backslash G/C}\ \varinjlim_{\substack{L\subset C\cdot K\\ \text{compact}\\ C\cdot L=C\cdot K}} \mathrm{KK}_i^H\big(C_0(Hg\cdot(L\times \overset{\circ}{S})),B\big).$$

Hence, in order to prove the first part of the lemma, it is enough to show the bijectivity of

$$\varinjlim_{\substack{K\subset\mathcal{E}(G)\\ \text{compact}}}\bigoplus_{\ddot g}\varinjlim_{\substack{L\subset C\cdot K\\ \text{compact}\\ C\cdot L=C\cdot K}}\mathrm{KK}_i^H\big(C_0(Hg\cdot(L\times\overset{\circ}{S})),B\big)\xrightarrow{\mathrm{Ind}_H^G}\varinjlim_{\substack{K\subset\mathcal{E}(G)\\ \text{compact}}}\mathrm{KK}_i^G\big(C_0(G\cdot(K\times\overset{\circ}{S})),\mathrm{Ind}_H^G B\big). \tag{6.6}$$

We already noticed that $Y_0 = G\cdot(K\times\overset{\circ}{S})$ is canonically G-homeomorphic to the induced space $Y_0 = G\times_C(C\cdot K\times\overset{\circ}{S})$. Correspondingly, we now check that $X_0^{\ddot g} = Hg\cdot(L\times\overset{\circ}{S})$ is also an induced space. The composition

$$\begin{array}{ccccc} X_0^{\ddot g} = Hg\cdot(L\times\overset{\circ}{S}) & \overset{\pi_2}{\to} & Hg\cdot\overset{\circ}{S} & \to & H/(gCg^{-1}\cap H)\\ hg\cdot(l,s) & \mapsto & hgs & \mapsto & h(gCg{-}1\cap H)\end{array}$$

is an H-equivariant map, and the pre-image of the coset $gCg{-}1\cap H$ of the identity is $\pi_2^{-1}(g\cdot\overset{\circ}{S}) = g\cdot(C_H^g\cdot L\times\overset{\circ}{S})$, where C_H^g is the group $C\cap g^{-1}Hg$. Applying Proposition 5.12, we see that $X_0^{\ddot g}$ is H-homeomorphic to the induced space $H\times_{gCg^{-1}\cap H}\big(g\cdot(C_H^g\cdot L\times\overset{\circ}{S})\big)$, with H-homeomorphism given by

$$\begin{array}{ccc} X_0^{\ddot g} = Hg\cdot(L\times\overset{\circ}{S}) & \to & H\times_{gCg^{-1}\cap H}\big(g\cdot(C_H^g\cdot L\times\overset{\circ}{S})\big)\\ hg\cdot(l,s) & \mapsto & \big(h, g\cdot(l,s)\big)\end{array}$$

Let $\varphi_H := \mathrm{Bott}\circ\mathrm{comp}_{C_H^g}^H$ be the composition of the sequence of isomorphisms

$$\begin{array}{rl}\mathrm{KK}_i^H\big(C_0(Hg\cdot(L\times\overset{\circ}{S})),B\big) \xrightarrow{\mathrm{comp}_{C_H^g}^H} & \mathrm{KK}_i^{C_H^g}\big(C_0(C_H^g\cdot L\times\overset{\circ}{S}),B\big)\\ \xrightarrow{\mathrm{Bott}} & \mathrm{KK}_{i+n}^{C_H^g}\big(C_0(C_H^g\cdot L),B\big),\end{array}$$

where $n=\dim(S)$. Here the compression isomorphism $\mathrm{comp}_{C_H^g}^H$ has to be understood as the composition of the compression

$$\mathrm{comp}_{H\cap gCg^{-1}}^H : \mathrm{KK}_*^H\big(C_0(X_0^{\ddot g}),B\big)\to\mathrm{KK}_*^{H\cap gCg^{-1}}\big(C_0(g\cdot(C_H^g\cdot L\times\overset{\circ}{S}),B\big),$$

and then making the identification

$$\mathrm{KK}_*^{H\cap gCg^{-1}}\big(C_0(g\cdot(C_H^g\cdot L\times \overset{\circ}{S}),B\big)\cong \mathrm{KK}_*^{C_H^g}\big(C_0(C_H^g\cdot L\times \overset{\circ}{S}),B\big),$$

which comes from identifying C_H^g with $H\cap gCg^{-1}$ via conjugation with g (please compare with the definition of the restriction map in Lemma 6.4). In particular, we regard B as a C_H^g-algebra by setting $g^{-1}hg\cdot b := h\cdot b$ for $h\in gCg^{-1}\cap H$. By first taking the direct limit over the compact subsets $L\subseteq C\cdot K$, then taking the algebraic direct sum over the double cosets of $H\backslash G/C$, and eventually taking the inductive limit over the compact subsets $K\subseteq \mathcal{E}(G)$, we then obtain an isomorphism

$$\varinjlim_{\substack{K\subset\mathcal{E}(G)\\ \text{compact}}}\bigoplus_{\ddot g}\varinjlim_{\substack{L\subset C\cdot K\\ \text{compact}\\ C\cdot L=C\cdot K}}\mathrm{KK}_i^H\big(C_0(X_0^{\ddot g}),B\big)\xrightarrow{\varphi_H}\varinjlim_{\substack{K\subset\mathcal{E}(G)\\ \text{compact}}}\bigoplus_{\ddot g}\varinjlim_{\substack{L\subset C\cdot K\\ \text{compact}\\ C\cdot L=C\cdot K}}\mathrm{KK}_{i+n}^{C_H^g}\big(C_0(C_H^G\cdot L),B\big).$$

Note that the direct limit over K and the direct sum over $\ddot g\in H\backslash G/C$ can be permuted in the right-hand side term. Thus we get

$$\varinjlim_{\substack{K\subset\mathcal{E}(G)\\ \text{compact}}}\bigoplus_{\ddot g\in H\backslash G/C}\varinjlim_{\substack{L\subset C\cdot K\\ \text{compact}\\ C\cdot L=C\cdot K}}\mathrm{KK}_{i+n}^{C_H^g}\big(C_0(C_H^G\cdot L),B\big) = \bigoplus_{\ddot g\in H\backslash G/C}\mathrm{K}_{i+n}^{\mathrm{top}}(C_H^g;B).$$

In the end, we see that the left-hand side of (6.6) is isomorphic to $\bigoplus_{\ddot g\in H\backslash G/C}\mathrm{K}_{i+n}^{\mathrm{top}}(C_H^g;B)$.

On the right-hand side of (6.6) we have a corresponding sequence of isomorphisms: We first consider the composition $\varphi_G := \mathrm{comp}_C^G\circ\mathrm{Bott}$ of the sequence of isomorphisms

$$\begin{aligned}\mathrm{KK}_i^G\big(C_0(Y_0),\mathrm{Ind}_H^G B\big)&\xrightarrow{\mathrm{comp}_C^G}\mathrm{KK}_i^C\big(C_0(C\cdot K\times \overset{\circ}{S}),\mathrm{Ind}_H^G B\big)\\ &\xrightarrow{\mathrm{Bott}}\mathrm{KK}_{i+n}^C\big(C_0(C\cdot K),\mathrm{Ind}_H^G B\big).\end{aligned}$$

Exactly as above, taking the direct limit over the compact subsets K of $\mathcal{E}(G)$, we obtain an isomorphism for the right-hand side of (6.6):

$$\begin{array}{ccc}\varinjlim\limits_{\substack{K\subset\mathcal{E}(G)\\ \text{compact}}}\mathrm{KK}_i^G\big(C_0(G\cdot(K\times\overset{\circ}{S})),\mathrm{Ind}_H^G B\big) & \xrightarrow{\varphi_G} & \varinjlim\limits_{\substack{K\subset\mathcal{E}(G)\\ \text{compact}}}\mathrm{KK}_{i+n}^C\big(C_0(C\cdot K),\mathrm{Ind}_H^G B\big)\\ & & \Big\downarrow\cong\\ & & \mathrm{K}_{i+n}^{\mathrm{top}}(C;\mathrm{Ind}_H^G B)\end{array}$$

On the other hand, we can use Lemma 6.3 to see that $\mathrm{Ind}_H^G B$ is C-isomorphic to the direct sum $\bigoplus_{\ddot{g}\in H\backslash G/C} \mathrm{Ind}_{C_H^g}^C B$. But C is a compact extension of G_0, so it is almost connected . Therefore, Lemma 6.5 implies that

$$\mathrm{K}_*^{\mathrm{top}}(C;\mathrm{Ind}_H^G B) \cong \mathrm{K}_*^{\mathrm{top}}(C;\bigoplus_{\ddot{g}\in H\backslash G/C} \mathrm{Ind}_{C_H^g}^C B) \cong \bigoplus_{\ddot{g}\in H\backslash G/C} \mathrm{K}_*^{\mathrm{top}}(C;\mathrm{Ind}_{C_H^g}^C B).$$

Let us now consider the following diagram, where the top line is the map (6.6)

$$\begin{array}{ccc}
\varinjlim_{Y} \bigoplus_{\ddot{g}} \varinjlim_{X^{\ddot{g}}\subset Y^{\ddot{g}}} \mathrm{KK}_i^H\big(C_0(X_0^{\ddot{g}}),B\big) & \xrightarrow{\mathrm{Ind}_H^G} & \varinjlim_{Y} \mathrm{KK}_i^G\big(C_0(Y_0),\mathrm{Ind}_H^G B\big) \\
 & & \downarrow{\scriptstyle \varphi_G} \\
\downarrow{\scriptstyle \varphi_H} & & \mathrm{K}_{i+n}^{\mathrm{top}}(C;\mathrm{Ind}_H^G B) \\
 & & \downarrow{\scriptstyle \cong} \\
\bigoplus_{\ddot{g}\in H\backslash G/C} \mathrm{K}_{i+n}^{\mathrm{top}}(C_H^g;B) & \xrightarrow[\bigoplus \mathrm{Ind}_{C_H^g}^C]{} & \bigoplus_{\ddot{g}\in H\backslash G/C} \mathrm{K}_{i+n}^{\mathrm{top}}(C;\mathrm{Ind}_{C_H^g}^C B).
\end{array} \tag{6.7}$$

The columns of (6.7) are bijections. The bottom line (obtained from the induction morphisms from C_H^g to C) is an isomorphism by Lemma 6.2, since C has a γ-element. Hence, to obtain that the map in (6.6) is an isomorphism, and thus to conclude the proof of the lemma, we just have to show that Diagram (6.7) commutes.

For this let g be any element in G, let K be a compact subset of $\mathcal{E}(G)$, and let L be a compact subset of $C\cdot K$ such that $C\cdot L = C\cdot K$. Let

$$X_0^{\ddot{g}} = Hg\cdot(L\times \overset{\circ}{S}) \quad \text{and} \quad Y_0 = G\times_C (C\cdot L\times \overset{\circ}{S}).$$

For each $g''\in G$ let $\rho_{g''} : \bigoplus \mathrm{Ind}_{C_H^{g'}}^C B \to \mathrm{Ind}_{C_H^{g''}}^C B$ denote the canonical projection. To see that (6.7) commutes, we have to verify the following two statements:

(i) The diagram

$$\begin{array}{ccc}
\mathrm{KK}_i^H\big(C_0(X_0^{\ddot g}),B\big) & \xrightarrow{\mathrm{Ind}_H^G} & \mathrm{KK}_i^G\big(C_0(Y_0),\mathrm{Ind}_H^G B\big)\\
{\scriptstyle \mathrm{comp}^H_{C_H^g}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{comp}_C^G}\\
\mathrm{KK}_i^{C_H^g}\big(C_0(C_H^g\cdot L\times \mathring{S}),B\big) & & \mathrm{KK}_i^C\big(C_0(C\cdot K\times \mathring{S}),\mathrm{Ind}_H^G B\big)\\
 & & \big\downarrow{\scriptstyle \otimes\,\mathrm{Bott}}\\
{\scriptstyle \otimes\,\mathrm{Bott}}\big\downarrow & & \mathrm{KK}_{i+n}^C\big(C_0(C\cdot K),\mathrm{Ind}_H^G B\big)\\
 & & \big\downarrow{\scriptstyle \rho_g^*}\\
\mathrm{KK}_{i+n}^{C_H^g}\big(C_0(C_H^g\cdot L),B\big) & \xrightarrow[\mathrm{Ind}^C_{C_H^g}]{} & \mathrm{KK}_{i+n}^C\big(C_0(C\cdot K),\mathrm{Ind}^C_{C_H^g} B\big)
\end{array}$$

commutes, and

(ii) The composition

$$\begin{aligned}
\mathrm{KK}_i^H\big(C_0(X_0^{\ddot g}),B\big) &\xrightarrow{\mathrm{Ind}_H^G} \mathrm{KK}_i^G\big(C_0(Y_0),\mathrm{Ind}_H^G B\big)\\
&\xrightarrow{\mathrm{comp}_C^G} \mathrm{KK}_i^C\big(C_0(C\cdot K\times \mathring{S}),\mathrm{Ind}_H^G B\big)\\
&\xrightarrow{\mathrm{Bott}} \mathrm{KK}_{i+n}^C\big(C_0(C\cdot K),\mathrm{Ind}_H^G B\big)\\
&\xrightarrow{\rho_{g''}} \mathrm{KK}_{i+n}^C\big(C_0(C\cdot K),\mathrm{Ind}^C_{C_H^{g''}} B\big)
\end{aligned}$$

is the zero homomorphism whenever $g''\notin HgC$.

If $\beta\in\mathrm{KK}_n(\mathbb{C},C_0(\mathring{S}))$ denotes the Bott-element, then the first condition is just the equation

$$\beta\otimes\mathrm{comp}_C^G\circ\mathrm{Ind}_H^G(\alpha)\otimes[\rho_g]=\mathrm{Ind}^C_{C_H^g}(\beta\otimes\mathrm{comp}^H_{C_H^g}(\alpha)),$$

for all $\alpha\in\mathrm{KK}_i^H\big(C_0(X_0^{\ddot g}),B\big)$. Since C acts trivially on $\mathring{S}$, we can permute $\mathrm{Ind}^C_{C_H^g}$ and the product with the Bott element β. Thus, the problem reduces to showing that

$$\mathrm{comp}_C^G\circ\mathrm{Ind}_H^G(\alpha)\otimes[\rho_g]=\mathrm{Ind}^H_{C_H^g}\circ\mathrm{comp}^H_{C_H^g}(\alpha). \tag{6.8}$$

In order to check this equation, it is useful to introduce the following notations:

- F_1 is the G-equivariant map appearing in the definition of Ind_H^G:

$$\begin{array}{rccc}
F_1: & G\times_H X_0^{\ddot g} & \to & Y_0\subset\mathcal{E}(G)\times\mathcal{E}(G/N)\\
 & [g_1,hg\cdot(l,s)] & \mapsto & g_1hg\cdot(l,s),
\end{array}$$

where we used the equation $X_0^{\ddot g}=Hg(L\times\mathring{S})$.

- i_1 is the C-equivariant inclusion used to define comp_C^G

$$i_1: \quad C \cdot L \times \overset{\circ}{S} \quad \to \quad G \times_C (C \cdot L \times \overset{\circ}{S}) \cong Y_0,$$

- i_2 is the C_H^g-equivariant inclusion in the definition of $\mathrm{comp}_{C_H^g}^H$

$$\begin{array}{rccc} i_2: & C_H^g \cdot L \times \overset{\circ}{S} & \to & X_0^{\ddot{g}} = Hg \cdot (L \times \overset{\circ}{S}) \\ & (c \cdot l, s) & \mapsto & gc \cdot (l, s), \end{array}$$

- F_2 is the C-equivariant map in the definition of $\mathrm{Ind}_{C_H^g}^C$

$$\begin{array}{rccc} F_2: & C \times_{C_H^g} (C_H^g \cdot L \times \overset{\circ}{S}) & \to & C \cdot L \times \overset{\circ}{S} \\ & [c, (c' \cdot l, s)] & \mapsto & (c'c \cdot l, s). \end{array}$$

We will also use that i_2 induces a C-equivariant injection

$$\begin{array}{rccc} I_2: & C \times_{C_H^g} (C_H^g \cdot L \times \overset{\circ}{S}) & \to & C \times_{C_H^g} (X_0^{\ddot{g}}) = C \times_{C_H^g} \big(Hg \cdot (L \times \overset{\circ}{S})\big) \\ & [c_1, (c_2 \cdot l, s)] & \mapsto & [c_1, gc_2 \cdot (l, s)]. \end{array}$$

and we will denote by i_3 the C-equivariant inclusion

$$\begin{array}{rccc} i_3: & C \times_{C_H^g} X_0^{\ddot{g}} & \to & G \times_H (X_0^{\ddot{g}}) \\ & [c, x] & \mapsto & [cg^{-1}, x]. \end{array}$$

Writing the KK-classes defined by these maps with the same letters, Equation (6.8) becomes:

$$[i_1] \otimes \mathrm{res}_C^G([F_1] \otimes i_H^G(\alpha)) \otimes [\rho_g] = [F_2] \otimes i_{C_H^g}^C([i_2] \otimes \mathrm{res}_{C_H^g}^H(\alpha)).$$

Since $[I_2] = \mathrm{Ind}_C^{C_H^g}([i_2])$, this is equivalent to:

$$[i_1] \otimes \mathrm{res}_C^G([F_1]) \otimes \mathrm{res}_C^G(\mathrm{Ind}_H^G(\alpha)) \otimes [\rho_g] = [F_2] \otimes [I_2] \otimes \mathrm{Ind}_{C_H^g}^C(\mathrm{res}_{C_H^G}^H(\alpha)). \tag{6.9}$$

A short computation shows that $[i_1] \otimes \mathrm{res}_C^G([F_1]) = [F_2] \otimes [I_2] \otimes [i_3]$ (just compute the compositions of the associated $*$-homomorphisms). Thus, Equation (6.9) reduces to:

$$[i_3] \otimes \mathrm{res}_C^G(i_H^G(\alpha)) \otimes [\rho_g] = i_{C_H^g}^C(\mathrm{res}_{C_H^G}^H(\alpha)).$$

Note now that $i_{3,*}$ coincides with the canonical inclusion

$$\lambda_g: \mathrm{Ind}_{C_H^g}^C C_0(X_0^{\ddot{g}}) \to \bigoplus_{g'} \mathrm{Ind}_{C_H^{g'}}^C C_0(X_0^{\ddot{g}}) \cong \mathrm{res}_C^G(\mathrm{Ind}_H^G C_0(X_0^{\ddot{g}})),$$

of Lemma 6.4, so that the last equality follows from the statement of that lemma.

We now have verified statement (i). Using the above computations, the proof of statement (ii) follows from the equation $[i_3] \otimes \mathrm{res}_C^G(i_H^G(\alpha)) \otimes [\rho''_g] = 0$, for $g'' \notin HgC$, which is also a consequence of Lemma 6.4. □

7 Continuity of topological K-theory

The aim of this short section is to state and prove a generalization of Lemma 6.5 which is used in [7]. In a similar way as in §5-6 above, we obtain the result by using $\mathcal{E}(G) \times \mathcal{E}(G/G_0)$ as a universal example for the proper actions of G, where we assume that $\mathcal{E}(G/G_0)$ is a simplicial complex (compare with the discussions preceding Proposition 6.6).

Proposition 7.1. *Let G be a group, let (B_i, f_{ij}) be an inductive system of G-algebras, and let $B = \lim B_i$. Then*

$$\mathrm{K}_*^{\mathrm{top}}(G;B) \cong \lim_i \mathrm{K}_*^{\mathrm{top}}(G;B_i),$$

where the isomorphism is obtained from the morphisms $f_{i,*} : \mathrm{K}_*^{\mathrm{top}}(G;B_i) \to \mathrm{K}_*^{\mathrm{top}}(G;B)$ induced by the canonical maps $f_i : B_i \to B$.

Proof. Let $f^* : \lim_i \mathrm{K}_*^{\mathrm{top}}(G;B_i) \to \mathrm{K}_*^{\mathrm{top}}(G;B)$ be the homomorphism induced by the morphisms $f_i : B_i \to B$, using the covariance of the topological K-theory groups as a functor on the category of G-C^*-algebras and the universal property of the inductive limit. We want to show that f^* is an isomorphism. For every proper G-space X, let

$$f_X^* : \lim_i \mathrm{KK}_*^G\big(C_0(X), B_i\big) \to \mathrm{KK}_*^G\big(C_0(X), B\big)$$

denote the canonical morphism on the level of X. Since the structure maps for taking the limits over X are given by left Kasparov products and the structure maps for taking limits over the B_i are given by right Kasparov products, it follows from the associativity of the Kasparov product that the limits can be permuted. Thus, the map f^* can be computed via the maps f_X^* by

$$\lim_X \Big(\lim_i \mathrm{KK}_*^G(C_0(X), B_i)\Big) \xrightarrow{f_X^*} \lim_X \mathrm{KK}_*^G(C_0(X), B), \tag{7.1}$$

where X runs through the G-compact subsets of $\mathcal{E}(G) \times \mathcal{E}(G/G_0)$, which we use as a realization of the universal example for the proper actions of G.
As before, let $\mathcal{F}$ denote the family of all G-saturations Z of finite unions of simplices in $\mathcal{E}(G/G_0)$. It follows then from Lemma 5.6 that

$$\mathrm{K}_*^{\mathrm{top}}(G;B_i) = \lim_{Z\in\mathcal{F}} \lim_X \mathrm{KK}_*^G\big(C_0(X), B_i\big) \quad \text{and}$$
$$\mathrm{K}_*^{\mathrm{top}}(G;B) = \lim_{Z\in\mathcal{F}} \lim_X \mathrm{KK}_*^G\big(C_0(X), B\big),$$

where X runs through the G-compact subsets of $\mathcal{E}(G)\times Z$ such that $\pi_2(X) = Z$, and where $\pi_2 : \mathcal{E}(G) \times \mathcal{E}(G/G_0) \to \mathcal{E}(G/G_0)$ denotes the projection onto the second factor. Combining these formulas with (7.1), the result will follow if we can show that for each $Z \in \mathcal{F}$ the map

$$\lim_{\substack{X\subseteq \mathcal{E}(G)\times Z\\ X\ G-\text{compact}\\ \pi_2(X)=Z}} \lim_i \mathrm{KK}_*^G(C_0(X), B_i) \xrightarrow{f_X^*} \lim_{\substack{X\subseteq \mathcal{E}(G)\times Z\\ X\ G-\text{compact}\\ \pi_2(X)=Z}} \mathrm{KK}_*^G(C_0(X), B) \tag{7.2}$$

is an isomorphism. We write Z_0 for the union of the interiors $\overset{o}{S}$ of simplices S in Z of maximal dimension, and we put $Z_1 = Z \setminus Z_0$,

$$X_0 = X \cap \big(\mathcal{E}(G) \times Z_0\big) \qquad \text{and} \qquad X_1 = X \cap \big(\mathcal{E}(G) \times Z_1\big).$$

Doing a similar, but much easier Five-Lemma argument as in the previous sections (compare with the discussions preceding Lemma 6.8), the result will follow if we can show that the following two statements are true:

(i) Assume that Z is generated by a single simplex S in $\mathcal{E}(G/G_0)$ with $\dim(S) > 0$. Then

$$\lim_{\substack{X \subseteq \mathcal{E}(G)\times Z\\ X\ G-\text{compact}\\ \pi_2(X)=Z}} \lim_i \mathrm{KK}_*^G(C_0(X_0), B_i) \stackrel{f_{X_0}^*}{\longrightarrow} \lim_{\substack{X \subseteq \mathcal{E}(G)\times Z\\ X\ G-\text{compact}\\ \pi_2(X)=Z}} \mathrm{KK}_*^G(C_0(X_0), B)$$

is bijective.

(ii) Assume that Z is the orbit of a single vertex in $\mathcal{E}(G/G_0)$. Then the map in (7.2) is bijective.

Again, the proof of (ii) is slightly easier than the proof of (i) (because we don't have to deal with the Bott-map), so we concentrate on (i). By the structure of Z, we have

$$Z_0 = G/G_0 \cdot \overset{o}{S} \cong (G/G_0) \times_{\dot{C}} \overset{o}{S},$$

where $\overset{o}{S}$ denotes the interior of the single simplex S generating Z and $\dot{C} \subseteq G/G_0$ denotes the stabilizer of S. Thus, if X is a G-compact subset of $\mathcal{E}(G) \times Z$ such that $\pi_2(X) = Z$, then it follows from Proposition 5.12 that X_0 is G-homeomorphic to the induced space $G \times_C \big(X \cap \pi_2^{-1}(\overset{o}{S})\big)$, where $C := q^{-1}(\dot{C}) \subseteq G$. Enlarging X, if necessary, we may further assume that $X \cap \pi_2^{-1}(\overset{o}{S}) = C \cdot K \times \overset{o}{S}$ for some compact subset $K \subseteq \mathcal{E}(G)$. Thus, using compression and Bott-periodicity, and taking the limit over X, we obtain the following commutative diagram

$$\begin{array}{ccc}
\lim\limits_{K,i} \mathrm{KK}_*^G\big(C_0\big(G \times_C (C{\cdot}K \times \overset{o}{S})\big), B_i\big) & \xrightarrow{f_{X_0}^*} & \lim\limits_{K} \mathrm{KK}_*^G\big(C_0\big(G \times_C (C{\cdot}K \times \overset{o}{S})\big), B\big) \\
{\scriptstyle \mathrm{Bott}\circ\mathrm{comp}_C^G}\downarrow\cong & & \cong\downarrow{\scriptstyle \mathrm{Bott}\circ\mathrm{comp}_C^G} \\
\lim\limits_{K} \lim\limits_{i} \mathrm{KK}_*^C\big(C_0(C \cdot K), B_i\big) & \xrightarrow{f_{C\cdot K}^*} & \lim\limits_{K} \mathrm{KK}_*^C\big(C_0(C \cdot K), B\big) \\
\cong\downarrow & & \downarrow\cong \\
\lim\limits_{i} \mathrm{K}_*^{\mathrm{top}}(C; B_i) & \xrightarrow{f^*} & \mathrm{K}_*^{\mathrm{top}}(C; B),
\end{array}$$

where K runs through the compact subsets of $\mathcal{E}(G)$. Note that the left-hand lower vertical isomorphism is given by permuting the limits. The top horizontal line coincides with the map in Item (i) above. Thus, since the bottom horizontal row is an isomorphism by Lemma 6.5 (again, here C is almost connected), the result follows. □

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