C^*-Algebras Associated with Presentations of Subshifts

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Abstract. A λ-graph system is a labeled Bratteli diagram with an upward shift except the top vertices. We construct a continuous graph in the sense of V. Deaconu from a λ-graph system. It yields a Renault's groupoid C^*-algebra by following Deaconu's construction. The class of these C^*-algebras generalize the class of C^*-algebras associated with subshifts and hence the class of Cuntz-Krieger algebras. They are unital, nuclear, unique C^*-algebras subject to operator relations encoded in the structure of the λ-graph systems among generating partial isometries and projections. If the λ-graph systems are irreducible (resp. aperiodic), they are simple (resp. simple and purely infinite). K-theory formulae of these C^*-algebras are presented so that we know an example of a simple and purely infinite C^*-algebra in the class of these C^*-algebras that is not stably isomorphic to any Cuntz-Krieger algebra.

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1. Introduction

In [CK], J. Cuntz-W. Krieger have presented a class of C^*-algebras associated to finite square matrices with entries in $\{0, 1\}$. The C^*-algebras are simple if the matrices satisfy condition (I) and irreducible. They are also purely infinite if the matrices are aperiodic. There are many directions to generalize the Cuntz-Krieger algebras (cf. [An],[De],[De2],[EL],[KPRR],[KPW],[Pi], [Pu],[T], etc.). The Cuntz-Krieger algebras have close relationships to topological Markov shifts by Cuntz-Krieger's observation in [CK]. Let Σ be a finite set, and let σ be the shift on the infinite product space $\Sigma^{\mathbb{Z}}$ defined by

$\sigma((x_n)_{n\in\mathbb{Z}}) = (x_{n+1})_{n\in\mathbb{Z}}, (x_n)_{n\in\mathbb{Z}} \in \Sigma^{\mathbb{Z}}$. For a closed σ-invariant subset Λ of $\Sigma^{\mathbb{Z}}$, the topological dynamical system Λ with σ is called a subshift. The topological Markov shifts form a class of subshifts. In [Ma], the author has generalized the class of Cuntz-Krieger algebras to a class of C^*-algebras associated with subshifts. He has formulated several topological conjugacy invariants for subshifts by using the K-theory for these C^*-algebras ([Ma5]). He has also introduced presentations of subshifts, that are named symbolic matrix system and λ-graph system ([Ma5]). They are generalized notions of symbolic matrix and λ-graph (= labeled graph) for sofic subshifts respectively.
We henceforth denote by $\mathbb{Z}_+$ and $\mathbb{N}$ the set of all nonnegative integers and the set of all positive integers respectively. A symbolic matrix system $(\mathcal{M}, I)$ over a finite set Σ consists of two sequences of rectangular matrices $(\mathcal{M}_{l,l+1}, I_{l,l+1}), l \in \mathbb{Z}_+$. The matrices $\mathcal{M}_{l,l+1}$ have their entries in the formal sums of Σ and the matrices $I_{l,l+1}$ have their entries in $\{0,1\}$. They satisfy the following relations

$$I_{l,l+1}\mathcal{M}_{l+1,l+2} = \mathcal{M}_{l,l+1}I_{l+1,l+2}, \qquad l \in \mathbb{Z}_+. \tag{1.1}$$

It is assumed for $I_{l,l+1}$ that for i there exists j such that the (i,j)-component $I_{l,l+1}(i,j) = 1$ and that for j there uniquely exists i such that $I_{l,l+1}(i,j) = 1$. A λ-graph system $\mathfrak{L} = (V, E, \lambda, \iota)$ consists of a vertex set $V = V_0 \cup V_1 \cup V_2 \cup \cdots$, an edge set $E = E_{0,1} \cup E_{1,2} \cup E_{2,3} \cup \cdots$, a labeling map $\lambda : E \to \Sigma$ and a surjective map $\iota_{l,l+1} : V_{l+1} \to V_l$ for each $l \in \mathbb{Z}_+$. It naturally arises from a symbolic matrix system. For a symbolic matrix system $(\mathcal{M}, I)$, a labeled edge from a vertex $\mathfrak{v}_i^l \in V_l$ to a vertex $\mathfrak{v}_j^{l+1} \in V_{l+1}$ is given by a symbol appearing in the (i,j)-component $\mathcal{M}_{l,l+1}(i,j)$ of the matrix $\mathcal{M}_{l,l+1}$. The matrix $I_{l,l+1}$ defines a surjection $\iota_{l,l+1}$ from V_{l+1} to V_l for each $l \in \mathbb{Z}_+$. The symbolic matrix systems and the λ-graph systems are the same objects. They give rise to subshifts by looking the set of all label sequences appearing in the labeled Bratteli diagram (V, E, λ). A canonical method to construct a symbolic matrix system and a λ-graph system from an arbitrary subshift has been introduced in [Ma5]. The obtained symbolic matrix system and the λ-graph system are said to be canonical for the subshift. For a symbolic matrix system $(\mathcal{M}, I)$, let $A_{l,l+1}$ be the nonnegative rectangular matrix obtained from $\mathcal{M}_{l,l+1}$ by setting all the symbols equal to 1 for each $l \in \mathbb{Z}_+$. The resulting pair (A, I) satisfies the following relations from (1.1)

$$I_{l,l+1}A_{l+1,l+2} = A_{l,l+1}I_{l+1,l+2}, \qquad l \in \mathbb{Z}_+. \tag{1.2}$$

We call (A, I) the nonnegative matrix system for $(\mathcal{M}, I)$.
In the present paper, we introduce C^*-algebras from λ-graph systems. If a λ-graph system is the canonical λ-graph system for a subshift Λ, the C^*-algebra coincides with the C^*-algebra $\mathcal{O}_\Lambda$ associated with the subshift. Hence the class of the C^*-algebras in this paper generalize the class of Cuntz-Krieger algebras. Let $\mathfrak{L} = (V, E, \lambda, \iota)$ be a λ-graph system over alphabet Σ. We first construct a continuous graph from $\mathfrak{L}$ in the sense of V. Deaconu ([D2],[De3],[De4]). We

then define the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ associated with $\mathfrak{L}$ as the Renault's C^*-algebra of a groupoid constructed from the continuous graph. For an edge $e \in E_{l,l+1}$, we denote by $s(e) \in V_l$ and $t(e) \in V_{l+1}$ its source vertex and its terminal vertex respectively. Let Λ^l be the set of all words of length l of symbols appearing in the labeled Bratteli diagram of $\mathfrak{L}$. We put $\Lambda^* = \cup_{l=0}^{\infty}\Lambda^l$ where Λ^0 denotes the empty word. Let $\{\mathfrak{v}_1^l, \dots, \mathfrak{v}_{m(l)}^l\}$ be the vertex set V_l. We denote by $\Gamma_l^-(\mathfrak{v}_i^l)$ the set of all words in Λ^l presented by paths starting at a vertex of V_0 and terminating at the vertex $\mathfrak{v}_i^l$. $\mathfrak{L}$ is said to be left-resolving if there are no distinct edges with the same label and the same terminal vertex. $\mathfrak{L}$ is said to be predecessor-separated if $\Gamma_l^-(\mathfrak{v}_i^l) \neq \Gamma_l^-(\mathfrak{v}_j^l)$ for distinct i, j and for all $l \in \mathbb{N}$. Assume that $\mathfrak{L}$ is left-resolving and satisfies condition (I), a mild condition generalizing Cuntz-Krieger's condition (I). We then prove:

Theorem A (Theorem 3.6 and Theorem 4.3). *Suppose that a λ-graph system $\mathfrak{L}$ satisfies condition (I). Then the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ is the universal concrete unique C^*-algebra generated by partial isometries $S_\alpha, \alpha \in \Sigma$ and projections $E_i^l, i = 1, 2, \dots, m(l),\ l \in \mathbb{Z}_+$ satisfying the following operator relations:*

$$\sum_{\alpha\in\Sigma} S_\alpha S_\alpha^* = 1, \tag{1.3}$$

$$\sum_{i=1}^{m(l)} E_i^l = 1, \qquad E_i^l = \sum_{j=1}^{m(l+1)} I_{l,l+1}(i,j) E_j^{l+1}, \tag{1.4}$$

$$S_\alpha S_\alpha^* E_i^l = E_i^l S_\alpha S_\alpha^*, \tag{1.5}$$

$$S_\alpha^* E_i^l S_\alpha = \sum_{j=1}^{m(l+1)} A_{l,l+1}(i,\alpha,j) E_j^{l+1}, \tag{1.6}$$

for $i = 1, 2, \dots, m(l), l \in \mathbb{Z}_+, \alpha \in \Sigma$, where

$$A_{l,l+1}(i,\alpha,j) = \begin{cases} 1 & \text{if } s(e) = \mathfrak{v}_i^l, \lambda(e) = \alpha, t(e) = \mathfrak{v}_j^{l+1} \text{ for some } e \in E_{l,l+1}, \\ 0 & \text{otherwise,} \end{cases}$$

$$I_{l,l+1}(i,j) = \begin{cases} 1 & \text{if } \iota_{l,l+1}(\mathfrak{v}_j^{l+1}) = \mathfrak{v}_i^l, \\ 0 & \text{otherwise} \end{cases}$$

for $i = 1, 2, \dots, m(l),\ j = 1, 2, \dots, m(l+1),\ \alpha \in \Sigma$.

If $\mathfrak{L}$ is predecessor-separated, the following relations:

$$E_i^l = \prod_{\substack{\mu,\nu\in\Lambda^l \\ \mu\in\Gamma_l^-(\mathfrak{v}_i^l), \nu\notin\Gamma_l^-(\mathfrak{v}_i^l)}} S_\mu^* S_\mu (1 - S_\nu^* S_\nu), \qquad l \in \mathbb{N}, \tag{1.7}$$

$$E_i^0 = \sum_{\alpha\in\Sigma}\sum_{j=1}^{m(1)} A_{0,1}(i,\alpha,j) S_\alpha E_j^1 S_\alpha^*$$

hold for $i = 1,2,\dots,m(l)$, where $S_\mu = S_{\mu_1}\cdots S_{\mu_k}$ for $\mu = (\mu_1,\dots,\mu_k), \mu_1,\dots,\mu_k \in \Sigma$. In this case, $\mathcal{O}_{\mathfrak{L}}$ is generated by only the partial isometries $S_\alpha, \alpha \in \Sigma$.
If $\mathfrak{L}$ comes from a finite directed graph G, the algebra $\mathcal{O}_{\mathfrak{L}}$ becomes the Cuntz-Krieger algebra $\mathcal{O}_{A_G}$ associated to its adjacency matrix A_G with entries in $\{0,1\}$.
We generalize irreducibility and aperiodicity for finite directed graphs to λ-graph systems. Then simplicity arguments of the Cuntz algebras in [C], the Cuntz-Krieger algebras in [CK] and the C^*-algebras associated with subshifts in [Ma] are generalized to our C^*-algebras $\mathcal{O}_{\mathfrak{L}}$ so that we have

Theorem B (Theorem 4.7 and Proposition 4.9). *If $\mathfrak{L}$ satisfies condition (I) and is irreducible, the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ is simple. In particular if $\mathfrak{L}$ is aperiodic, $\mathcal{O}_{\mathfrak{L}}$ is simple and purely infinite.*

There exists an action $\alpha_{\mathfrak{L}}$ of the torus group $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ on the algebra $\mathcal{O}_{\mathfrak{L}}$ that is called the gauge action. It satisfies $\alpha_{\mathfrak{L}z}(S_\alpha) = zS_\alpha, \alpha \in \Sigma$ for $z \in \mathbb{T}$. The fixed point subalgebra $\mathcal{O}_{\mathfrak{L}}{}^{\alpha_{\mathfrak{L}}}$ of $\mathcal{O}_{\mathfrak{L}}$ under $\alpha_{\mathfrak{L}}$ is an AF-algebra $\mathcal{F}_{\mathfrak{L}}$, that is stably isomorphic to the crossed product $\mathcal{O}_{\mathfrak{L}} \rtimes_{\alpha_{\mathfrak{L}}} \mathbb{T}$. Let $(A,I) = (A_{l,l+1}, I_{l,l+1})_{l\in\mathbb{Z}_+}$ be the nonnegative matrix system for the symbolic matrix system corresponding to the λ-graph system $\mathfrak{L}$. In [Ma5], its dimension group $(\Delta_{(A,I)}, \Delta^+_{(A,I)}, \delta_{(A,I)})$, its Bowen-Franks groups $BF^i(A,I), i = 0,1$ and its K-groups $K_i(A,I), i = 0,1$ have been formulated. They are related to topological conjugacy invariants of subshifts. The following K-theory formulae are generalizations of the K-theory formulae for the Cuntz-Krieger algebras and the C^*-algebras associated with subshifts ([Ma2],[Ma4],[Ma5],[Ma6], cf.[C2],[C3],[CK]).

Theorem C (Proposition 5.3, Theorem 5.5 and Theorem 5.9).

$$(K_0(\mathcal{F}_{\mathfrak{L}}), K_0(\mathcal{F}_{\mathfrak{L}})_+, \widehat{\alpha_{\mathfrak{L}}}_*) \cong (\Delta_{(A,I)}, \Delta^+_{(A,I)}, \delta_{(A,I)}),$$
$$K_i(\mathcal{O}_{\mathfrak{L}}) \cong K_i(A,I), \qquad i = 0,1,$$
$$\mathrm{Ext}^{i+1}(\mathcal{O}_{\mathfrak{L}}) \cong BF^i(A,I), \qquad i = 0,1$$

where $\widehat{\alpha_{\mathfrak{L}}}$ denotes the dual action of the gauge action $\alpha_{\mathfrak{L}}$ on $\mathcal{O}_{\mathfrak{L}}$.

We know that the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ is nuclear and satisfies the Universal Coefficient Theorem (UCT) in the sense of Rosenberg and Schochet (Proposition 5.7)([RS], cf. [Bro2]). Hence, if $\mathfrak{L}$ is aperiodic, $\mathcal{O}_{\mathfrak{L}}$ is a unital, separable, nuclear, purely infinite, simple C^*-algebra satisfying the UCT, that lives in a classifiable class by K-theory of E. Kirchberg [Kir] and N. C. Phillips [Ph]. By Rørdam's result [Rø;Proposition 6.7], one sees that $\mathcal{O}_{\mathfrak{L}}$ is isomorphic to the C^*-algebra of an inductive limit of a sequence $B_1 \to B_2 \to B_3 \to \cdots$ of simple Cuntz-Krieger algebras (Corollary 5.8).
We finally present an example of a λ-graph system for which the associated C^*-algebra is not stably isomorphic to any Cuntz-Krieger algebra $\mathcal{O}_A$ and any Cuntz-algebra $\mathcal{O}_n$ for $n = 2,3,\dots,\infty$. The example is a λ-graph system $\mathfrak{L}(\mathcal{S})$ constructed from a certain Shannon graph $\mathcal{S}$ (cf.[KM]). We obtain

Theorem D (Theorem 7.7). *The C^*-algebra $\mathcal{O}_{\mathfrak{L}(\mathcal{S})}$ is unital, simple, purely infinite, nuclear and generated by five partial isometries with mutually orthogonal ranges. Its K-groups are*

$$K_0(\mathcal{O}_{\mathfrak{L}(\mathcal{S})}) = 0, \qquad K_1(\mathcal{O}_{\mathfrak{L}(\mathcal{S})}) = \mathbb{Z}.$$

In [Ma7], among other things, relationships between ideals of $\mathcal{O}_{\mathfrak{L}}$ and sub λ-graph systems of $\mathfrak{L}$ are studied so that the class of C^*-algebras associated with λ-graph systems is closed under quotients by its ideals.

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2. Continuous graphs constructed from λ-graph systems

We will construct Deaconu's continuous graphs from λ-graph systems. They yield Renault's r-discrete groupoid C^*-algebras by Deaconu ([De],[De2],[De3]). Following V. Deaconu in [De3], by a continuous graph we mean a closed subset $\mathcal{E}$ of $\mathcal{V} \times \Sigma \times \mathcal{V}$ where $\mathcal{V}$ is a compact metric space and Σ is a finite set. If in particular $\mathcal{V}$ is zero-dimensional, that is, the set of all clopen sets form a basis of the open sets, we say $\mathcal{E}$ to be zero-dimensional or Stonean.

Let $\mathfrak{L} = (V, E, \lambda, \iota)$ be a λ-graph system over Σ with vertex set $V = \cup_{l \in \mathbb{Z}_+} V_l$ and edge set $E = \cup_{l \in \mathbb{Z}_+} E_{l,l+1}$ that is labeled with symbols in Σ by $\lambda : E \to \Sigma$, and that is supplied with surjective maps $\iota (= \iota_{l,l+1}) : V_{l+1} \to V_l$ for $l \in \mathbb{Z}_+$. Here the vertex sets $V_l, l \in \mathbb{Z}_+$ are finite disjoint sets. Also $E_{l,l+1}, l \in \mathbb{Z}_+$ are finite disjoint sets. An edge e in $E_{l,l+1}$ has its source vertex $s(e)$ in V_l and its terminal vertex $t(e)$ in V_{l+1} respectively. Every vertex in V has a successor and every vertex in V_l for $l \in \mathbb{N}$ has a predecessor. It is then required that there exists an edge in $E_{l,l+1}$ with label α and its terminal is $v \in V_{l+1}$ if and only if there exists an edge in $E_{l-1,l}$ with label α and its terminal is $\iota(v) \in V_l$. For $u \in V_{l-1}$ and $v \in V_{l+1}$, we put

$$E^{\iota}(u, v) = \{e \in E_{l,l+1} \mid t(e) = v, \iota(s(e)) = u\},$$
$$E_{\iota}(u, v) = \{e \in E_{l-1,l} \mid s(e) = u, t(e) = \iota(v)\}.$$

Then there exists a bijective correspondence between $E^{\iota}(u, v)$ and $E_{\iota}(u, v)$ that preserves labels for each pair of vertices u, v. We call this property the local property of $\mathfrak{L}$. Let $\Omega_{\mathfrak{L}}$ be the projective limit of the system $\iota_{l,l+1} : V_{l+1} \to V_l, l \in \mathbb{Z}_+$, that is defined by

$$\Omega_{\mathfrak{L}} = \{(v^l)_{l \in \mathbb{Z}_+} \in \prod_{l \in \mathbb{Z}_+} V_l \mid \iota_{l,l+1}(v^{l+1}) = v^l, l \in \mathbb{Z}_+\}.$$

We endow $\Omega_{\mathfrak{L}}$ with the projective limit topology so that it is a compact Hausdorff space. An element v in $\Omega_{\mathfrak{L}}$ is called an ι-orbit or also a vertex. Let $E_{\mathfrak{L}}$ be the set of all triplets $(u,\alpha,v) \in \Omega_{\mathfrak{L}} \times \Sigma \times \Omega_{\mathfrak{L}}$ such that for each $l \in \mathbb{Z}_+$, there exists $e_{l,l+1} \in E_{l,l+1}$ satisfying

$$u^l = s(e_{l,l+1}), \quad v^{l+1} = t(e_{l,l+1}) \quad \text{and} \quad \alpha = \lambda(e_{l,l+1})$$

where $u = (u^l)_{l\in\mathbb{Z}_+}, v = (v^l)_{l\in\mathbb{Z}_+} \in \Omega_{\mathfrak{L}}$.

PROPOSITION 2.1. *The set $E_{\mathfrak{L}} \subset \Omega_{\mathfrak{L}}\times\Sigma\times\Omega_{\mathfrak{L}}$ is a zero-dimensional continuous graph.*

Proof. It suffices to show that $E_{\mathfrak{L}}$ is closed. For $(u,\beta,v) \in \Omega_{\mathfrak{L}} \times \Sigma \times \Omega_{\mathfrak{L}}$ with $(u,\beta,v) \notin E_{\mathfrak{L}}$, one finds $l \in \mathbb{N}$ such that there does not exist any edge e in $E_{l,l+1}$ with $s(e) = u^l, t(e) = v^{l+1}$ and $\lambda(e) = \beta$. Put

$$U_{u^l} = \{(w^i)_{i\in\mathbb{Z}_+} \in \Omega_{\mathfrak{L}} \mid w^l = u^l\}, \quad U_{v^{l+1}} = \{(w^i)_{i\in\mathbb{Z}_+} \in \Omega_{\mathfrak{L}} \mid w^{l+1} = v^{l+1}\}.$$

They are open sets in $\Omega_{\mathfrak{L}}$. Hence $U_{u^l} \times \{\beta\} \times U_{v^{l+1}}$ is an open neighborhood of (u,β,v) that does not intersect with $E_{\mathfrak{L}}$ so that $E_{\mathfrak{L}}$ is closed. □

We denote by $\{\mathfrak{v}_1^l,\dots,\mathfrak{v}_{m(l)}^l\}$ the vertex set V_l. Put for $\alpha \in \Sigma, i = 1,\dots,m(1)$

$$U_i^1(\alpha) = \{(u,\alpha,v) \in E_{\mathfrak{L}} \mid v^1 = \mathfrak{v}_i^1 \text{ where } v = (v^l)_{l\in\mathbb{Z}_+} \in \Omega_{\mathfrak{L}}\}.$$

Then $U_i^1(\alpha)$ is a clopen set in $E_{\mathfrak{L}}$ such that

$$\cup_{\alpha\in\Sigma} \cup_{i=1}^{m(1)} U_i^1(\alpha) = E_{\mathfrak{L}}, \qquad U_i^1(\alpha) \cap U_j^1(\beta) = \emptyset \quad \text{if } (i,\alpha) \neq (j,\beta).$$

Put $t(u,\alpha,v) = v$ for $(u,\alpha,v) \in E_{\mathfrak{L}}$. Suppose that $\mathfrak{L}$ is left-resolving. It is easy to see that if $U_i^1(\alpha) \neq \emptyset$, the restriction of t to $U_i^1(\alpha)$ is a homeomorphism onto $U_{\mathfrak{v}_i^1} = \{(v^l)_{l\in\mathbb{Z}_+} \in \Omega_{\mathfrak{L}} \mid v^1 = \mathfrak{v}_i^1\}$. Hence $t : E_{\mathfrak{L}} \to \Omega_{\mathfrak{L}}$ is a local homeomorphism.

Following Deaconu [De3], we consider the set $X_{\mathfrak{L}}$ of all one-sided paths of $E_{\mathfrak{L}}$:

$$X_{\mathfrak{L}} = \{(\alpha_i,u_i)_{i=1}^\infty \in \prod_{i=1}^\infty(\Sigma\times\Omega_{\mathfrak{L}}) \mid (u_i,\alpha_{i+1},u_{i+1}) \in E_{\mathfrak{L}} \text{ for all } i \in \mathbb{N} \text{ and } (u_0,\alpha_1,u_1) \in E_{\mathfrak{L}} \text{ for some } u_0 \in \Omega_{\mathfrak{L}}\}.$$

The set $X_{\mathfrak{L}}$ has the relative topology from the infinite product topology of $\Sigma \times \Omega_{\mathfrak{L}}$. It is a zero-dimensional compact Hausdorff space. The shift map $\sigma : (\alpha_i,u_i)_{i=1}^\infty \in X_{\mathfrak{L}} \to (\alpha_{i+1},u_{i+1})_{i=1}^\infty \in X_{\mathfrak{L}}$ is continuous. For $v = (v^l)_{l\in\mathbb{Z}_+} \in \Omega_{\mathfrak{L}}$ and $\alpha \in \Sigma$, the local property of $\mathfrak{L}$ ensures that if there exists $e_{0,1} \in E_{0,1}$ satisfying $v^1 = t(e_{0,1}), \alpha = \lambda(e_{0,1})$, there exist $e_{l,l+1} \in E_{l,l+1}$ and $u = (u^l)_{l\in\mathbb{Z}_+} \in \Omega_{\mathfrak{L}}$ satisfying $u^l = s(e_{l,l+1}), v^{l+1} = t(e_{l,l+1}), \alpha = \lambda(e_{l,l+1})$ for each $l \in \mathbb{Z}_+$. Hence if $\mathfrak{L}$ is left-resolving, for any $x = (\alpha_i,v_i)_{i=1}^\infty \in X_{\mathfrak{L}}$, there uniquely exists $v_0 \in \Omega_{\mathfrak{L}}$ such that $(v_0,\alpha_1,v_1) \in E_{\mathfrak{L}}$. Denote by $v(x)_0$ the unique vertex v_0 for $x \in X_{\mathfrak{L}}$.

Lemma 2.2. *For a λ-graph system $\mathfrak{L}$, consider the following conditions*
(i) *$\mathfrak{L}$ is left-resolving.*
(ii) *$E_{\mathfrak{L}}$ is left-resolving, that is, for $(u,\alpha,v),(u',\alpha,v') \in E_{\mathfrak{L}}$, the condition $v = v'$ implies $u = u'$.*
(iii) *σ is a local homeomorphism on $X_{\mathfrak{L}}$.*

Then we have
$$(i) \Leftrightarrow (ii) \Rightarrow (iii).$$

Proof. The implications $(i) \Leftrightarrow (ii)$ are direct. We will see that $(ii) \Rightarrow (iii)$. Suppose that $\mathfrak{L}$ is left-resolving. Let $\{\gamma_1,\dots,\gamma_m\} = \Sigma$ be the list of the alphabet. Put
$$X_{\mathfrak{L}}(k) = \{(\alpha_i, v_i)_{i=1}^{\infty} \in X_{\mathfrak{L}} \mid \alpha_1 = \gamma_k\}$$
that is a clopen set of $X_{\mathfrak{L}}$. Since the family $X_{\mathfrak{L}}(k), k = 1,\dots,m$ is a disjoint covering of $X_{\mathfrak{L}}$ and the restriction of σ to each of them $\sigma|_{X_{\mathfrak{L}}(k)} : X_{\mathfrak{L}}(k) \to X_{\mathfrak{L}}$ is a homeomorphism, the continuous surjection σ is a local homeomorphism on $X_{\mathfrak{L}}$. $\square$

Remark. We will remark that a continuous graph coming from a left-resolving, predecessor-separated λ-graph system is characterized as in the following way. Let $\mathcal{E} \subset \mathcal{V} \times \Sigma \times \mathcal{V}$ be a continuous graph. Following [KM], we define the l-past context of $v \in \mathcal{V}$ as follows:
$$\Gamma_l^-(v) = \{(\alpha_1,\dots,\alpha_l) \in \Sigma^l \mid \exists v_0, v_1, \dots, v_{l-1} \in \mathcal{V}; \\ (v_{i-1}, \alpha_i, v_i) \in \mathcal{E}, i = 1,2,\dots,l-1, (v_{l-1}, \alpha_l, v) \in \mathcal{E}\}.$$
We say $\mathcal{E}$ to be predecessor-separated if for two vertices $u, v \in \mathcal{V}$, there exists $l \in \mathbb{N}$ such that $\Gamma_l^-(u) \neq \Gamma_l^-(v)$. The following proposition can be directly proved by using an idea of [KM]. Its result will not be used in our further discussions so that we omit its proof.

Proposition 2.3. *Let $\mathcal{E} \subset \mathcal{V} \times \Sigma \times \mathcal{V}$ be a zero-dimensional continuous graph such that $\mathcal{E}$ is left-resolving, predecessor-separated. If the map $t : \mathcal{E} \to \mathcal{V}$ defined by $t(u,\alpha,v) = v$ is a surjective open map, there exists a λ-graph system $\mathcal{L}^{\mathcal{E}}$ over Σ and a homeomorphism Φ from $\mathcal{V}$ onto $\Omega_{\mathfrak{L}^{\mathcal{E}}}$ such that the map $\Phi \times \mathrm{id} \times \Phi : \mathcal{V} \times \Sigma \times \mathcal{V} \to \Omega_{\mathfrak{L}^{\mathcal{E}}} \times \Sigma \times \Omega_{\mathfrak{L}^{\mathcal{E}}}$ satisfies $(\Phi \times \mathrm{id} \times \Phi)(\mathcal{E}) = E_{\mathfrak{L}^{\mathcal{E}}}$.*

3. The C^*-algebra $\mathcal{O}_{\mathfrak{L}}$.

In what follows we assume $\mathfrak{L}$ to be left-resolving. Following V. Deaconu [De2],[De3],[De4], one may construct a locally compact r-discrete groupoid from a local homeomorphism σ on $X_{\mathfrak{L}}$ as in the following way (cf. [An],[Re]). Set
$$G_{\mathfrak{L}} = \{(x,n,y) \in X_{\mathfrak{L}} \times \mathbb{Z} \times X_{\mathfrak{L}} \mid \exists k, l \geq 0;\ \sigma^k(x) = \sigma^l(y), n = k - l\}.$$
The range map and the domain map are defined by
$$r(x,n,y) = x, \qquad d(x,n,y) = y.$$

The multiplication and the inverse operation are defined by

$$(x,n,y)(y,m,z) = (x,n+m,z), \qquad (x,n,y)^{-1} = (y,-n,x).$$

The unit space $G^0_{\mathfrak{L}}$ is defined to be the space $X_{\mathfrak{L}} = \{(x,0,x) \in G_{\mathfrak{L}} \mid x \in X_{\mathfrak{L}}\}$. A basis of the open sets for $G_{\mathfrak{L}}$ is given by

$$Z(U,V,k,l) = \{(x, k-l, (\sigma^l|_V)^{-1} \circ (\sigma^k(x)) \in G_{\mathfrak{L}} \mid x \in U\}$$

where U,V are open sets of $X_{\mathfrak{L}}$, and $k,l \in \mathbb{N}$ are such that $\sigma^k|_U$ and $\sigma^l|_V$ are homeomorphisms with the same open range. Hence we see

$$Z(U,V,k,l) = \{(x,k-l,y) \in G_{\mathfrak{L}} \mid x \in U, y \in V, \sigma^k(x) = \sigma^l(y)\}.$$

The groupoid C^*-algebra $C^*(G_{\mathfrak{L}})$ for the groupoid $G_{\mathfrak{L}}$ is defined as in the following way ([Re], cf. [An],[De2],[De3],[De4]). Let $C_c(G_{\mathfrak{L}})$ be the set of all continuous functions on $G_{\mathfrak{L}}$ with compact support that has a natural product structure of $*$-algebra given by

$$(f*g)(s) = \sum_{\substack{t \in G_{\mathfrak{L}},\\ r(t)=r(s)}} f(t)g(t^{-1}s) = \sum_{\substack{t_1,t_2 \in G_{\mathfrak{L}},\\ s=t_1t_2}} f(t_1)g(t_2),$$

$$f^*(s) = \overline{f(s^{-1})}, \qquad f,g \in C_c(G_{\mathfrak{L}}), \quad s \in G_{\mathfrak{L}}.$$

Let $C_0(G^0_{\mathfrak{L}})$ be the C^*-algebra of all continuous functions on $G^0_{\mathfrak{L}}$ that vanish at infinity. The algebra $C_c(G_{\mathfrak{L}})$ is a $C_0(G^0_{\mathfrak{L}})$-module, endowed with a $C_0(G^0_{\mathfrak{L}})$-valued inner product by

$$(\xi f)(x,n,y) = \xi(x,n,y)f(y), \qquad \xi \in C_c(G_{\mathfrak{L}}), \quad f \in C_0(G^0_{\mathfrak{L}}), \quad (x,n,y) \in G_{\mathfrak{L}},$$

$$< \xi, \eta > (y) = \sum_{\substack{x,n\\ (x,n,y) \in G_{\mathfrak{L}}}} \overline{\xi(x,n,y)}\eta(x,n,y), \qquad \xi,\eta \in C_c(G_{\mathfrak{L}}), \quad y \in X_{\mathfrak{L}}.$$

Let us denote by $l^2(G_{\mathfrak{L}})$ the completion of the inner product $C_0(G^0_{\mathfrak{L}})$-module $C_c(G_{\mathfrak{L}})$. It is a Hilbert C^*-right module over the commutative C^*-algebra $C_0(G^0_{\mathfrak{L}})$. We denote by $B(l^2(G_{\mathfrak{L}}))$ the C^*-algebra of all bounded adjointable $C_0(G^0_{\mathfrak{L}})$-module maps on $l^2(G_{\mathfrak{L}})$. Let π be the $*$-homomorphism of $C_c(G_{\mathfrak{L}})$ into $B(l^2(G_{\mathfrak{L}}))$ defined by $\pi(f)\xi = f * \xi$ for $f,\xi \in C_c(G_{\mathfrak{L}})$. Then the closure of $\pi(C_c(G_{\mathfrak{L}}))$ in $B(l^2(G_{\mathfrak{L}}))$ is called the (reduced) C^*-algebra of the groupoid $G_{\mathfrak{L}}$, that we denote by $C^*(G_{\mathfrak{L}})$.

Definition. *The C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ associated with λ-graph system $\mathfrak{L}$ is defined to be the C^*-algebra $C^*(G_{\mathfrak{L}})$ of the groupoid $G_{\mathfrak{L}}$.*

We will study the algebraic structure of the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$. Recall that Λ^k denotes the set of all words of Σ^k that appear in $\mathfrak{L}$. For $x = (\alpha_n, u_n)_{n=1}^\infty \in X_{\mathfrak{L}}$, we put $\lambda(x)_n = \alpha_n \in \Sigma$, $v(x)_n = u_n \in \Omega_{\mathfrak{L}}$ respectively. The ι-orbit $v(x)_n$

is written as $v(x)_n = (v(x)^l_n)_{l\in\mathbb{Z}_+} \in \Omega_{\mathfrak{L}} = \varprojlim V_l$. Now $\mathfrak{L}$ is left-resolving so that there uniquely exists $v(x)_0 \in \Omega_{\mathfrak{L}}$ satisfying $(v(x)_0, \alpha_1, u_1) \in E_{\mathfrak{L}}$. Set for $\mu = (\mu_1, \dots, \mu_k) \in \Lambda^k$,

$$U(\mu) = \{(x, k, y) \in G_{\mathfrak{L}} \mid \sigma^k(x) = y, \lambda(x)_1 = \mu_1, \dots, \lambda(x)_k = \mu_k\}$$

and for $\mathfrak{v}^l_i \in V_l$,

$$U(\mathfrak{v}^l_i) = \{(x, 0, x) \in G_{\mathfrak{L}} \mid v(x)^l_0 = \mathfrak{v}^l_i\}$$

where $v(x)_0 = (v(x)^l_0)_{l\in\mathbb{Z}_+} \in \Omega_{\mathfrak{L}}$. They are clopen sets of $G_{\mathfrak{L}}$. We set

$$S_\mu = \pi(\chi_{U(\mu)}), \qquad E^l_i = \pi(\chi_{U(\mathfrak{v}^l_i)}) \qquad \text{in} \qquad \pi(C_c(G_{\mathfrak{L}}))$$

where $\chi_F \in C_c(G_{\mathfrak{L}})$ denotes the characteristic function of a clopen set F on the space $G_{\mathfrak{L}}$. Then it is straightforward to see the following lemmas.

Lemma 3.1.

(i) *S_μ is a partial isometry satisfying $S_\mu = S_{\mu_1} \cdots S_{\mu_k}$, where $\mu = (\mu_1, \dots, \mu_k) \in \Lambda^k$.*

(ii) *$\sum_{\mu\in\Lambda^k} S_\mu S^*_\mu = 1$ for $k \in \mathbb{N}$. We in particular have*

$$\sum_{\alpha\in\Sigma} S_\alpha S^*_\alpha = 1. \tag{3.1}$$

(iii) *E^l_i is a projection such that*

$$\sum_{i=1}^{m(l)} E^l_i = 1, \qquad E^l_i = \sum_{j=1}^{m(l+1)} I_{l,l+1}(i,j) E^{l+1}_j, \tag{3.2}$$

where $I_{l,l+1}$ is the matrix defined in Theorem A in Section 1, corresponding to the map $\iota_{l,l+1} : V_{l+1} \to V_l$.

Take $\mu = (\mu_1, \dots, \mu_k) \in \Lambda^k$, $\nu = (\nu_1, \dots, \nu_{k'}) \in \Lambda^{k'}$ and $\mathfrak{v}^l_i \in V_l$ with $k, k' \le l$ such that there exist paths ξ, η in $\mathfrak{L}$ satisfying $\lambda(\xi) = \mu, \lambda(\eta) = \nu$ and $t(\xi) = t(\eta) = \mathfrak{v}^l_i$. We set

$$\begin{aligned} U(\mu, \mathfrak{v}^l_i, \nu) =& \{(x, k-k', y) \in G_{\mathfrak{L}} \mid \sigma^k(x) = \sigma^{k'}(y),\ v(x)^l_k = v(y)^l_{k'} = \mathfrak{v}^l_i, \\ & \lambda(x)_1 = \mu_1, \dots, \lambda(x)_k = \mu_k,\ \lambda(y)_1 = \nu_1, \dots, \lambda(y)_{k'} = \nu_{k'}\ \}. \end{aligned}$$

The sets $U(\mu, \mathfrak{v}^l_i, \nu), \mu \in \Lambda^k, \nu \in \Lambda^{k'}, i = 1, \dots, m(l)$ are clopen sets and generate the topology of $G_{\mathfrak{L}}$.

Lemma 3.2.

$$S_\mu E^l_i S^*_\nu = \pi(\chi_{U(\mu,\mathfrak{v}^l_i,\nu)}) \in \pi(C_c(G_{\mathfrak{L}})).$$

Hence the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ is generated by $S_\alpha, \alpha \in \Sigma$ and $E^l_i, i = 1, \dots, m(l), l \in \mathbb{Z}_+$.

The generators S_α, E^l_i satisfy the following operator relations, that are straightforwardly checked.

LEMMA 3.3.

$$S_\alpha S_\alpha^* E_i^l = E_i^l S_\alpha S_\alpha^*, \tag{3.3}$$

$$S_\alpha^* E_i^l S_\alpha = \sum_{j=1}^{m(l+1)} A_{l,l+1}(i,\alpha,j) E_j^{l+1}, \tag{3.4}$$

for $\alpha \in \Sigma$, $i = 1, 2, \dots, m(l), l \in \mathbb{Z}_+$, *where* $A_{l,l+1}(i,\alpha,j)$ *is defined in Theorem A in Section 1.*

The four operator relations (3.1),(3.2),(3.3),(3.4) are called the relations $(\mathfrak{L})$. Let $\mathcal{A}_l, l \in \mathbb{Z}_+$ be the C^*-subalgebra of $\mathcal{O}_\mathfrak{L}$ generated by the projections $E_i^l, i = 1, \dots, m(l)$, that is,

$$\mathcal{A}_l = \mathbb{C}E_1^l \oplus \cdots \oplus \mathbb{C}E_{m(l)}^l.$$

The projections $S_\alpha^* S_\alpha, \alpha \in \Sigma$ and $S_\mu^* S_\mu, \mu \in \Lambda^k, k \le l$ belong to $\mathcal{A}_l, l \in \mathbb{N}$ by (3.4) and the first relation of (3.2). Let $\mathcal{A}_\mathfrak{L}$ be the C^*-subalgebra of $\mathcal{O}_\mathfrak{L}$ generated by all the projections $E_i^l, i = 1, \dots, m(l), l \in \mathbb{Z}_+$. By the second relation of (3.2), the algebra $\mathcal{A}_l$ is naturally embedded in $\mathcal{A}_{l+1}$ so that $\mathcal{A}_\mathfrak{L}$ is a commutative AF-algebra. We note that there exists an isomorphism between $\mathcal{A}_l$ and $C(V_l)$ for each $l \in \mathbb{Z}_+$ that is compatible with the embeddings $\mathcal{A}_l \hookrightarrow \mathcal{A}_{l+1}$ and $I_{l,l+1}^t (= \iota_{l,l+1}^*) : C(V_l) \hookrightarrow C(V_{l+1})$. Hence there exists an isomorphism between $\mathcal{A}_\mathfrak{L}$ and $C(\Omega_\mathfrak{L})$. Let k, l be natural numbers with $k \le l$. We set

$\mathcal{D}_\mathfrak{L}$ =The C^*-subalgebra of $\mathcal{O}_\mathfrak{L}$ generated by $S_\mu a S_\mu^*, \mu \in \Lambda^*, a \in \mathcal{A}_\mathfrak{L}$.

$\mathcal{F}_k^l$ =The C^*-subalgebra of $\mathcal{O}_\mathfrak{L}$ generated by $S_\mu a S_\nu^*, \mu, \nu \in \Lambda^k, a \in \mathcal{A}_l$.

$\mathcal{F}_k^\infty$ =The C^*-subalgebra of $\mathcal{O}_\mathfrak{L}$ generated by $S_\mu a S_\nu^*, \mu, \nu \in \Lambda^k, a \in \mathcal{A}_\mathfrak{L}$.

$\mathcal{F}_\mathfrak{L}$ =The C^*-subalgebra of $\mathcal{O}_\mathfrak{L}$ generated by $S_\mu a S_\nu^*, \mu, \nu \in \Lambda^*$, $|\mu| = |\nu|, a \in \mathcal{A}_\mathfrak{L}$.

The algebra $\mathcal{D}_\mathfrak{L}$ is isomorphic to $C(X_\mathfrak{L})$. It is obvious that the algebra $\mathcal{F}_k^l$ is finite dimensional and there exists an embedding $\iota_{l,l+1} : \mathcal{F}_k^l \hookrightarrow \mathcal{F}_k^{l+1}$ through the preceding embedding $\mathcal{A}_l \hookrightarrow \mathcal{A}_{l+1}$. Define a homomorphism $c : (x, n, y) \in G_\mathfrak{L} \to n \in \mathbb{Z}$. We denote by $F_\mathfrak{L}$ the subgroupoid $c^{-1}(0)$ of $G_\mathfrak{L}$. Let $C^*(F_\mathfrak{L})$ be its groupoid C^*-algebra. It is also immediate that the algebra $\mathcal{F}_\mathfrak{L}$ is isomorphic to $C^*(F_\mathfrak{L})$. By (3.1),(3.3),(3.4), the relations:

$$E_i^l = \sum_{\alpha \in \Sigma} \sum_{j=1}^{m(l+1)} A_{l,l+1}(i,\alpha,j) S_\alpha E_j^{l+1} S_\alpha^*, \qquad i = 1, 2, \dots, m(l)$$

hold. They yield

$$S_\mu E_i^l S_\nu^* = \sum_{\alpha \in \Sigma} \sum_{j=1}^{m(l+1)} A_{l,l+1}(i,\alpha,j) S_{\mu\alpha} E_j^{l+1} S_{\nu\alpha}^* \qquad \text{for } \mu, \nu \in \Lambda^k,$$

that give rise to an embedding $\mathcal{F}_k^l \hookrightarrow \mathcal{F}_{k+1}^{l+1}$. It induces an embedding of $\mathcal{F}_k^\infty$ into $\mathcal{F}_{k+1}^\infty$ that we denote by $\lambda_{k,k+1}$.

Proposition 3.4.

(i) $\mathcal{F}_k^\infty$ *is an AF-algebra defined by the inductive limit of the embeddings* $\iota_{l,l+1} : \mathcal{F}_k^l \hookrightarrow \mathcal{F}_k^{l+1}, l \in \mathbb{N}$.

(ii) $\mathcal{F}_\mathfrak{L}$ *is an AF-algebra defined by the inductive limit of the embeddings* $\lambda_{k,k+1} : \mathcal{F}_k^\infty \hookrightarrow \mathcal{F}_{k+1}^\infty, k \in \mathbb{Z}_+$.

Let $U_z, z \in \mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ be an action of $\mathbb{T}$ to the unitary group of $B(l^2(G_\mathfrak{L}))$ defined by

$$(U_z\xi)(x,n,y) = z^n\xi(x,n,y) \quad \text{for} \quad \xi \in l^2(G_\mathfrak{L}), (x,n,y) \in G_\mathfrak{L}.$$

The action $Ad(U_z)$ on $B(l^2(G_\mathfrak{L}))$ leaves $\mathcal{O}_\mathfrak{L}$ globally invariant. It gives rise to an action on $\mathcal{O}_\mathfrak{L}$. We denote it by $\alpha_\mathfrak{L}$ and call it the gauge action. Let $E_\mathfrak{L}$ be the expectation from $\mathcal{O}_\mathfrak{L}$ onto the fixed point subalgebra ${\mathcal{O}_\mathfrak{L}}^{\alpha_\mathfrak{L}}$ under $\alpha_\mathfrak{L}$ defined by

$$E_\mathfrak{L}(X) = \int_{z\in\mathbb{T}} {\alpha_\mathfrak{L}}_z(X)dz, \qquad X \in \mathcal{O}_\mathfrak{L}. \tag{3.5}$$

Let $\mathcal{P}_\mathfrak{L}$ be the $*$-algebra generated algebraically by $S_\alpha, \alpha \in \Sigma$ and $E_i^l, i = 1,\dots,m(l)$, $l \in \mathbb{Z}_+$. For $\mu = (\mu_1,\dots,\mu_k) \in \Lambda^k$, it follows that by (3.3), $E_i^l S_{\mu_1}\cdots S_{\mu_k} = S_{\mu_1}S_{\mu_1}^* E_i^l S_{\mu_1}\cdots S_{\mu_k}$. As $S_{\mu_1}^* E_i^l S_{\mu_1}$ is a linear combination of $E_j^{l+1}, j = 1,\dots,m(l+1)$ by (3.4), one sees $S_{\mu_1}^* E_i^l S_{\mu_1} S_{\mu_2} = S_{\mu_2}S_{\mu_2}^* S_{\mu_1}^* E_i^l S_{\mu_1} S_{\mu_2}$ and inductively

$$E_i^l S_\mu = S_\mu S_\mu^* E_i^l S_\mu, \qquad E_i^l S_\mu S_\mu^* = S_\mu S_\mu^* E_i^l. \tag{3.6}$$

By the relations (3.6), each element $X \in \mathcal{P}_\mathfrak{L}$ is expressed as a finite sum

$$X = \sum_{|\nu|\ge 1} X_{-\nu}S_\nu^* + X_0 + \sum_{|\mu|\ge 1} S_\mu X_\mu \qquad \text{for some} \qquad X_{-\nu}, X_0, X_\mu \in \mathcal{F}_\mathfrak{L}.$$

Then the following lemma is routine.

Lemma 3.5. *The fixed point subalgebra* ${\mathcal{O}_\mathfrak{L}}^{\alpha_\mathfrak{L}}$ *of* $\mathcal{O}_\mathfrak{L}$ *under* $\alpha_\mathfrak{L}$ *is the AF-algebra* $\mathcal{F}_\mathfrak{L}$.

We can now prove a universal property of $\mathcal{O}_\mathfrak{L}$.

Theorem 3.6. *The* C^**-algebra* $\mathcal{O}_\mathfrak{L}$ *is the universal* C^**-algebra subject to the relations* $(\mathfrak{L})$.

Proof. Let $\mathcal{O}_{[\mathfrak{L}]}$ be the universal C^*-algebra generated by partial isometries $s_\alpha, \alpha \in \Sigma$ and projections $e_i^l, i = 1,\dots,m(l), l \in \mathbb{Z}_+$ subject to the operator relations $(\mathfrak{L})$. This means that $\mathcal{O}_{[\mathfrak{L}]}$ is generated by $s_\alpha, \alpha \in \Sigma$ and $e_i^l, i = 1,\dots,m(l), l \in \mathbb{Z}_+$, that have only operator relations $(\mathfrak{L})$. The C^*-norm of $\mathcal{O}_{[\mathfrak{L}]}$ is given by the universal C^*-norm. Let us denote by $\mathcal{F}_{[k]}^{[l]}, \mathcal{F}_{[\mathfrak{L}]}$

the similarly defined subalgebras of $\mathcal{O}_{[\mathfrak{L}]}$ to $\mathcal{F}_k^l, \mathcal{F}_{\mathfrak{L}}$ respectively. The algebra $\mathcal{F}_{[k]}^{[l]}$ as well as $\mathcal{F}_k^l$ is a finite dimensional algebra. Since $s_\mu e_i^l s_\nu^* \neq 0$ if and only if $S_\mu E_i^l S_\nu^* \neq 0$, the correspondence $s_\mu e_i^l s_\nu^* \to S_\mu E_i^l S_\nu^*, |\mu| = |\nu| = k \leq l$ yields an isomorphism from $\mathcal{F}_{[k]}^{[l]}$ to $\mathcal{F}_k^l$. It induces an isomorphism from $\mathcal{F}_{[\mathfrak{L}]}$ to $\mathcal{F}_{\mathfrak{L}}$. By the universality, for $z \in \mathbb{C}, |z| = 1$ the correspondence $s_\alpha \to z s_\alpha, \alpha \in \Sigma$, $e_i^l \to e_i^l, i = 1, \dots, m(l), l \in \mathbb{Z}_+$ gives rise to an action of the torus group $\mathbb{T}$ on $\mathcal{O}_{[\mathfrak{L}]}$, which we denote by $\alpha_{[\mathfrak{L}]}$. Let $E_{[\mathfrak{L}]}$ be the expectation from $\mathcal{O}_{[\mathfrak{L}]}$ onto the fixed point subalgebra ${\mathcal{O}_{[\mathfrak{L}]}}^{\alpha_{[\mathfrak{L}]}}$ under $\alpha_{[\mathfrak{L}]}$ similarly defined to (3.5). The algebra ${\mathcal{O}_{[\mathfrak{L}]}}^{\alpha_{[\mathfrak{L}]}}$ is nothing but the algebra $\mathcal{F}_{[\mathfrak{L}]}$. By the universality of $\mathcal{O}_{[\mathfrak{L}]}$, the correspondence $s_\alpha \to S_\alpha, \alpha \in \Sigma$, $e_i^l \to E_i^l, i = 1, \dots, m(l), l \in \mathbb{Z}_+$ extends to a surjective homomorphism from $\mathcal{O}_{[\mathfrak{L}]}$ to $\mathcal{O}_{\mathfrak{L}}$, which we denote by $\pi_{\mathfrak{L}}$. The restriction of $\pi_{\mathfrak{L}}$ to $\mathcal{F}_{[\mathfrak{L}]}$ is the preceding isomorphism. As we see that $E_{\mathfrak{L}} \circ \pi_{\mathfrak{L}} = \pi_{\mathfrak{L}} \circ E_{[\mathfrak{L}]}$ and $E_{[\mathfrak{L}]}$ is faithful, we conclude that $\pi_{\mathfrak{L}}$ is isomorphic by a similar argument to [CK; 2.12. Proposition]. □

4. UNIQUENESS AND SIMPLICITY

We will prove that $\mathcal{O}_{\mathfrak{L}}$ is the unique C^*-algebra subject to the operator relations $(\mathfrak{L})$ under a mild condition on $\mathfrak{L}$, called (I). The condition (I) is a generalization of condition (I) for a finite square matrix with entries in $\{0, 1\}$ defined by Cuntz-Krieger in [CK] and condition (I) for a subshift defined in [Ma4]. A related condition for a Hilbert C^*-bimodule has been introduced by Kajiwara-Pinzari-Watatani in [KPW]. For an infinite directed graph, such a condition is defined by Kumjian-Pask-Raeburn-Renault in [KPRR]. For a vertex $\mathfrak{v}_i^l \in V_l$, let $\Gamma^+(\mathfrak{v}_i^l)$ be the set of all label sequences in $\mathfrak{L}$ starting at $\mathfrak{v}_i^l$. That is,

$$\begin{aligned}\Gamma^+(\mathfrak{v}_i^l) = \{(\alpha_1, \alpha_2, \dots,) \in \Sigma^{\mathbb{N}} \mid \exists e_{n,n+1} \in E_{n,n+1} \text{ for } n = l, l+1, \dots;\\ \mathfrak{v}_i^l = s(e_{l,l+1}),\, t(e_{n,n+1}) = s(e_{n+1,n+2}),\, \lambda(e_{n,n+1}) = \alpha_{n-l+1}\}.\end{aligned}$$

DEFINITION. *A λ-graph system $\mathfrak{L}$ satisfies condition (I) if for each $\mathfrak{v}_i^l \in V$, the set $\Gamma^+(\mathfrak{v}_i^l)$ contains at least two distinct sequences.*

For $\mathfrak{v}_i^l \in V_l$ set $F_i^l = \{x \in X_{\mathfrak{L}} \mid v(x)_0^l = \mathfrak{v}_i^l\}$ where $v(x)_0 = (v(x)_0^l)_{l \in \mathbb{Z}_+} \in \Omega_{\mathfrak{L}} = \varprojlim V_l$ is the unique ι-orbit for $x \in X_{\mathfrak{L}}$ such that $(v(x)_0, \lambda(x)_1, v(x)_1) \in E_{\mathfrak{L}}$ as in the preceding section. By a similar discussion to [Ma4; Section 5] (cf.[CK; 2.6.Lemma]), we know that if $\mathfrak{L}$ satisfies (I), for $l, k \in \mathbb{N}$ with $l \geq k$, there exists $y_i^l \in F_i^l$ for each $i = 1, 2, \dots, m(l)$ such that

$$\sigma^m(y_i^l) \neq y_j^l \qquad \text{for all} \quad 1 \leq i, j \leq m(l), \quad 1 \leq m \leq k.$$

By the same manner as the proof of [Ma4;Lemma 5.3], we obtain

LEMMA 4.1. *Suppose that $\mathfrak{L}$ satisfies condition (I). Then for $l, k \in \mathbb{N}$ with $l \geq k$, there exists a projection $q_k^l \in \mathcal{D}_{\mathfrak{L}}$ such that*

(i) $q_k^l a \neq 0$ *for all nonzero* $a \in \mathcal{A}_l$,
(ii) $q_k^l \phi_{\mathfrak{L}}^m(q_k^l) = 0$ *for all* $m = 1, 2, \dots, k$, *where* $\phi_{\mathfrak{L}}^m(X) = \sum_{\mu \in \Lambda^m} S_\mu X S_\mu^*$.

Now we put $Q_k^l = \phi_{\mathfrak{L}}^k(q_k^l)$ a projection in $\mathcal{D}_{\mathfrak{L}}$. Note that each element of $\mathcal{D}_{\mathfrak{L}}$ commutes with elements of $\mathcal{A}_{\mathfrak{L}}$. As we see $S_\mu \phi_{\mathfrak{L}}^j(X) = \phi_{\mathfrak{L}}^{j+|\mu|}(X)S_\mu$ for $X \in \mathcal{D}_{\mathfrak{L}}, j \in \mathbb{Z}_+, \mu \in \Lambda^*$, a similar argument to [CK;2.9.Proposition] leads to the following lemma.

LEMMA 4.2.

(i) *The correspondence:* $X \in \mathcal{F}_k^l \longrightarrow Q_k^l X Q_k^l \in Q_k^l \mathcal{F}_k^l Q_k^l$ *extends to an isomorphism from* $\mathcal{F}_k^l$ *onto* $Q_k^l \mathcal{F}_k^l Q_k^l$.
(ii) $Q_k^l X - X Q_k^l \to 0$, $\|Q_k^l X\| \to \|X\|$ *as* $k, l \to \infty$ *for* $X \in \mathcal{F}_{\mathfrak{L}}$.
(iii) $Q_k^l S_\mu Q_k^l, Q_k^l S_\mu^* Q_k^l \to 0$ *as* $k, l \to \infty$ *for* $\mu \in \Lambda^*$.

We then prove the uniqueness of the algebra $\mathcal{O}_{\mathfrak{L}}$ subject to the relations $(\mathfrak{L})$.

THEOREM 4.3. *Suppose that* $\mathfrak{L}$ *satisfies condition (I). Let* $\widehat{S}_\alpha, \alpha \in \Sigma$ *and* $\widehat{E}_i^l, i = 1, 2, \dots, m(l), l \in \mathbb{Z}_+$ *be another family of nonzero partial isometries and nonzero projections satisfying the relations* $(\mathfrak{L})$. *Then the map* $S_\alpha \to \widehat{S}_\alpha, \alpha \in \Sigma$, $E_i^l \to \widehat{E}_i^l, i = 1, \dots, m(l), l \in \mathbb{Z}_+$ *extends to an isomorphism from* $\mathcal{O}_{\mathfrak{L}}$ *onto the* C^**-algebra* $\widehat{\mathcal{O}}_{\mathfrak{L}}$ *generated by* $\widehat{S}_\alpha, \alpha \in \Sigma$ *and* $\widehat{E}_i^l, i = 1, \dots, m(l), l \in \mathbb{Z}_+$.

Proof. We may define C^*-subalgebras $\widehat{\mathcal{D}}_{\mathfrak{L}}, \widehat{\mathcal{F}}_k^l, \widehat{\mathcal{F}}_{\mathfrak{L}}$ of $\widehat{\mathcal{O}}_{\mathfrak{L}}$ by using the elements $\widehat{S}_\mu \widehat{E}_i^l \widehat{S}_\nu^*$ by the same manners as the constructions of the C^*-subalgebras $\mathcal{D}_{\mathfrak{L}}, \mathcal{F}_k^l, \mathcal{F}_{\mathfrak{L}}$ of $\mathcal{O}_{\mathfrak{L}}$ respectively. As in the proof of Theorem 3.6, the map $S_\mu E_i^l S_\nu^* \in \mathcal{F}_k^l \to \widehat{S}_\mu \widehat{E}_i^l \widehat{S}_\nu^* \in \widehat{\mathcal{F}}_k^l, |\mu| = |\nu| = k \le l$ extends to an isomorphism from the AF-algebra $\mathcal{F}_{\mathfrak{L}}$ onto the AF-algebra $\widehat{\mathcal{F}}_{\mathfrak{L}}$. By Theorem 3.6, the algebra $\mathcal{O}_{\mathfrak{L}}$ has a universal property subject to the relations $(\mathfrak{L})$ so that there exists a surjective homomorphism $\hat{\pi}$ from $\mathcal{O}_{\mathfrak{L}}$ onto $\widehat{\mathcal{O}}_{\mathfrak{L}}$ satisfying $\hat{\pi}(S_\alpha) = \widehat{S}_\alpha$ and $\hat{\pi}(E_i^l) = \widehat{E}_i^l$. The restriction of $\hat{\pi}$ to $\mathcal{F}_{\mathfrak{L}}$ is the preceding isomorphism onto $\widehat{\mathcal{F}}_{\mathfrak{L}}$. Now $\mathfrak{L}$ satisfies (I). Let Q_k^l be the sequence of projections as in Lemma 4.2. We put $\widehat{Q}_k^l = \hat{\pi}(Q_k^l) \in \widehat{\mathcal{D}}_{\mathfrak{L}}$ that has the corresponding properties to Lemma 4.2 for the algebra $\widehat{\mathcal{F}}_{\mathfrak{L}}$. Let $\widehat{\mathcal{P}}_{\mathfrak{L}}$ be the $*$-algebra generated algebraically by $\widehat{S}_\alpha, \alpha \in \Sigma$ and $\widehat{E}_i^l, i = 1, \dots, m(l), l \in \mathbb{Z}_+$. By the relations $(\mathfrak{L})$, each element $X \in \widehat{\mathcal{P}}_{\mathfrak{L}}$ is expressed as a finite sum

$$X = \sum_{|\nu| \ge 1} X_{-\nu} \widehat{S}_\nu^* + X_0 + \sum_{|\mu| \ge 1} \widehat{S}_\mu X_\mu \qquad \text{for some} \qquad X_{-\nu}, X_0, X_\mu \in \widehat{\mathcal{F}}_{\mathfrak{L}}.$$

By a similar argument to [CK;2.9.Proposition], it follows that the map $X \in \widehat{\mathcal{P}}_{\mathfrak{L}} \to X_0 \in \widehat{\mathcal{F}}_{\mathfrak{L}}$ extends to an expectation $\widehat{E}_{\mathfrak{L}}$ from $\widehat{\mathcal{O}}_{\mathfrak{L}}$ onto $\widehat{\mathcal{F}}_{\mathfrak{L}}$, that satisfies $\widehat{E}_{\mathfrak{L}} \circ \hat{\pi} = \hat{\pi} \circ E_{\mathfrak{L}}$. As $E_{\mathfrak{L}}$ is faithful, we conclude that $\hat{\pi}$ is isomorphic. □

REMARK. Let e_x be a vector assigned to $x \in X_{\mathfrak{L}}$. Let $\mathfrak{H}_{\mathfrak{L}}$ be the Hilbert space spanned by the vectors $e_x, x \in X_{\mathfrak{L}}$ such that the vectors $e_x, x \in X_{\mathfrak{L}}$ form its complete orthonormal basis. For $x = (\alpha_i, v_i)_{i=1}^\infty \in X_{\mathfrak{L}}$, take $v_0 = v(x)_0 \in \Omega_{\mathfrak{L}}$. For a symbol $\beta \in \Sigma$, if there exists a vertex $v_{-1} \in \Omega_{\mathfrak{L}}$ such that $(v_{-1}, \beta, v_0) \in E_{\mathfrak{L}}$, we define $\beta x \in X_{\mathfrak{L}}$, by putting $\alpha_0 = \beta$, as

$$\beta x = (\alpha_{i-1}, v_{i-1})_{i=1}^\infty \in X_{\mathfrak{L}}.$$

Put

$$\Gamma_1^-(x) = \{\gamma \in \Sigma \mid (v_{-1}, \gamma, v(x)_0) \in E_{\mathfrak{L}} \text{ for some } v_{-1} \in \Omega_{\mathfrak{L}}\}.$$

We define the creation operators $\widetilde{S}_\beta, \beta \in \Sigma$ on $\mathfrak{H}_{\mathfrak{L}}$ by

$$\widetilde{S}_\beta e_x = \begin{cases} e_{\beta x} & \text{if } \beta \in \Gamma_1^-(x), \\ 0 & \text{if } \beta \notin \Gamma_1^-(x). \end{cases}$$

PROPOSITION 4.4. *Suppose that $\mathfrak{L}$ satisfies condition (I). If $\mathfrak{L}$ is predecessor-separated, $\mathcal{O}_{\mathfrak{L}}$ is isomorphic to the C^*-algebra $C^*(\widetilde{S}_\beta, \beta \in \Sigma)$ generated by the partial isometries $\widetilde{S}_\beta, \beta \in \Sigma$ on the Hilbert space $\mathfrak{H}_{\mathfrak{L}}$.*

Proof. Suppose that $\mathfrak{L}$ is predecessor-separated. Define a sequence of projections $\widetilde{E}_i^l, i = 1, \dots, m(l), l \in \mathbb{N}$ and $\widetilde{E}_i^0, i = 1, \dots, m(0)$ by using the formulae (1.7) from the partial isometries $\widetilde{S}_\beta, \beta \in \Sigma$. It is straightforward to see that $\widetilde{E}_i^l, i = 1, \dots, m(l), l \in \mathbb{Z}_+$ are nonzero. The partial isometries $\widetilde{S}_\beta$ and the projections $\widetilde{E}_i^l$ satisfy the relations ($\mathfrak{L}$). □

Let Λ be a subshift and $\mathfrak{L}^\Lambda$ its canonical λ-graph system, that is left-resolving and predecessor-separated. It is easy to see that Λ satisfies condition (I) in the sense of [Ma4] if and only if $\mathfrak{L}^\Lambda$ satisfies condition (I).

COROLLARY 4.5(CF.[MA],[CAM]). *The C^*-algebra $\mathcal{O}_{\mathfrak{L}^\Lambda}$ associated with λ-graph system $\mathfrak{L}^\Lambda$ is canonically isomorphic to the C^*-algebra $\mathcal{O}_\Lambda$ associated with subshift Λ.*

We next refer simplicity and purely infiniteness of the algebra $\mathcal{O}_{\mathfrak{L}}$. We introduce the notions of irreducibility and aperiodicity for λ-graph system

DEFINITION.

(i) *A λ-graph system $\mathfrak{L}$ is said to be irreducible if for a vertex $v \in V_l$ and $x = (x_1, x_2, \dots) \in \Omega_{\mathfrak{L}} = \varprojlim V_l$, there exists a path in $\mathfrak{L}$ starting at v and terminating at x_{l+N} for some $N \in \mathbb{N}$.*

(ii) *A λ-graph system $\mathfrak{L}$ is said to be aperiodic if for a vertex $v \in V_l$ there exists an $N \in \mathbb{N}$ such that there exist paths in $\mathfrak{L}$ starting at v and terminating at all the vertices of V_{l+N}.*

Aperiodicity automatically implies irreducibility. Define a positive operator $\lambda_{\mathfrak{L}}$ on $\mathcal{A}_{\mathfrak{L}}$ by

$$\lambda_{\mathfrak{L}}(X) = \sum_{\alpha \in \Sigma} S_\alpha^* X S_\alpha \qquad \text{for} \qquad X \in \mathcal{A}_{\mathfrak{L}}.$$

We say that $\lambda_{\mathfrak{L}}$ is *irreducible* if there exists no non-trivial ideal of $\mathcal{A}_{\mathfrak{L}}$ invariant under $\lambda_{\mathfrak{L}}$, and $\lambda_{\mathfrak{L}}$ is *aperiodic* if for a projection $E_i^l \in \mathcal{A}_l$ there exists $N \in \mathbb{N}$ such that $\lambda_{\mathfrak{L}}^N(E_i^l) \geq 1$. The following lemma is easy to prove (cf.[Ma4]).

Lemma 4.6.

(i) *A λ-graph system $\mathfrak{L}$ is irreducible if and only if $\lambda_{\mathfrak{L}}$ is irreducible.*
(ii) *A λ-graph system $\mathfrak{L}$ is aperiodic if and only if $\lambda_{\mathfrak{L}}$ is aperiodic.*

We thus obtain

Theorem 4.7. *Suppose that a λ-graph system $\mathfrak{L}$ satisfies condition (I). If $\mathfrak{L}$ is irreducible, $\mathcal{O}_{\mathfrak{L}}$ is simple.*

Proof. Suppose that there exists a nonzero ideal $\mathcal{I}$ of $\mathcal{O}_{\mathfrak{L}}$. As $\mathfrak{L}$ satisfies condition (I), by uniqueness of the algebra $\mathcal{O}_{\mathfrak{L}}$, $\mathcal{I}$ must contain a projection E_i^l for some l, i. Hence $\mathcal{I} \cap \mathcal{A}_{\mathfrak{L}}$ is a nonzero ideal of $\mathcal{A}_{\mathfrak{L}}$ that is invariant under $\lambda_{\mathfrak{L}}$. This leads to $\mathcal{I} = \mathcal{A}_{\mathfrak{L}}$ so that $\mathcal{O}_{\mathfrak{L}}$ is simple. $\square$

The above theorem is a generalization of [CK; 2.14.Theorem] and [Ma;Theorem 6.3]. We next see that $\mathcal{O}_{\mathfrak{L}}$ is purely infinite (and simple) if $\mathfrak{L}$ is aperiodic. Assume that the subshift presented by λ-graph system $\mathfrak{L}$ is not a single point. Note that if $\mathfrak{L}$ is aperiodic, it satisfies condition (I). By [Bra;Corollary 3.5], the following lemma is straightforward.

Lemma 4.8. *A λ-graph system $\mathfrak{L}$ is aperiodic if and only if the AF-algebra $\mathcal{F}_{\mathfrak{L}}$ is simple.*

As in the proof of [C3;1.6 Proposition], we conclude

Proposition 4.9 (cf.[C;1.13 Theorem]). *If a λ-graph system $\mathfrak{L}$ is aperiodic, $\mathcal{O}_{\mathfrak{L}}$ is simple and purely infinite.*

5. K-Theory

The K-groups for the C^*-algebras associated with subshifts have been computed in [Ma2] by using an analogous idea to the Cuntz's paper [C3]. The discussion given in [Ma2] well works for our algebras $\mathcal{O}_{\mathfrak{L}}$ associated with λ-graph systems. Let (A, I) be the nonnegative matrix system of the symbolic matrix system for $\mathfrak{L}$. We first study the K_0-group for the AF-algebra $\mathcal{F}_{\mathfrak{L}}$. We denote by $\Lambda^k(\mathfrak{v}_i^l)$ the set of words of length k that terminate at the vertex $\mathfrak{v}_i^l$. Let $\mathcal{F}_k^{l,i}$ be the C^*-subalgebra of $\mathcal{F}_k^l$ generated by the elements $S_\mu E_i^l S_\nu^*, \mu, \nu \in \Lambda^k$. It is isomorphic to the full matrix algebra $M_{n_i^l(k)}(\mathbb{C})$ of size $n_i^l(k)$ where $n_i^l(k)$ denotes the number of the set $\Lambda^k(\mathfrak{v}_i^l)$, so that one sees

$$\mathcal{F}_k^l \cong M_{n_1^l(k)}(\mathbb{C}) \oplus \cdots \oplus M_{n_{m(l)}^l(k)}(\mathbb{C}).$$

The map $\Phi_k^l : [S_\mu E_i^l S_\mu^*] \in K_0(\mathcal{F}_k^l) \to [E_i^l] \in K_0(\mathcal{A}_l)$ for $i = 1, 2, \dots, m(l), \mu \in \Lambda^k(\mathfrak{v}_i^l)$ yields an isomorphism between $K_0(\mathcal{F}_k^l)$ and $K_0(\mathcal{A}_l) = \mathbb{Z}^{m(l)} = \sum_{i=1}^{m(l)} \mathbb{Z}[E_i^l]$. The isomorphisms $\Phi_k^l, l \in \mathbb{N}$ induce an isomorphism $\Phi_k = \varinjlim_l \Phi_k^l$ from $K_0(\mathcal{F}_k^\infty) = \varinjlim_{\iota_{l,l+1*}} K_0(\mathcal{F}_k^l)$ onto $K_0(\mathcal{A}_{\mathfrak{L}}) = \varinjlim_{\iota_{l,l+1*}} K_0(\mathcal{A}_l)$ in a natural way.

The latter group is denoted by $\mathbb{Z}_{I^t}$, that is isomorphic to the abelian group $\varinjlim_l \{\mathbb{Z}^{m(l)}, I^t_{l,l+1}\}$ of the inductive limit of the homomorphisms $I^t_{l,l+1} : \mathbb{Z}^{m(l)} \longrightarrow \mathbb{Z}^{m(l+1)}, l \in \mathbb{N}$. The embedding $\lambda_{k,k+1}$ of $\mathcal{F}_k^\infty$ into $\mathcal{F}_{k+1}^\infty$ given in Proposition 3.4 (ii) induces a homomorphism $\lambda_{k,k+1_*}$ from $K_0(\mathcal{F}_k^\infty)$ to $K_0(\mathcal{F}_{k+1}^\infty)$ that satisfies

$$\lambda_{k,k+1_*}([S_\mu E_i^l S_\mu^*]) = \sum_{\alpha\in\Sigma}[S_{\mu\alpha}S_\alpha^* E_i^l S_\alpha S_{\mu\alpha}^*], \quad \mu \in \Lambda^k(\mathfrak{v}_i^l), \quad i = 1,2,\dots,m(l).$$

Define a homomorphism λ_l from $K_0(\mathcal{A}_l)$ to $K_0(\mathcal{A}_{l+1})$ by

$$\lambda_l([E_i^l]) = \sum_{j=1}^{m(l+1)} A_{l,l+1}(i,j)[E_j^{l+1}].$$

As $A_{l,l+1}(i,j) = \sum_{\alpha\in\Sigma} A_{l,l+1}(i,\alpha,j)$, we see $\lambda_l([E_i^l]) = \sum_{\alpha\in\Sigma}[S_\alpha^* E_i^l S_\alpha]$ by (3.4). The homomorphisms $\lambda_l : K_0(\mathcal{A}_l) \to K_0(\mathcal{A}_{l+1}), l \in \mathbb{N}$ act as the transposes $A^t_{l,l+1}$ of the matrices $A_{l,l+1} = [A_{l,l+1}(i,j)]_{i,j}$, that are compatible with the embeddings $\iota_{l,l+1_*}(= I^t_{l,l+1}) : K_0(\mathcal{A}_l) \to K_0(\mathcal{A}_{l+1})$ by (1.2). They define an endomorphism on $\mathbb{Z}_{I^t}(\cong K_0(\mathcal{A}_\mathfrak{L})$. We denote it by $\lambda_{(A,I)}$. Since the diagram

$$\begin{array}{ccc} K_0(\mathcal{F}_k^\infty) & \xrightarrow{\lambda_{k,k+1_*}} & K_0(\mathcal{F}_{k+1}^\infty) \\ {\scriptstyle \Phi_k}\big\downarrow & & \big\downarrow{\scriptstyle \Phi_{k+1}} \\ K_0(\mathcal{A}_\mathfrak{L}) & \xrightarrow[\lambda_{(A,I)}]{} & K_0(\mathcal{A}_\mathfrak{L}) \end{array}$$

is commutative, one obtains

Proposition 5.1. $K_0(\mathcal{F}_\mathfrak{L}) = \varinjlim\{\mathbb{Z}_{I^t}, \lambda_{(A,I)}\}$.

The group $\varinjlim\{\mathbb{Z}_{I^t}, \lambda_{(A,I)}\}$ is the dimension group $\Delta_{(A,I)}$ for the nonnegative matrix system (A,I) defined in [Ma5]. The dimension group for a nonnegative square finite matrix has been introduced by W. Krieger in [Kr] and [Kr2]. It is realized as the K_0-group for the canonical AF-algebra inside the Cuntz-Krieger algebra associated with the matrix ([C2],[C3]). If a λ-graph system $\mathfrak{L}$ is arising from the finite directed graph associated with the matrix, the C^*-algebras $\mathcal{O}_\mathfrak{L}$ and $\mathcal{F}_\mathfrak{L}$ coincide with the Cuntz-Krieger algebra and the canonical AF-algebra respectively (cf. Section 7). Hence in this case, $K_0(\mathcal{F}_\mathfrak{L})$ coincides with the Krieger's dimension group for the matrix.

Let $p_0 : \mathbb{T} \to \mathcal{O}_\mathfrak{L}$ be the constant function whose value everywhere is the unit 1 of $\mathcal{O}_\mathfrak{L}$. It belongs to the algebra $L^1(\mathbb{T}, \mathcal{O}_\mathfrak{L})$ and hence to the crossed product $\mathcal{O}_\Lambda \rtimes_{\alpha_\mathfrak{L}} \mathbb{T}$. By [Ro], the fixed point subalgebra ${\mathcal{O}_\mathfrak{L}}^{\alpha_\mathfrak{L}}$ is isomorphic to the algebra $p_0(\mathcal{O}_\mathfrak{L} \rtimes_{\alpha_\mathfrak{L}} \mathbb{T})p_0$ through the correspondence : $x \in {\mathcal{O}_\mathfrak{L}}^{\alpha_\mathfrak{L}} \to \hat{x} \in L^1(\mathbb{T}, \mathcal{O}_\mathfrak{L}) \subset \mathcal{O}_\mathfrak{L} \rtimes_{\alpha_\mathfrak{L}} \mathbb{T}$ where the function $\hat{x}$ is defined by $\hat{x}(t) = x, t \in \mathbb{T}$. Then as in [Ma2;Section 4], the projection p_0 is full in $\mathcal{O}_\mathfrak{L} \rtimes_{\alpha_\mathfrak{L}} \mathbb{T}$. Since the AF-algebra $\mathcal{F}_\mathfrak{L}$ is realized as ${\mathcal{O}_\mathfrak{L}}^{\alpha_\mathfrak{L}}$, one sees, by [Bro;Corollary 2.6]

Lemma 5.2. *$\mathcal{O}_{\mathfrak{L}} \rtimes_{\alpha_{\mathfrak{L}}} \mathbb{T}$ is stably isomorphic to $\mathcal{F}_{\mathfrak{L}}$.*

The natural inclusion $\iota : p_0(\mathcal{O}_{\mathfrak{L}} \rtimes_{\alpha_{\mathfrak{L}}} \mathbb{T})p_0 \to \mathcal{O}_{\mathfrak{L}} \rtimes_{\alpha_{\mathfrak{L}}} \mathbb{T}$ induces an isomorphism $\iota_* : K_0(p_0(\mathcal{O}_{\mathfrak{L}} \rtimes_{\alpha_{\mathfrak{L}}} \mathbb{T})p_0) \to K_0(\mathcal{O}_{\mathfrak{L}} \rtimes_{\alpha_{\mathfrak{L}}} \mathbb{T})$ on K-theory (cf.[Ri;Proposition 2.4]). Denote by $\widehat{\alpha_{\mathfrak{L}}}$ the dual action of $\alpha_{\mathfrak{L}}$ on $\mathcal{O}_{\mathfrak{L}} \rtimes_{\alpha_{\mathfrak{L}}} \mathbb{T}$. Under the identification between $\mathcal{F}_{\mathfrak{L}}$ and $p_0(\mathcal{O}_{\mathfrak{L}} \rtimes_{\alpha_{\mathfrak{L}}} \mathbb{T})p_0$, we define an automorphism β on $K_0(\mathcal{F}_{\mathfrak{L}})$ by $\beta = {\iota_*}^{-1} \circ \widehat{\alpha_{\mathfrak{L}}}_* \circ \iota_*$. By a similar argument to [Ma2;Lemma 4.5], [Ma2; Lemma 4.6] and [Ma2;Corollary 4.7], the automorphism $\beta^{-1} : K_0(\mathcal{F}_{\mathfrak{L}}) \to K_0(\mathcal{F}_{\mathfrak{L}})$ corresponds to the shift σ on $\varinjlim\{\mathbb{Z}_{I^t}, \lambda_{(A,I)}\}$. That is, if $x = (x_1, x_2, \dots)$ is a sequence representing an element of $\varinjlim\{\mathbb{Z}_{I^t}, \lambda_{(A,I)}\}$, then $\beta^{-1}x$ is represented by $\sigma(x) = (x_2, x_3, \dots)$. As the dimension automorphism $\delta_{(A,I)}$ of $\Delta_{(A,I)}$ is defined to be the shift of the inductive limit $\varinjlim\{\mathbb{Z}_{I^t}, \lambda_{(A,I)}\}$ ([Ma5]), we obtain

Proposition 5.3. $(K_0(\mathcal{F}_{\mathfrak{L}}), K_0(\mathcal{F}_{\mathfrak{L}})_+, \widehat{\alpha_{\mathfrak{L}}}_*) \cong (\Delta_{(A,I)}, \Delta^+_{(A,I)}, \delta_{(A,I)})$.

We will next present the K-theory formulae for $\mathcal{O}_{\mathfrak{L}}$. As $K_1(\mathcal{O}_{\mathfrak{L}} \rtimes_{\alpha_{\mathfrak{L}}} \mathbb{T}) = 0$, the Pimsner-Voiculescu's six term exact sequence of the K-theory for the crossed product $(\mathcal{O}_{\mathfrak{L}} \rtimes_{\alpha_{\mathfrak{L}}} \mathbb{T}) \rtimes_{\widehat{\alpha_{\mathfrak{L}}}} \mathbb{Z}$ [PV] says the following lemma:

Lemma 5.4.

(i) $K_0(\mathcal{O}_{\mathfrak{L}}) \cong K_0(\mathcal{O}_{\mathfrak{L}} \rtimes_{\alpha_{\mathfrak{L}}} \mathbb{T})/(\mathrm{id} - \widehat{\alpha_{\mathfrak{L}}}_*^{-1})K_0(\mathcal{O}_{\mathfrak{L}} \rtimes_{\alpha_{\mathfrak{L}}} \mathbb{T})$.

(ii) $K_1(\mathcal{O}_{\mathfrak{L}}) \cong \mathrm{Ker}(\mathrm{id} - \widehat{\alpha_{\mathfrak{L}}}_*^{-1})$ *on* $K_0(\mathcal{O}_{\mathfrak{L}} \rtimes_{\alpha_{\mathfrak{L}}} \mathbb{T})$.

Therefore we have the K-theory formulae for $\mathcal{O}_{\mathfrak{L}}$ by a similar argument to [C3;3.1.Proposition].

Theorem 5.5.

(i)

$$\begin{aligned} K_0(\mathcal{O}_{\mathfrak{L}}) &\cong \mathbb{Z}_{I^t}/(\mathrm{id} - \lambda_{(A,I)})\mathbb{Z}_{I^t} \\ &\cong \varinjlim\{\mathbb{Z}^{m(l+1)}/(I^t_{l,l+1} - A^t_{l,l+1})\mathbb{Z}^{m(l)}; \bar{I}^t_{l,l+1}\}, \end{aligned}$$

(ii)

$$\begin{aligned} K_1(\mathcal{O}_{\mathfrak{L}}) &\cong \mathrm{Ker}(\mathrm{id} - \lambda_{(A,I)}) \ \textit{in}\ \mathbb{Z}_{I^t} \\ &\cong \varinjlim\{\mathrm{Ker}(I^t_{l,l+1} - A^t_{l,l+1}) \ \textit{in}\ \mathbb{Z}^{m(l)}; I^t_{l,l+1}\} \end{aligned}$$

where $\bar{I}^t_{l,l+1}$ *is the homomorphism from* $\mathbb{Z}^{m(l)}/(I^t_{l-1,l} - A^t_{l-1,l})\mathbb{Z}^{m(l-1)}$ *to* $\mathbb{Z}^{m(l+1)}/(I^t_{l,l+1} - A^t_{l,l+1})\mathbb{Z}^{m(l)}$ *induced by* $I^t_{l,l+1}$. *More precisely, for the minimal projections* $E^l_1, \dots, E^l_{m(l)}$ *of* $\mathcal{A}_l$ *with* $\sum_{i=1}^{m(l)} E^l_i = 1$ *and the canonical basis* $e^l_1, \dots, e^l_{m(l)}$ *of* $\mathbb{Z}^{m(l)}$, *the map* $[E^l_i] \to e^l_i$ *extends to an isomorphism from* $K_0(\mathcal{O}_{\mathfrak{L}})$ *onto* $\varinjlim\{\mathbb{Z}^{m(l+1)}/(I^t_{l,l+1} - A^t_{l,l+1})\mathbb{Z}^{m(l)}; \bar{I}^t_{l,l+1}\}$. *Hence we have*

$$K_i(\mathcal{O}_{\mathfrak{L}}) \cong K_i(A, I) \qquad i = 0, 1.$$

Since the double crossed product $(\mathcal{O}_{\mathfrak{L}} \rtimes_{\alpha_{\mathfrak{L}}} \mathbb{T}) \times_{\widehat{\alpha_{\mathfrak{L}}}} \mathbb{Z}$ is stably isomorphic to $\mathcal{O}_{\mathfrak{L}}$, the following proposition is immediate from Lemma 5.2 (cf.[RS],[Bl;p.287]).

Proposition 5.6. *The C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ is nuclear and satisfies the Universal Coefficient Theorem in the sense of Rosenberg and Schochet [RS] (also [Bro2]).*

Hence, for an aperiodic λ-graph system $\mathfrak{L}$, $\mathcal{O}_{\mathfrak{L}}$ is a unital, separable, nuclear, purely infinite, simple C^*-algebra satisfying the UCT so that it lives in a classifiable class of nuclear C^*-algebras by Kirchberg [Kir] and Phillips [Ph]. As the K-groups $K_0(\mathcal{O}_{\mathfrak{L}}), K_1(\mathcal{O}_{\mathfrak{L}})$ are countable abelian groups with $K_1(\mathcal{O}_{\mathfrak{L}})$ torsion free by Theorem 5.5, Rørdam's result [Rø;Proposition 6.7] says the following:

Corollary 5.7. *For an aperiodic λ-graph system $\mathfrak{L}$, the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ is isomorphic to the C^*-algebra of an inductive limit of a sequence $B_1 \to B_2 \to B_3 \to \cdots$ of simple Cuntz-Krieger algebras.*

Set the Ext-groups

$$\mathrm{Ext}^1(\mathcal{O}_{\mathfrak{L}}) = \mathrm{Ext}(\mathcal{O}_{\mathfrak{L}}), \qquad \mathrm{Ext}^0(\mathcal{O}_{\mathfrak{L}}) = \mathrm{Ext}(\mathcal{O}_{\mathfrak{L}} \otimes C_0(\mathbb{R})).$$

As the UCT holds for our algebras as in the lemma below, it is now easy to compute the Ext-groups by using Theorem 5.5.

Lemma 5.8([RS],[Bro2]). *There exist short exact sequences*

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(K_0(\mathcal{O}_{\mathfrak{L}}), \mathbb{Z}) \longrightarrow \mathrm{Ext}^1(\mathcal{O}_{\mathfrak{L}}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(K_1(\mathcal{O}_{\mathfrak{L}}), \mathbb{Z}) \longrightarrow 0,$$
$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(K_1(\mathcal{O}_{\mathfrak{L}}), \mathbb{Z}) \longrightarrow \mathrm{Ext}^0(\mathcal{O}_{\mathfrak{L}}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(K_0(\mathcal{O}_{\mathfrak{L}}), \mathbb{Z}) \longrightarrow 0$$

that split unnaturally.

We denote by $\mathbb{Z}_I$ the abelian group defined by the projective limit $\varprojlim_l \{I_{l,l+1} : \mathbb{Z}^{m(l+1)} \to \mathbb{Z}^{m(l)}\}$. The sequence $A_{l,l+1}, l \in \mathbb{Z}_+$ naturally acts on $\mathbb{Z}_I$ as an endomorphism that we denote by A. The identity on $\mathbb{Z}_I$ is denoted by I. Then the cokernel and the kernel of the endomorphism $I - A$ on $\mathbb{Z}_I$ are the Bowen-Franks groups $BF^0(A, I)$ and $BF^1(A, I)$ for (A, I) respectively ([Ma5]). By [Ma5;Theorem 9.6], there exists a short exact sequence

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(K_0(A, I), \mathbb{Z}) \longrightarrow BF^0(A, I) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(K_1(A, I), \mathbb{Z}) \longrightarrow 0$$

that splits unnaturally. And also

$$BF^1(A, I) \cong \mathrm{Hom}_{\mathbb{Z}}(K_0(A, I), \mathbb{Z}).$$

As in the proof of [Ma5;Lemma 9.7], we see that $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_{I^t}, \mathbb{Z}) = 0$ so that $\mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{Ker}(\mathrm{id} - \lambda_{(A,I)}) \text{ in } \mathbb{Z}_{I^t}, \mathbb{Z}) = 0$. This means that $\mathrm{Ext}^1_{\mathbb{Z}}(K_1(A, I)), \mathbb{Z}) = 0$. Theorem 5.5 says that $K_i(A, I) \cong K_i(\mathcal{O}_{\mathfrak{L}})$ so that we conclude by Lemma 5.8,

Theorem 5.9.

(i) $\mathrm{Ext}(\mathcal{O}_{\mathfrak{L}}) = \mathrm{Ext}^1(\mathcal{O}_{\mathfrak{L}}) \cong BF^0(A, I) = \mathbb{Z}_I/(I - A)\mathbb{Z}_I$,
(ii) $\mathrm{Ext}^0(\mathcal{O}_{\mathfrak{L}}) \cong BF^1(A, I) = \mathrm{Ker}(I - A)$ *in* $\mathbb{Z}_I$.

Theorem 5.9 is a generalization of [CK;5.3 Theorem] and [Ma6].

6. Realizations as endomorphism crossed products and Hilbert C^*-bimodule algebras

Following Deaconu's discussions in [De2],[De3],[De4], we will realize the algebra $\mathcal{O}_{\mathfrak{L}}$ as an endomorphism crossed product $\mathcal{F}_{\mathfrak{L}} \times_{\beta_{\mathfrak{L}}} \mathbb{N}$. Recall that the algebra $\mathcal{F}_{\mathfrak{L}}$ is isomorphic to the C^*-algebra $C^*(F_{\mathfrak{L}})$ of the groupoid $F_{\mathfrak{L}}$. The groupoid $F_{\mathfrak{L}}$ is written as

$$\{(x,y) \in X_{\mathfrak{L}} \times X_{\mathfrak{L}} \mid \sigma^k(x) = \sigma^k(y) \text{ for some } k \in \mathbb{Z}_+\}.$$

Put

$$\beta_{\mathfrak{L}}(f)(x,y) = \frac{1}{\sqrt{p(\sigma(x))p(\sigma(y))}} f(\sigma(x),\sigma(y)), \quad f \in C_c(F_{\mathfrak{L}}), x, y \in X_{\mathfrak{L}}$$

where $p(x)$ is the number of the paths z such that $\sigma(z) = x$, and for $(x,n,y) \in G_{\mathfrak{L}}$

$$v(x,n,y) = \begin{cases} \frac{1}{\sqrt{p(\sigma(x))}}, & \text{if } n = 1 \text{ and } y = \sigma(x), \\ 0 & \text{otherwise.} \end{cases}$$

Regarding $C^*(F_{\mathfrak{L}})$ as a subalgebra of $C^*(G_{\mathfrak{L}})$, one sees that v is a nonunitary isometry satisfying $\beta_{\mathfrak{L}}(f) = vfv^*$ ([De2],[De3],[De4]). Then $\beta_{\mathfrak{L}}$ is a proper corner endomorphism of $C^*(F_{\mathfrak{L}})$ such that $C^*(G_{\mathfrak{L}})$ is isomorphic to the crossed product $C^*(F_{\mathfrak{L}}) \times_{\beta_{\mathfrak{L}}} \mathbb{N}$ (cf.[Rø2]). We will write the isometry v in terms of the generators S_α, E_i^l. For $l \in \mathbb{N}, i = 1,\dots,m(l)$, we denote by n_i^l the number of the edges e in $E_{l-1,l}$ such that $t(e) = \mathfrak{v}_i^l$. As $\mathfrak{L}$ is left-resolving, it is the number of the symbols $\alpha \in \Sigma$ such that $S_\alpha^* S_\alpha E_i^l \neq 0$. It follows that $n_i^l E_i^l = \sum_{\alpha\in\Sigma} S_\alpha^* S_\alpha E_i^l$. Note that if $I_{l,l+1}(i,j) = 1$, then $n_i^l = n_j^{l+1}$. Then one obtains

$$v = \sum_{i=1}^{m(l)} \frac{1}{\sqrt{n_i^l}} \sum_{\alpha\in\Sigma} S_\alpha E_i^l \tag{6.1}$$

where the right-hand side does not depend on the choice of $l \in \mathbb{N}$. We can immediately see that $\mathcal{O}_{\mathfrak{L}}$ is generated by the C^*-algebra $\mathcal{F}_{\mathfrak{L}}$ and the above isometry v, that satisfies

$$v^*v = 1, \quad v\mathcal{F}_{\mathfrak{L}}v^* \subset \mathcal{F}_{\mathfrak{L}}, \quad v^*\mathcal{F}_{\mathfrak{L}}v \subset \mathcal{F}_{\mathfrak{L}}. \tag{6.2}$$

The universality (Theorem 3.6) of the algebra $\mathcal{O}_{\mathfrak{L}}$ corresponds to the universality of the crossed product $C^*(F_{\mathfrak{L}}) \times_{\beta_{\mathfrak{L}}} \mathbb{N}$. It needs however a slightly complicated argument to directly determine the operator relations ($\mathfrak{L}$) by using (6.1) and (6.2), as it is possible. There are some merits to realize $\mathcal{O}_{\mathfrak{L}}$ as $\mathcal{F}_{\mathfrak{L}} \times_{\beta_{\mathfrak{L}}} \mathbb{N}$. One is the fact that its purely infiniteness is immediately deduced from Rørdam's result [Rø2;Theorem 3.1] under the condition that $\mathcal{F}_{\mathfrak{L}}$ is simple. The other one is K-theory formulae. Rørdam also in [Rø2; Corollary 2.2] showed that

(i) $K_0(\mathcal{F}_{\mathfrak{L}} \times_{\beta_{\mathfrak{L}}} \mathbb{N}) \cong K_0(\mathcal{F}_{\mathfrak{L}})/(\mathrm{id} - \beta_{\mathfrak{L}*})K_0(\mathcal{F}_{\mathfrak{L}})$.
(ii) $K_1(\mathcal{F}_{\mathfrak{L}} \times_{\beta_{\mathfrak{L}}} \mathbb{N}) \cong \mathrm{Ker}(\mathrm{id} - \beta_{\mathfrak{L}*})$ on $K_0(\mathcal{F}_{\mathfrak{L}})$

(cf.Paschke [Pa], Deaconu [De3]). These are precisely the formulae of Lemma 5.4 with Lemma 5.2.

In [De3; Section 3], Deaconu showed that the groupoid C^*-algebras of continuous graphs are realized as C^*-algebras constructed from Hilbert C^*-bimodules defined in [Pi] (see also [Kat]). A special case of continuous graphs was studied in Kajiwara-Watatani [KW]. We identify the algebra $C(\Omega_{\mathfrak{L}})$ of all continuous functions on $\Omega_{\mathfrak{L}}$ with the commutative C^*-algebra $\mathcal{A}_{\mathfrak{L}}$. Let $X_{\mathcal{A}_{\mathfrak{L}}}$ be the set $C(E_{\mathfrak{L}})$ of all continuous functions on $E_{\mathfrak{L}}$, that is identified with $\sum_{\alpha\in\Sigma}^{\oplus}\mathbb{C}S_\alpha\mathcal{A}_{\mathfrak{L}}$, because $\mathfrak{L}$ is left-resolving. We endow $X_{\mathcal{A}_{\mathfrak{L}}}$ with a Hilbert C^*-bimodule structure over $\mathcal{A}_{\mathfrak{L}}$ defined by

$$(S_\alpha a)\cdot b = S_\alpha \cdot ab, \qquad < S_\alpha a, S_\beta b >_{\mathcal{A}_{\mathfrak{L}}} = a^* S_\alpha^* S_\beta b,$$
$$\phi_{\mathfrak{L}}(b) S_\alpha a = b S_\alpha a = S_\alpha \cdot S_\alpha^* b S_\alpha a$$

for $a, b \in \mathcal{A}_{\mathfrak{L}}, \alpha, \beta \in \Sigma$. A special case of this construction of the Hilbert C^*-bimodules is seen in the proof of [PWY;Theorem 4.2] for the C^*-algebras associated with subshifts. The above mentioned Deaconu's result says the following proposition:

Proposition 6.1. *The C^*-algebra constructed from the Hilbert C^*-bimodule $(\phi_{\mathfrak{L}}, X_{\mathcal{A}_{\mathfrak{L}}})$ over $\mathcal{A}_{\mathfrak{L}}$ is isomorphic to the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$.*

7. Examples

In this section, we give two kinds of examples of λ-graph systems and study their associated C^*-algebras. The first ones appear as presentations of sofic shifts. The second one is defined by a Shannon graph with countable infinite vertices.

Presentations of sofic shifts come from labeled graphs with finite vertices that are called λ-graphs (cf. [Fi],[Kr4],[Kr5],[LM],[We], ...). Let $G = (V, E)$ be a finite directed graph with finite vertex set V and finite edge set E. Let $\mathcal{G} = (G, \lambda)$ be a labeled graph over Σ defined by G and a labeling map $\lambda : E \to \Sigma$. Suppose that it is left-resolving and predecessor-separated. Let A_G be the adjacency matrix of G, that is defined by

$$A_G(e,f) = \begin{cases} 1 & \text{if } t(e) = s(e), \\ 0 & \text{otherwise} \end{cases}$$

for $e, f \in E$. The matrix A_G defines a shift of finite type by regarding its edges as its alphabet. Since the matrix A_G is of entries in $\{0,1\}$, we have the Cuntz-Krieger algebra $\mathcal{O}_{A_G}$ defined by A_G ([CK] cf.[KPPR],[Rø]). By putting $V_l^{\mathcal{G}} = V$, $E_{l,l+1}^{\mathcal{G}} = E$ for $l \in \mathbb{Z}_+$, and $\lambda^{\mathcal{G}} = \lambda, \iota^{\mathcal{G}} = \mathrm{id}$, we have a λ-graph system $\mathfrak{L}_{\mathcal{G}} = (V^{\mathcal{G}}, E^{\mathcal{G}}, \lambda^{\mathcal{G}}, \iota^{\mathcal{G}})$. Then we have

Proposition 7.1. *The C^*-algebra $\mathcal{O}_{\mathfrak{L}_{\mathcal{G}}}$ is isomorphic to the Cuntz-Krieger algebra $\mathcal{O}_{A_G}$.*

Proof. Let $V = \{v_1, \dots, v_m\}$ be the vertex set of G. Let $S_\alpha, \alpha \in \Sigma$ be the canonical generating partial isometries of $\mathcal{O}_{\mathfrak{L}_{\mathcal{G}}}$. We denote by $E_1, E_2, \dots, E_m$ the set of all minimal projections of $\mathcal{A}_l = \mathcal{A}_{\mathfrak{L}_{\mathcal{G}}}, l \in \mathbb{N}$ corresponding to the vertices $v_1, \dots, v_m$. As the labeled graph $\mathcal{G}$ is predecessor-separated, they are written in terms of $S_\alpha, \alpha \in \Sigma$ as in (1.7). Note that in the algebra $\mathcal{O}_{\mathfrak{L}_{\mathcal{G}}}$, $S_\alpha E_i \neq 0$ if and only if there exists an edge $e \in E$ satisfying $\lambda(e) = \alpha$ and $t(e) = v_i$. As $\mathcal{G}$ is left-resolving, the correspondence

$$e \in E \longleftrightarrow (\lambda(e), t(e)) \in \{(\alpha, v_i) \in \Sigma \times V \mid S_\alpha E_i \neq 0\}$$

is bijective. For $e \in E$, put $s_e = S_{\lambda(e)} E_{t(e)} \in \mathcal{O}_{\mathfrak{L}_{\mathcal{G}}}$, where E_{v_i} denotes E_i. As $E_{t(e)} = s_e^* s_e$ for $e \in E$ and $S_\alpha = \sum_{\substack{e \in E, \\ \lambda(e) = \alpha}} s_e$ for $\alpha \in \Sigma$, the algebra $\mathcal{O}_{\mathfrak{L}_{\mathcal{G}}}$ is generated by the partial isometries $s_e, e \in E$. It is immediate to see that the following relations hold:

$$\sum_{e \in E} s_e s_e^* = 1, \qquad s_e^* s_e = \sum_{f \in E} A_G(e,f) s_f s_f^*.$$

This means that the C^*-algebra generated by $s_e, e \in E$ is the Cuntz-Krieger algebra $\mathcal{O}_{A_G}$ defined by the matrix A_G. □

If, in particular, a labeled graph $\mathcal{G} = (G, \lambda)$ has different labels for different edges, it defines a shift of finite type. In this case, one may identify the edge set E with the alphabet Σ. Let $\mathfrak{L}_G$ be the λ-graph system $\mathfrak{L}_{\mathcal{G}}$ as in the above one. Let $S_\alpha, \alpha \in \Sigma$ be the generating partial isometries of $\mathcal{O}_{\mathfrak{L}_G}$. It is obvious that the relations (1.4),(1.5) and (1.6) give rise to the following relations:

$$S_\alpha^* S_\alpha = \sum_{\beta \in E} A_G(\alpha, \beta) S_\beta S_\beta^*, \qquad \alpha \in \Sigma.$$

Remark. While completing this paper, Toke M. Carlsen let the author know his preprint [Ca], where he shows that the C^*-algebra associated with sofic shifts are isomorphic to the Cuntz-Krieger algebras of their left Krieger cover graphs. His result is a special case of the above proposition.

We will next present a λ-graph system for which the associated C^*-algebra is not stably isomorphic to any Cuntz-Krieger algebra and any Cuntz algebra. There is a method introduced in [KM] to construct λ-graph systems from Shannon graphs. By a Shannon graph we mean here a left-resolving labeled directed graph with countable vertices and finite labels.
Let us consider a Shannon graph defined as follows: Let $V = \{v_1, v_2, \dots, \}$ be its countable infinite vertex set. Its alphabet Σ consist of the five symbols $\{\alpha, \beta, \gamma, \delta, \epsilon\}$. The edges labeled α are from v_{n+1} to v_n for $n = 1, 2, \dots$. The

edges labeled β are from v_1 to v_2 and from v_{2n} to v_{2n+2} for $n = 1, 2, \ldots$. The edges labeled γ are self-loops at v_n for $n = 2, 3, \ldots$. The edge labeled δ is a self-loop at v_1. The edges labeled ϵ are from v_1 to v_n for $n = 1, 2, \ldots$. The resulting labeled graph is left-resolving and hence it is a Shannon graph. We denote it by $\mathcal{S}$. We will construct a λ-graph system $\mathfrak{L}(\mathcal{S})$ from the Shannon graph $\mathcal{S}$ by a method introduced in [KM] as in the following way. For a vertex $v \in V$ and for $l \in \mathbb{N}$, let $\Gamma_l^-(v)$ be the set of all label sequences of length l terminating at v. Define an equivalence relation $v \approx_{(l)} v'$ for vertices $v, v' \in V$ by $\Gamma_l^-(v) = \Gamma_l^-(v')$. For $l = 0$, define $v \approx_{(l)} v'$ for all $v, v' \in V$. The vertex set V_l is then defined by the set of $\approx_{(l)}$-equivalence classes of V. We denote by $V_l = \{\mathfrak{v}_1^l, \ldots, \mathfrak{v}_{m(l)}^l\}$. The vertices $\mathfrak{v}_i^l, i = 1, \ldots, m(l)$ of V_l may be identified with $\{\Gamma_l^-(v) : v \in V\}$. We define a map $\iota_{l,l+1} : V_{l+1} \to V_l$ by $\iota_{l,l+1}(\mathfrak{v}_j^{l+1}) = \mathfrak{v}_i^l$ if $\mathfrak{v}_j^{l+1} \subset \mathfrak{v}_i^l$. We define an edge labeled $\omega \in \Sigma$ from $\mathfrak{v}_i^l$ to $\mathfrak{v}_j^{l+1}$ if there exists an edge labeled ω in $\mathcal{S}$ from a vertex in $\mathfrak{v}_i^l$ to a vertex in $\mathfrak{v}_j^{l+1}$. Then the resulting labeled graph with vertex sets V_l, labeled edges from V_l to V_{l+1} and surjective maps $\iota_{l,l+1} : V_{l+1} \to V_l$ for $l \in \mathbb{Z}_+$ defines a λ-graph system over Σ ([KM]). We denote it by $\mathfrak{L}(\mathcal{S})$.

The vertex sets $V_l, l \in \mathbb{Z}_+$ are written as in the following way:

$$
\begin{aligned}
&V_0 : \mathfrak{v}_1^0 = \{v_n \mid n = 1, 2, \ldots\}.\\
&V_1 : \mathfrak{v}_1^1 = \{v_1\}, \mathfrak{v}_2^1 = \{v_{2n} \mid n = 1, 2, \ldots\}, \mathfrak{v}_3^1 = \{v_{2n+1} \mid n = 1, 2, \ldots\}.\\
&V_2 : \mathfrak{v}_1^2 = \{v_1\}, \mathfrak{v}_2^2 = \{v_2\}, \mathfrak{v}_3^2 = \{v_{2n} \mid n = 2, 3, \ldots\},\\
&\quad \mathfrak{v}_4^2 = \{v_{2n+1} \mid n = 1, 2, \ldots\}.\\
&V_3 : \mathfrak{v}_1^3 = \{v_1\}, \mathfrak{v}_2^3 = \{v_2\}, \mathfrak{v}_3^3 = \{v_4\}, \mathfrak{v}_4^3 = \{v_{2n} \mid n = 3, 4, \ldots\},\\
&\quad \mathfrak{v}_5^3 = \{v_{2n+1} \mid n = 1, 2, \ldots\}.\\
&V_4 : \mathfrak{v}_1^4 = \{v_1\}, \mathfrak{v}_2^4 = \{v_2\}, \mathfrak{v}_3^4 = \{v_4\}, \mathfrak{v}_4^4 = \{v_6\}, \mathfrak{v}_5^4 = \{v_{2n} \mid n = 4, 5, \ldots\},\\
&\quad \mathfrak{v}_6^4 = \{v_3\}, \mathfrak{v}_7^4 = \{v_{2n+1} \mid n = 2, 3, \ldots\}.\\
&V_5 : \mathfrak{v}_1^5 = \{v_1\}, \mathfrak{v}_2^5 = \{v_2\}, \mathfrak{v}_3^5 = \{v_4\}, \mathfrak{v}_4^5 = \{v_6\}, \mathfrak{v}_5^5 = \{v_8\},\\
&\quad \mathfrak{v}_6^5 = \{v_{2n} \mid n = 5, 6, \ldots\}, \mathfrak{v}_7^5 = \{v_3\}, \mathfrak{v}_8^5 = \{v_5\},\\
&\quad \mathfrak{v}_9^5 = \{v_{2n+1} \mid n = 3, 4, \ldots\}.\\
&V_6 : \mathfrak{v}_1^6 = \{v_1\}, \mathfrak{v}_2^6 = \{v_2\}, \mathfrak{v}_3^6 = \{v_4\}, \mathfrak{v}_4^6 = \{v_6\}, \mathfrak{v}_5^6 = \{v_8\}, \mathfrak{v}_6^6 = \{v_{10}\},\\
&\quad \mathfrak{v}_7^6 = \{v_{2n} \mid n = 6, 7, \ldots\}, \mathfrak{v}_8^6 = \{v_3\}, \mathfrak{v}_9^6 = \{v_5\}, \mathfrak{v}_{10}^6 = \{v_7\},\\
&\quad \mathfrak{v}_{11}^6 = \{v_{2n+1} \mid n = 4, 5, \ldots\}.\\
&V_7 : \mathfrak{v}_1^7 = \{v_1\}, \mathfrak{v}_2^7 = \{v_2\}, \mathfrak{v}_3^7 = \{v_4\}, \mathfrak{v}_4^7 = \{v_6\}, \mathfrak{v}_5^7 = \{v_8\}, \mathfrak{v}_6^7 = \{v_{10}\},\\
&\quad \mathfrak{v}_7^7 = \{v_{12}\}, \mathfrak{v}_8^7 = \{v_{2n} \mid n = 7, 8, \ldots\}, \mathfrak{v}_9^7 = \{v_3\}, \mathfrak{v}_{10}^7 = \{v_5\}, \mathfrak{v}_{11}^7 = \{v_7\},\\
&\quad \mathfrak{v}_{12}^7 = \{v_9\}, \mathfrak{v}_{13}^7 = \{v_{2n+1} \mid n = 5, 6, \ldots\}.\\
&\quad \cdots\cdots\cdots.
\end{aligned}
$$

Lemma 7.2. *The λ-graph system $\mathfrak{L}(\mathcal{S})$ is aperiodic.*

Proof. Take and fix an arbitrary vertex $\mathfrak{v}_i^l \in V_l$, let k be the minimum number such that $v_k \in \mathfrak{v}_i^l$. There exists a path of length k in $\mathfrak{L}(\mathcal{S})$ that starts at $\mathfrak{v}_i^l$ and ends at $\mathfrak{v}_1^{l+k} = \{v_1\} \in V_{l+k}$ whose label is $(\alpha, \alpha, \dots, \alpha)$. There exist edges from $\mathfrak{v}_1^{l+k}$ to $\mathfrak{v}_j^{l+k+1}$ for all $j = 1, 2, \dots, m(l+1)$ whose labels are ϵ. This means that $\mathfrak{L}(\mathcal{S})$ is aperiodic. $\square$

Therefore the associated C^*-algebra $\mathcal{O}_{\mathfrak{L}(\mathcal{S})}$ is simple and purely infinite.

Let us compute its K-groups. We first present the matrices $A_{l,l+1}^t$ and $I_{l,l+1}^t$ for all $l \geq 5$. As in the preceding table, we see that $m(l) = 2l - 1$ for $l \geq 5$. We write the matrices along the ordered basis $\mathfrak{v}_1^l, \mathfrak{v}_2^l, \dots, \mathfrak{v}_{m(l)}^l$, where blanks denote zeros. For $l = 5$, we have

$$A_{5,6}^t = \begin{bmatrix} 2 & 1 & & & & & & & \\ 2 & 1 & 1 & & & & & & \\ 1 & & 1 & 1 & & & & & \\ 1 & 1 & & 1 & 1 & & & & \\ 1 & & & & 1 & 1 & & & \\ 1 & & & 1 & & 1 & & 1 & \\ 1 & & & & & & 1 & 1 & \\ 1 & & & & & 1 & 1 & 1 & \\ 1 & & & & & & 1 & 1 & 1 \\ 1 & & & & & & & 1 & 1 \\ 1 & & & & & & & 1 & 2 \end{bmatrix}, \quad I_{5,6}^t = \begin{bmatrix} 1 & & & & & & & & \\ & 1 & & & & & & & \\ & & 1 & & & & & & \\ & & & 1 & & & & & \\ & & & & 1 & & & & \\ & & & & & 1 & & & \\ & & & & & & & 1 & \\ & & & & & & 1 & & \\ & & & & & & & & 1 \\ & & & & & & & 1 & \\ & & & & & & & & 1 \end{bmatrix},$$

so that

$$A_{5,6}^t - I_{5,6}^t = \begin{bmatrix} 1 & 1 & & & & & & & \\ 2 & & 1 & & & & & & \\ 1 & & & 1 & & & & & \\ 1 & 1 & & & 1 & & & & \\ 1 & & & & & 1 & & & \\ 1 & & & 1 & & & & 1 & \\ 1 & & & & & & 1 & & \\ 1 & & & & & 1 & & 1 & \\ 1 & & & & & & 1 & 1 & \\ 1 & & & & & & & & 1 \\ 1 & & & & & & & 1 & 1 \end{bmatrix}.$$

For general $l \geq 5$, we see

$$A^t_{l,l+1} = \begin{bmatrix}
2 & 1 & & & & & & & & & & & & & \\
2 & 1 & 1 & & & & & & & & & & & & \\
1 & & 1 & 1 & & & & & & & & & & & \\
1 & 1 & & 1 & 1 & & & & & & & & & & \\
1 & & & & 1 & 1 & & & & & & & & & \\
1 & & & 1 & & 1 & 1 & & & & & & & & \\
1 & & & & & & 1 & 1 & & & & & & & \\
1 & & & & & 1 & & 1 & 1 & & & & & & \\
\vdots & & & & & & & & \ddots & \ddots & & & & & \\
\vdots & & & & & & & \ddots & & \ddots & \ddots & & & & \\
1 & & & & & & & & & & 1 & 1 & & & \\
1 & & & & & & & & & 1 & & 1 & & 1 & \\
1 & & & & & & & & & & & & 1 & 1 & \\
1 & & & & & & & & & & & 1 & 1 & 1 & \\
1 & & & & & & & & & & & & 1 & 1 & 1 \\
1 & & & & & & & & & & & & & 1 & 1 \\
1 & & & & & & & & & & & & & 1 & 2
\end{bmatrix},$$

$$I^t_{l,l+1} = \begin{bmatrix}
1 & & & & & & & & & & & & & & \\
& 1 & & & & & & & & & & & & & \\
& & 1 & & & & & & & & & & & & \\
& & & 1 & & & & & & & & & & & \\
& & & & 1 & & & & & & & & & & \\
& & & & & 1 & & & & & & & & & \\
& & & & & & 1 & & & & & & & & \\
& & & & & & & 1 & & & & & & & \\
& & & & & & & & \ddots & & & & & & \\
& & & & & & & & & \ddots & & & & & \\
& & & & & & & & & & 1 & & & & \\
& & & & & & & & & & & 1 & & & \\
& & & & & & & & & & & & & 1 & \\
& & & & & & & & & & & & 1 & & \\
& & & & & & & & & & & & & & 1 \\
& & & & & & & & & & & & & 1 & \\
& & & & & & & & & & & & & & 1
\end{bmatrix},$$

and hence

$$A^t_{l,l+1} - I^t_{l,l+1} = \begin{bmatrix}
1 & 1 & & & & & & & & & & & & & \\
2 & & 1 & & & & & & & & & & & & \\
1 & & & 1 & & & & & & & & & & & \\
1 & 1 & & & 1 & & & & & & & & & & \\
1 & & & & & 1 & & & & & & & & & \\
1 & & & 1 & & & 1 & & & & & & & & \\
1 & & & & & & & 1 & & & & & & & \\
1 & & & & & 1 & & & 1 & & & & & & \\
\vdots & & & & & & & & & \ddots & & & & & \\
\vdots & & & & & & & \ddots & & & \ddots & & & & \\
1 & & & & & & & & & & & 1 & & & \\
1 & & & & & & & & & 1 & & & & 1 & \\
1 & & & & & & & & & & & & 1 & & \\
1 & & & & & & & & & & & 1 & & 1 & \\
1 & & & & & & & & & & & & 1 & 1 & \\
1 & & & & & & & & & & & & & & 1 \\
1 & & & & & & & & & & & & & 1 & 1
\end{bmatrix}.$$

Lemma 7.3. *For $l \geq 4$ and $z = \begin{bmatrix} z_1 \\ \vdots \\ z_{2l+1} \end{bmatrix} \in \mathbb{Z}^{2l+1}$, put*

$$\begin{aligned}
\varphi_l([z_i]_{i=1}^{2l+1}) &= z_{2l-7} - z_{2l-5} - z_{2l-4} + z_{2l-2}, \\
\psi_l([z_i]_{i=1}^{2l+1}) &= z_{2l-7} - z_{2l-4} - z_{2l-3} + z_{2l-1}, \\
\xi_l([z_i]_{i=1}^{2l+1}) &= z_{2l-7} - z_{2l-4} - z_{2l} + z_{2l+1}.
\end{aligned}$$

Then there exists $x = \begin{bmatrix} x_1 \\ \vdots \\ x_{2l-1} \end{bmatrix} \in \mathbb{Z}^{2l-1}$ such that

$$\begin{bmatrix} z_1 \\ \vdots \\ z_{2l-1} \\ z_{2l} \\ z_{2l+1} \end{bmatrix} = (A^t_{l,l+1} - I^t_{l,l+1}) \begin{bmatrix} x_1 \\ \vdots \\ x_{2l-1} \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \varphi_l(z) \\ \psi_l(z) \\ 0 \\ \xi_l(z) \end{bmatrix}.$$

Lemma 7.4. *The map $z = [z_i]_{i=1}^{2l+1} \in \mathbb{Z}^{2l+1} \to (\varphi_l(z), \psi_l(z), \xi_l(z)) \in \mathbb{Z}^3$ induces an isomorphism from $\mathbb{Z}^{2l+1}/(A^t_{l,l+1} - I^t_{l,l+1})\mathbb{Z}^{2l-1}$ onto $\mathbb{Z}^3$.*

Proof. It suffices to show the surjectivity of the induced map

$$z = [z_i]_{i=1}^{2l+1} \in \mathbb{Z}^{2l+1}/(A^t_{l,l+1} - I^t_{l,l+1})\mathbb{Z}^{2l-1} \longrightarrow (\varphi_l(z), \psi_l(z), \xi_l(z)) \in \mathbb{Z}^3.$$

For $(m, n, k) \in \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$, put $z = [z_i]_{i=1}^{2l+1}$ where

$$z_i = \begin{cases} 0 & \text{for } i = 1, 2, \dots, 2l-3, 2l, \\ m & \text{for } i = 2l-2, \\ n & \text{for } i = 2l-1, \\ k & \text{for } i = 2l+1. \end{cases}$$

Then we see that $\varphi_l(z) = m, \psi_l(z) = n, \xi_l(z) = k$. □

We denote by ρ_{l+1} the above isomorphism from $\mathbb{Z}^{2l+1}/(A^t_{l,l+1} - I^t_{l,l+1})\mathbb{Z}^{2l-1}$ onto $\mathbb{Z}^3$. Let L be the matrix $\begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix}$. Since the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{Z}^{2l-1}/(A^t_{l-1,l} - I^t_{l-1,l})\mathbb{Z}^{2l-3} & \xrightarrow{I^t_{l,l+1}} & \mathbb{Z}^{2l+1}/(A^t_{l,l+1} - I^t_{l,l+1})\mathbb{Z}^{2l-1} \\ \rho_l \downarrow & & \rho_{l+1} \downarrow \\ \mathbb{Z}^3 & \xrightarrow{L} & \mathbb{Z}^3, \end{array}$$

we obtain

PROPOSITION 7.5. $K_0(\mathcal{O}_{\mathfrak{L}(\mathcal{S})}) \cong 0$.

Proof. As $L^3 = 0$, by Theorem 5.5, it follows that

$$K_0(\mathcal{O}_{\mathfrak{L}(\mathcal{S})}) = \varinjlim\{\mathbb{Z}^{2l+1}/(A^t_{l,l+1} - I^t_{l,l+1})\mathbb{Z}^{2l-1}, \bar{I}^t_{l+1,l+2}\} = \varinjlim\{\mathbb{Z}^3, L\} \cong 0.$$

□

Concerning the group $K_1(\mathcal{O}_{\mathfrak{L}(\mathcal{S})})$, one sees

PROPOSITION 7.6. $K_1(\mathcal{O}_{\mathfrak{L}(\mathcal{S})}) \cong \mathbb{Z}$.

Proof. For $l \geq 5$, put $x(l) = [x_i]_{i=1}^{2l-1} \in \mathbb{Z}^{2l-1}$ where

$$x_1 = 1, \quad x_2 = -1, \quad x_3 = -2,$$
$$x_i = -1 \quad \text{for} \quad i = 4, 6, 8, \dots, 2l-4, 2l-3, 2l-1,$$
$$x_i = 0 \quad \text{for} \quad i = 5, 7, 9, \dots, 2l-5, 2l-2.$$

It is easy to see that

$$\operatorname{Ker}(A^t_{l,l+1} - I^t_{l,l+1}) = \mathbb{Z}x(l), \qquad I^t_{l,l+1}x(l) = x(l+1).$$

Hence we obtain $K_1(\mathcal{O}_{\mathfrak{L}(\mathcal{S})}) \cong \mathbb{Z}$ by Theorem 5.5. □

Therefore we conclude

Theorem 7.7. *The C^*-algebra $\mathcal{O}_{\mathfrak{L}(\mathcal{S})}$ is unital, simple, purely infinite, nuclear and generated by five partial isometries with mutually orthogonal ranges. Its K-groups are*

$$K_0(\mathcal{O}_{\mathfrak{L}(\mathcal{S})}) \cong 0, \qquad K_1(\mathcal{O}_{\mathfrak{L}(\mathcal{S})}) \cong \mathbb{Z}.$$

As the K_1-group of a Cuntz-Krieger algebra is the torsion-free part of its K_0-group, the algebra $\mathcal{O}_{\mathfrak{L}(\mathcal{S})}$ lives outside the Cuntz-Krieger algebras (cf.[Ma3]).

Remark. M. Tomforde in [T] considered C^*-algebras associated to labeled graphs as a generalization of Cuntz-Krieger algebras (cf.[T2]). He deals with labeled directed graphs with (generally) infinite vertices. If the labeled graphs have finite vertices, the resulting graphs are ones in the first examples of this section. In this case, his C^*-algebras coincide with our C^*-algebras. The referee informed to the author that his algebras in general are not ours of λ-graph systems.

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