Tight Embeddings of Simply Connected 4-Manifolds

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Dedicated to Thomas F. Banchoff on the occasion of his 65th birthday[1]

Abstract. The classification of compact and simply connected PL 4-manifolds states that the homeomorphism classes coincide with the homotopy classes, and that these are classified by the intersection form. We show here that "most" of these classes with an indefinite intersection form can be represented by a tight polyhedral embedding into some Euclidean space. It remains open which of the PL structures can be realized in such a way.

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Introduction and Result

An embedding $M \to \mathbb{E}^N$ of a compact manifold into Euclidean space is called *tight*, if for any open half space $E_+ \subset \mathbb{E}^N$ the induced homomorphism

$$H_*(M \cap E_+) \longrightarrow H_*(M)$$

is injective where H_* denotes an appropriate homology theory with coefficients in a certain field. In the smooth case (and, with certain modifications, also in the polyhedral case) this is equivalent to the condition that almost all height functions on M are perfect functions, i.e., have the minimum number of critical points which coincides with the sum of the Betti numbers. For a survey on tightness see [14] or [3].

[1]The main result of this paper was first presented at the TFB conference on this occasion at Providence, RI, Nov. 1, 2003.

For compact 2-manifolds without boundary this is equivalent to the Two-piece property (TPP) which states that the intersection of M with any (open or closed) halfspace is connected. Smooth tight surfaces were investigated by N.H.Kuiper [13] and others, the study of tight polyhedral surfaces was initiated by T.F.Banchoff [1]. One of the results is that any given closed surface admits a tight polyhedral embedding into some Euclidean space. For obtaining this, it is sufficient to start with the three cases of the sphere, the real projective plane and the Klein bottle [2] and then to attach handles tightly. For tight polyhedral immersions into 3-space the situation is the following: Any given closed surface (except for the real projective plane and the Klein bottle) admits a tight polyhedral immersion into 3-space. The crucial and most difficult case $\chi = -1$ had been open for many years and was solved only recently by D.Cervone [5]. Smooth tight immersions into 3-space exist for all surfaces except for the real projective plane, the Klein bottle and the surface with $\chi = -1$. The latter is again the most crucial case and was solved by F.Haab [7]. There is a smooth tight embedding $\mathbb{R}P^2 \to \mathbb{E}^4$ as a suitable linear projection of the Veronese surface. The cases of the Klein bottle and $\chi = -1$ seem still to be open. One approach might be to attach a handle tightly to the Veronese surface in 4-space (or a slightly distorted version of it) but that has so far turned out to be unmanageable. In 5-space the only smooth tight surface is the classical Veronese surface itself by a theorem of N.H.Kuiper.

In the case of compact 3-manifolds not too much seems to be known at all. Smooth tight examples include the Veronese embedding $\mathbb{R}P^3 \to \mathbb{E}^9$, connected sums of handles $S^1 \times S^2$ and cartesian products of a circle with tight surfaces as well as tubes around embedded tight surfaces in 4-space. The more restrictive class of taut 3-manifolds was classified in [17]. In particular it includes an embedding of the twisted product $S^1 \times_h S^2$ as a "complexified 2-sphere" and the quaternion space as Cartan's isoparametric hypersurface in S^4. There are a number of constructions for tight polyhedral embeddings of 3-manifolds, compare [9]. However, we are far from being able to cover major parts of the class of all 3-manifolds. It seems that we do not know a tight embedding of any Lens space (except for $\mathbb{R}P^3$) and it seems also that we do not even know a tight polyhedral embedding of $\mathbb{R}P^3$. For any given tight polyhedral 3-manifold it is easy to attach handles tightly but that procedure does not help too much if the other building blocks are missing. Unfortunately there is no simple combinatorial condition which implies the tightness. Instead one has to check all the homology classes in all the open halfspaces, just by applying the definition above. This is better in the case of simply connected 4-manifolds.

For compact and simply connected 4-manifolds without boundary the tightness is equivalent to the requirement that $M \cap E_+$ is always connected and simply connected. The only smooth tight immersions of simply connected 4-manifolds which are known are spheres as convex hypersurfaces in 5-space, the Veronese-type embedding of $\mathbb{C}P^2$ into 8-space [13] and certain embeddings of arbitrary connected sums of copies of $S^2 \times S^2$ in 5-space [8]. G.Thorbergsson [18] found topological obstructions to the existence of smooth tight immersions in terms of the intersection form and Stiefel-Whitney classes. This leads to restrictions for the existence of smooth tight immersions of connected sums of copies of $\mathbb{C}P^2$ and $-\mathbb{C}P^2$. In particular, it turned out that the $K3$ surface does not admit any smooth tight immersion. The obstruction is that it does not admit a splitting as a connected sum of two smooth manifolds even though the intersection form splits as a connected sum.

This is much different in the polyhedral case because the same type of topological obstruction is not there. The polyhedral tight embedding $\mathbb{C}P^2 \to \mathbb{E}^8$ [10] leads to tight embeddings $\mathbb{C}P^2 \# k(-\mathbb{C}P^2) \to \mathbb{E}^8$ for any number k, see [9, Sect.6C], and a tight embedding of the $K3$ surface into 15-space was recently found in [4]. We use them as building blocks and show in our Theorem 7 below how these – together with attaching 2-handles of type $S^2 \times S^2$ – lead to polyhedral tight embeddings of any given topological type of a simply connected PL 4-manifold, subject to a certain extra assumption on the intersection form. Our proof relies on the following results from the classification of 4-manifolds. For an outline of them see [16, Sect.5].

Definition The *intersection form* Q of a compact 4-manifold M is the symmetric bilinear form $Q\colon H_2(M;\mathbb{Z}) \times H_2(M;\mathbb{Z}) \to \mathbb{Z}$ which is dual to the cup product defined on the cohomology $H^2(M;\mathbb{Z})$. It satisfies the equation $Q(M_1 \# M_2) \cong Q(M_1) \oplus Q(M_2)$. If we represent the intersection form in a basis over the integers then the corresponding matrix is invertible and hence *unimodular*, i.e., it has determinant ± 1. The *rank* of Q is the rank of $H_2(M;\mathbb{Z})$ as a $\mathbb{Z}$-module, also known as the second Betti number, the *signature* is the number of negative eigenvalues minus the number of positive eigenvalues. A quadratic form is called *odd* if some diagonal entry in the representing integer matrix is odd, otherwise it is called *even*. It is known from algebra [15] that an indefinite quadratic form over the integers is uniquely classified by its rank, its signature and by its type (even or odd).

Theorem 1 (S.S.Cairns 1940)
The equivalence classes of smooth 4-manifolds and PL 4-manifolds are in $(1-1)$-correspondence. More precisely, every smooth 4-manifold induces precisely one PL manifold (up to PL-homeomorphism) and, vice versa, every PL 4-manifold admits exactly one smoothing (up to diffeomorphism).

Theorem 2 (V.A.Rohlin 1952)
The signature of any simply connected smooth or PL 4-manifold with an even intersection form is an integer multiple of 16.

Theorem 3 (S.Donaldson 1983)
If the intersection form of a simply connected PL 4-manifold is definite then it is diagonalizable over the integers and, in particular, odd.

THEOREM 4 (J.Milnor 1958)
The homotopy classes of simply connected 4-manifolds are uniquely classified by their intersection forms.

The topological classification turned out to be much harder, and it took almost 25 more years until this problem was solved by M.Freedman. The smooth (or PL) classification appears still to be open.

THEOREM 5 (M.Freedman 1982)
The homeomorphism classes of simply connected PL *4-manifolds are uniquely classified by their intersection forms. More precisely: Two such* PL *manifolds* $M, \widetilde{M}$ *are homeomorphic (not necessarily* PL homeomorphic*) if and only if their intersection forms* $Q, \widetilde{Q}$ *are equivalent over the integers.*

There is an algebraic classification of indefinite unimodular quadratic forms as follows:

THEOREM 6 (see [16, Sect.5])

1. *Any indefinite, odd and unimodular quadratic form over the integers is equivalent to* $l(+1) \oplus k(-1)$.
2. *Any indefinite, even and unimodular quadratic form over the integers is equivalent to* $n(\mp E_8) \oplus m\binom{0\ 1}{1\ 0}$.

The rank is $k + l$ or $8n + 2m$, respectively, the signature is $k - l$ or $\pm 8n$, respectively. Conversely, rank and signature of the quadratic form determine these numbers k, l, m, n uniquely. Here E_8 denotes the following unimodular and positive definite matrix:

$$E_8 = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$$

COROLLARY
Let $K3$ *denote the* $K3$ *surface with its intersection form* $(-E_8)\oplus(-E_8)\oplus 3\binom{0\ 1}{1\ 0}$. *Then the manifolds*

$$l(\mathbb{C}P^2)\#k(-\mathbb{C}P^2) \text{ with } k, l \geq 0$$

and

$$n(\pm K3)\#m(S^2 \times S^2) \text{ with } m, n \geq 0$$

cover all homotopy classes (and, in fact, homeomorphism classes) of simply connected PL *4-manifolds with intersection forms*

$$l(+1) \oplus k(-1) \quad \text{or} \quad 2\nu(\mp E_8) \oplus \mu\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where $k, l \geq 0$ and $\mu \geq 3\nu \geq 0$, respectively.

Remark For the intersection form of the $K3$ surface compare [15]. Theorem 3 together with the $\frac{11}{8}$-conjecture [6] implies that no other quadratic form can occur as the intersection form of any simply connected PL 4-manifold. In more detail this conjecture states that for an even intersection form Q the rank of Q is always at least $\frac{11}{8}$ times the absolute value of the signature of Q. It is easily seen that for the form $Q = 2\nu(\mp E_8) \oplus \mu\binom{0\ 1}{1\ 0}$ we have

$$\frac{\text{rank}(Q)}{|\text{sign}(Q)|} = \frac{16\nu + 2\mu}{16\nu} \geq \frac{11}{8} \quad \text{if and only if} \quad \mu \geq 3\nu.$$

Our main result is the following Theorem 7 which provides a construction of polyhedral tight embeddings for a large class of simply connected 4-manifolds. This follows the pattern in the case of 2-manifolds which was mentioned at the very beginning above: Start with certain building blocks and then attach handles tightly.

Theorem 7 *Let M be a simply connected* PL *4-manifold with an indefinite intersection form Q. Assume further that* $\text{rank(Q)} \geq \frac{11}{8}|\text{sign(Q)}| + 44$ *in case that Q is even with* $|\text{sign(Q)}| \geq 32$*. Then there exists a tight polyhedral embedding $\widetilde{M} \to \mathbb{E}^N$ for some N such that M and $\widetilde{M}$ are homeomorphic.*

Since this result relies on the classification in terms of the intersection form, we cannot obtain by this method that M and $\widetilde{M}$ are PL homeomorphic. However, by a theorem of C.T.C.Wall 1964 there is always a number $k \geq 0$ such that the manifolds $M\#k(S^2 \times S^2)$ and $\widetilde{M}\#k(S^2 \times S^2)$ in Theorem 7 are PL homeomorphic. So in some sense in "most" of the cases we can not only prescribe the topological type but also the PL type. Compare Remark 2 at the end of the paper. However, there are an infinite number of undecided cases left. In particular we do not have any example of a tight polyhedral realization of a manifold homeomorphic to $K3\#K3\#\cdots\#K3$. Such examples could remove the number 44 from the extra assumption in Theorem 7 which then would just transform into the hypothesis of the $\frac{11}{8}$-conjecture. For the case of a positive definite intersection form it would be sufficient – by Theorem 3 – to find a tight polyhedral embedding of $k(\mathbb{C}P^2)$ for arbitrary $k \geq 2$. However, such an example (for any $k \geq 2$) is still missing.

The building blocks and connected sums of them

First of all, there are tight triangulations of $\mathbb{C}P^2$ and of the $K3$ surface. This means, there is a triangulation of $\mathbb{C}P^2$ (with 9 vertices, see [10], [11]) and one of the $K3$ surface (with 16 vertices, see [4]) such that *any* simplexwise linear embedding into *any* Euclidean space is tight. In particular, we can regard $\mathbb{C}P^2_9$ as a tightly embedded subcomplex of the 8-simplex $\triangle^8$ and $(K3)_{16}$ as a tightly embedded subcomplex of the 15-simplex $\triangle^{15}$. In each case the manifold contains the complete 2-dimensional skeleton of the ambient 8-simplex or 15-simplex, respectively. This implies that the intersection with any open half-space is connected and simply connected. Compare [9] for general properties of tight triangulations and [12] for a list of known examples.

By truncating each of these subcomplexes at a vertex and by glueing in another copy of the same kind, one gets tight embeddings

$$\mathbb{C}P^2\#(-\mathbb{C}P^2) \to \mathbb{E}^8 \quad \text{and} \quad K3\#(-K3) \to \mathbb{E}^{15},$$

each with signature zero. This construction is quite similar to the original version [2] of Banchoff's tight Klein bottle in 5-space as a geometric connected sum $\mathbb{R}P^2\#\mathbb{R}P^2$. The process of truncating and glueing in additional copies of the same combinatorial type can be repeated arbitrarily often, as shown in [9, Sect.6C]. This implies that we can realize any quadratic form of type

$$(+1) \oplus k(-1), \quad k \geq 1$$

or

$$2(-E_8) \oplus 2n(E_8) \oplus 3(n+1)\begin{pmatrix}0&1\\1&0\end{pmatrix} \cong 2(n-1)E_8 \oplus (3n+19)\begin{pmatrix}0&1\\1&0\end{pmatrix}, \quad n \geq 1$$

by a tightly and polyhedrally embedded simply connected 4-manifold. In the latter case we have the equations rank $= 16(n-1) + 6n + 38 = 22(n+1)$ and $|\text{sign}| = 16(n-1)$, so in particular rank $\geq \frac{11}{8}|\text{sign}| + 44$.

In order to cover the other cases in Theorem 7, we have to attach handles, thus realizing the sum of a previously given quadratic form Q and copies of $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$.

Attaching handles tightly

There is an obvious procedure to attach a handle to a tight polyhedral surface in 3-space: Pick two faces opposite to one another (not necessarily in parallel planes), cut out a certain triangle in each of them, and glue in a polyhedral cylinder (as the boundary of a triangular prism), see Figure 1. It is, however, much less obvious how one can attach a 2-handle or a 3-handle tightly to a given polyhedron. One needs to fill in something within the convex hull of its boundary without hitting the rest of the manifold.

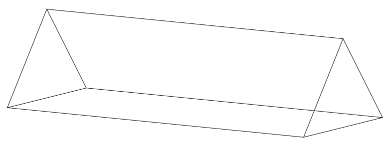

Figure 1: Attaching a 1-handle tightly

In general the procedure of attaching a k-handle of type $S^k \times S^{n-k}$ to an n-manifold is equivalent to cutting out a submanifold of type $S^{k-1} \times B^{n-k+1}$ inside a topological ball (e.g., a coordinate chart) and replacing it by $B^k \times S^{n-k}$. This is the classical *surgery* which we have to carry out in a polyhedral setting. The case $k = 1$ corresponds to attaching an ordinary 1-handle, like a bridge between two parts of the manifold. The case $k = 2, n = 4$ is crucial in the proof of our Theorem 7. For our purpose we have to realize this surgery within the class of tight polyhedral submanifolds. Therefore, we have to describe this process of *tight surgery* geometrically in the ambient space. It will always be carried out in some Euclidean $(n+1)$-space if the manifold is n-dimensional

Definition A *simple polyhedral sphere* Σ^{k-1} is a triangulation of the sphere S^{k-1} with $k+1$ vertices. This is nothing but the boundary complex of a k-dimensional simplex. A *short link* of a certain $(n-k)$-simplex in a triangulated n-manifold is a link which is combinatorially equivalent to a simple polyhedral sphere Σ^{k-1}. Notice that the link of a codimension–1 face is always short, the link of a codimension–2 face is short if and only if it has exactly 3 vertices and 3 edges. In the sequel let Δ^k denote a certain k-dimensional simplex in the simplicial complex which is considered, and let $\triangle^k$ denote an abstract k-simplex which is not necessarily in the complex.

Lemma *Assume that $M^n \subset \mathbb{E}^N$ is a simplicial submanifold containing a simplex Δ^{n-k} with a short link Σ^{k-1} such that the $n+2$ vertices of the star of Δ^{n-k} are in general position. Then there is a polyhedral "solid torus" of type $S^{k-1} \times B^{n-k+1}$ within the open star of Δ^{n-k} which is a tight submanifold-with-boundary in the affine subspace $\mathbb{E}^{n+1}$ of $\mathbb{E}^N$ which is spanned by the $n+2$ vertices of the star of Δ^{n-k}. Moreover, it can be arranged that the convex hull of the short link does not hit M except for its boundary. Therefore, we can choose the tight solid torus in such a way that its convex hull does not hit M either except for the solid torus itself.*

Proof. The procedure of attaching a handle will be carried out inside the open star of Δ^{n-k} without using any of the original vertices. Since the tightness is affinely invariant, we can assume that the $n+2$ vertices of the star of Δ^{n-k} form a regular simplex in $(n+1)$-space. In the classical case $k = 1$ we take the two barycenters of the two n-faces meeting at Δ^{n-1}. These form a 0-sphere. Then the procedure of attaching a handle tightly is suggested by Figure 1.

If $k = 2$ we take the three barycenters of the three $(n-1)$-faces meeting at Δ^{n-2}. Any two of them can be joined by a straight line segment inside of one of the three n-faces of M. The union of these three line segments is a simple polyhedral 1-sphere $\partial\triangle^2$ in M (but not as a subcomplex) such that its convex hull does not hit M except along exactly those three line segments. Then we construct a tight thickening of this polyhedral 1-sphere in the n-manifold as the union of three prisms of type $\triangle^1 \times \triangle^{n-1}$ such that they fit mutually together in $(n+1)$-space as an embedded solid handle $\partial \triangle^2 \times \triangle^{n-1}$.

In the general case for arbitrary k we proceed similarly: Take the $k+1$ barycenters of all the $(n-k+1)$-faces meeting at Δ^{n-k}. These span a regular k-simplex $\triangle^k$ in an $(n+1)$-dimensional Euclidean space. Its boundary is a simple polyhedral $(k-1)$-sphere inside the star of Δ^{n-k} in M (but not as a subcomplex). Then take a similar n-dimensional thickening of that simple sphere inside M and inside the same $(n+1)$-space. Then again replace the interior of a "solid torus" of type $\partial \triangle^k \times \triangle^{n-k+1}$ by the exterior of type $\triangle^k \times \partial\triangle^{n-k+1}$.

In order to describe this procedure in more detail we use the unique projective transformation $\Phi\colon \mathrm{star}(\Delta^{n-k}) \backslash \Delta^{n-k} \to \mathbb{E}^{n+1}$ which sends to infinity the hyperplane which contains Δ^{n-k} and the k-plane parallel to the opposite k-simplex in the star of Δ^{n-k}. Then the rest of the open star becomes an orthogonal cartesian product of the link of Δ^{n-k} with an open part of some Euclidean $\mathbb{E}^{n-k+1}$. Furthermore Φ maps the union of the $k+1$ open n-faces meeting at Δ^{n-k} to an open part of the orthogonal cartesian product $\partial \triangle^k \times \mathbb{E}^{n-k+1}$ in $\mathbb{E}^k \times \mathbb{E}^{n-k+1} = \mathbb{E}^{n+1}$. Hence the polyhedral thickening of $\partial\triangle^k$ can be defined as the cartesian product $\partial\triangle^k \times \triangle^{n-k+1}$ where $\triangle^{n-k+1}$ denotes a small simplex in $(n-k+1)$-space. This is tightly embedded since it is a product of two tight subsets. The boundary is the product $\partial \triangle^k \times \partial\triangle^{n-k+1}$ which is also a tight polyhedral embedding of $S^{k-1} \times S^{n-k}$.

By applying Φ^{-1} we obtain the tight solid torus in the actual open star of Δ^{n-k}. Note that projective transformations preserve tightness. For the surgery we cut out the interior of $\partial \triangle^k \times \triangle^{n-k+1}$ and replace it by the interior of $\triangle^k \times \partial\triangle^{n-k+1}$. A picture for $n = 3$ is shown in Figure 2. Note, however, that this is a 3-dimensional projection of a 3-dimensional solid torus in 4-space. It is not a solid torus in 3-space. □

Corollary *Given a tight triangulation of a* PL *n-manifold M where some $(n-k)$-simplex ($k \leq n/2$) has a short link, we can tightly attach arbitrarily many handles of type $S^k \times S^{n-k}$. Hence for any m we obtain a manifold of* PL *type $M\#m(S^k \times S^{n-k})$ tightly embedded into Euclidean space.*

Proof: We carry out the construction of the lemma above. It is quite clear that we can repeat it arbitrarily often within one star since these solid tori can be chosen arbitrarily thin and disjoint. It is not essential to use the exact barycenters in the construction. The tightness of the solid torus implies that

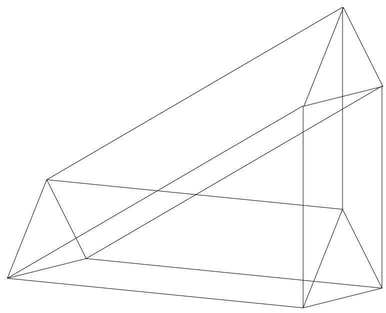

Figure 2: Attaching a 2-handle tightly; the case $k = 2, n = 3$

M minus the interior will still be tight. The same holds after the surgery. In the case of $n = 4$ and $k = 2$ which is most important for Theorem 7. We can easily see that the intersection with any halfspace is still connected and simply connected after the surgery if it was before. The tightness in the other cases follows similarly by considering the homology cycles created by the surgery. In any case the original S^{k-1} for starting the surgery is nullhomotopic in M, so that the topology after one step is that of $M\#(S^k \times S^{n-k})$. □

Proof of Theorem 7:
In case 1 we consider a simply connected 4-manifold M with an odd intersection form which is equivalent to $l(+1) \oplus k(-1)$. By assumption it is indefinite, so we can assume that the signature is nonnegative and thus $k \geq l \geq 1$. This quadratic form is also equivalent to

$$(+1) \oplus (k - l + 1)(-1) \oplus (l - 1)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We can realize this by starting with the tight $\mathbb{C}P^2_9$ in 8-space, by truncating vertices and by glueing in $k - l + 1$ combinatorially equivalent copies of $-\mathbb{C}P^2_9$ with an open vertex star removed (see [9, Sect.6C]), and finally by attaching $l - 1$ handles of type $S^2 \times S^2$. Here it is crucial that $\mathbb{C}P^2_9$ does contain triangles with a short link, e.g., the triangle $\Delta^2 = \langle 1, 2, 3\rangle$ in the labeling of [10]. Hence our lemma above is applicable.

In case 2 we consider an even intersection form with signature 0, 16 or $16m \geq 32$. If the signature is zero we just take the standard ladder construction of tight connected sums of $S^2 \times S^2$, as described in [3, Ex.2.6.4]. The case of the 4-sphere itself is realized by the boundary of any convex polyhedron. If the signature is 16 we start with the tight $K3$ surface in 15-space and attach handles of type $S^2 \times S^2$ tightly. Here it is crucial that this triangulation contains a triangle with a short link. In the labeling of Figure 1 in [4] this is the triangle $\Delta^2 = \left\langle \binom{0}{0}, \binom{1}{0}, \binom{x}{0} \right\rangle$.

If the signature is $16m \geq 32$ we first build a tight $(-K3)\#(K3)\#m(K3)$ by the truncation process from [9, Sect.6C] and then attach handles of type $S^2 \times S^2$ tightly. Here the extra assumption

$$\mathrm{rank}(Q) \geq \frac{11}{8}\mathrm{sign}(Q) + 44$$

comes in since the signature of $(-K3)\#(K3)\#m(K3)$ is $16m$ whereas the rank is $22(m+2)$, so it is our assumption which implies that we have a nonnegative number of handles to attach. In any case the resulting tightly embedded 4-manifold has the same intersection form as M and is, therefore, homeomorphic to M by Theorem 5. □

Remarks:

1. The cases of 4-manifolds which are not covered by Theorem 7 are $k(\mathbb{C}P^2)$ where $k \geq 2$ and $m(K3)\#n\binom{0\ 1}{1\ 0}$ where $m \geq 2$ and $n < 22$. Examples of that kind would imply that – modulo the validity of the $\frac{11}{8}$-conjecture – *every* homotopy (or homeomorphism) class of simply connected PL 4-manifolds would be realizable by a tight embedding into some Euclidean space .

2. It seems that no example is known of any pair $M, \widetilde{M}$ of PL manifolds which are homeomorphic to one another but not PL homeomorphic and where each admits a tight PL embedding. One might expect that the "standard structure" is preferred for tight polyhedral embeddings if there is any. This is true at least for the sphere and for any homology sphere: The image of any tight polyhedral embedding of a homology k-sphere is the boundary of a convex polyhedron in $(k+1)$-space, for a simple proof see [9, Cor.3.6].

3. The same construction of attaching handles can be applied to other classes of manifolds. In the case of simply connected 5-manifolds we have tight connected sums of $S^2 \times S^3$ on the one hand and also a tight 13-vertex triangulation of $SU(3)/SO(3)$ on the other, see [12, p.170]. Since the tetrahedron $\Delta^3 = \langle 0, 1, 4, 6\rangle$ in the latter one has a short 1-dimensional link, it is possible to attach 2-handles of type $S^2 \times S^3$ tightly.

4. The construction above of attaching handles does not raise the essential codimension of the embedding. In fact, reaching or estimating the maximum codimension is a different interesting problem. Here a conjecture states that for any simply connected 4-manifold M a tight polyhedral embedding into $\mathbb{E}^N$ (not in any hyperplane) can exist only if the Heawood type inequality

$$\binom{N-3}{3} \leq 10\beta_2(M)$$

is satisfied where β_2 denotes the second Betti number (similarly for $(k-1)$-connected $2k$-manifolds), see [9, Sect.4]. Equality is attained for the tight triangulations of $\mathbb{C}P^2$ and the $K3$ surface, perhaps also in other cases. By standard arguments this conjecture would follow if the following generalized van

Kampen–Flores theorem is true: *Assume that a simply connected* 4*-manifold* M *admits a polyhedral embedding of the complete* 2*-skeleton of the* N*-dimensional simplex. Then the inequality* $\binom{N-3}{3} \leq 10\beta_2(M)$ *holds.* The classical van Kampen–Flores theorem is nothing but the case of the 4-sphere.

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