

THE CENTRE OF COMPLETED GROUP ALGEBRAS
OF PRO- p GROUPS

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ABSTRACT. We compute the centre of the completed group algebra of an arbitrary countably based pro- p group with coefficients in \mathbb{F}_p or \mathbb{Z}_p . Some other results are obtained.

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1 INTRODUCTION

Let G be a pro- p group. In this paper we investigate some rings related to the completed group algebra of G over \mathbb{F}_p , which we denote by Ω_G :

$$\Omega_G = \mathbb{F}_p[[G]] := \varprojlim_{N \triangleleft_o G} \mathbb{F}_p[G/N].$$

When G is analytic in the sense of [3], Ω_G and its p -adic analogue Λ_G defined by

$$\Lambda_G = \mathbb{Z}_p[[G]] := \varprojlim_{N \triangleleft_o G} \mathbb{Z}_p[G/N]$$

are right and left Noetherian rings, which are in general noncommutative. If in addition G is torsion free, the results of Brumer, Neumann and others show that Ω_G and Λ_G have finite global dimension and have no zero-divisors; for an overview, see [1]. Moreover, under the name of Iwasawa algebras, these rings are frequently of interest to number theorists (see [2] for more details).

Our main result is

THEOREM A. *Let G be a countably based pro- p group. Then the centre of Ω_G is equal to the closure of the centre of $\mathbb{F}_p[G]$:*

$$Z(\Omega_G) = \overline{Z(\mathbb{F}_p[G])}.$$

Similarly, the centre of Λ_G is equal to the closure of the centre of $\mathbb{Z}_p[G]$:

$$Z(\Lambda_G) = \overline{Z(\mathbb{Z}_p[G])}.$$

When G is p -valued in the sense of Lazard [5, III.2.1.2], we obtain a cleaner result.

COROLLARY A. *Let G be a countably based p -valued pro- p group with centre Z . Then*

$$Z(\Omega_G) = \Omega_Z \quad \text{and} \quad Z(\Lambda_G) = \Lambda_Z.$$

The class of p -valued pro- p groups is rather large; for example, every closed subgroup of a uniform pro- p group [3, 4,1] is p -valued. Also, any pro- p subgroup of $GL_n(\mathbb{Z}_p)$ is p -valued when $p > n + 1$ [5, p. 101].

We remark that when G is an open pro- p subgroup of $GL_2(\mathbb{Z}_p)$, a version of the above result was proved by Howson [4, 4.2] using similar techniques.

We also use the method used in the proof of Theorem A to compute endomorphism rings of certain induced modules for Ω_G , when G is an analytic pro- p group.

THEOREM B. *Let H be a closed subgroup of an analytic pro- p group G . Let $M = \mathbb{F}_p \otimes_{\Omega_H} \Omega_G$ and write $R = \text{End}_{\Omega_G}(M)$. Then R is finite-dimensional over \mathbb{F}_p if and only if $N_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$, where \mathfrak{h} and \mathfrak{g} denote the \mathbb{Q}_p -Lie algebras of H and G , respectively.*

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2 FIXED POINTS

Let X be a group. For any (right) X -space S , let $S^X = \{s \in S : s.X = s\}$ denote the set of fixed points of X in S . Also, let $\mathcal{O}(S)$ denote the collection of all *finite* X -orbits on S , and for any orbit $\mathcal{C} \in \mathcal{O}(S)$, let $\hat{\mathcal{C}}$ denote the orbit sum

$$\hat{\mathcal{C}} = \sum_{s \in \mathcal{C}} s,$$

viewed as an element of the permutation module $\mathbb{F}_p[S]$. Thus, $\mathbb{F}_p[S]^X$ is spanned by all the $\hat{\mathcal{C}}$ as \mathcal{C} ranges over $\mathcal{O}(S)$: $\mathbb{F}_p[S]^X = \mathbb{F}_p[\hat{\mathcal{O}}(S)]$.

Now let X be a pro- p group. Assume we are given an inverse system

$$\dots \xrightarrow{\pi_{n+1}} A_n \xrightarrow{\pi_n} A_{n-1} \xrightarrow{\pi_{n-1}} \dots \xrightarrow{\pi_2} A_1$$

of finite X -spaces. We can consider the natural inverse system of permutation modules associated with the A_i :

$$\dots \xrightarrow{\pi_{n+1}} \mathbb{F}_p[A_n] \xrightarrow{\pi_n} \mathbb{F}_p[A_{n-1}] \xrightarrow{\pi_{n-1}} \dots \xrightarrow{\pi_2} \mathbb{F}_p[A_1]$$

where we keep the same notation for the connecting maps π_n . Now, form the inverse limit

$$Y = \varprojlim A_n,$$

this is clearly an X -space. We can also form the inverse limit

$$\Omega_Y = \varprojlim \mathbb{F}_p[A_n],$$

which is easily seen to be an Ω_X -module.

Note that Ω_Y is a compact metric space, with metric given by $d(\alpha, \beta) = \|\alpha - \beta\|$, where $\|\cdot\|$ is a norm on Ω_Y given by

$$\|(\alpha_n)_n\| = \sup\{p^{-n} : \alpha_n \neq 0\} \quad \text{and} \quad \|0\| = 0.$$

We are interested in the fixed points of Ω_Y , viewed as an X -space. It is straightforward to see that there is a natural embedding of $\mathbb{F}_p[Y]$ into Ω_Y and that $\mathbb{F}_p[Y]^X \subseteq \Omega_Y^X$.

PROPOSITION 2.1. *With the notations above, $\Omega_Y^X = \overline{\mathbb{F}_p[Y]^X} = \overline{\mathbb{F}_p[Y]^X}$.*

Proof. Because the action of X on Ω_Y is continuous, it is clear that Ω_Y^X is a closed subset of Ω_Y , so by the above remarks $\overline{\mathbb{F}_p[Y]^X} \subseteq \Omega_Y^X$. Let $\alpha = (\alpha_n)_n \in \Omega_Y^X$. Since the natural maps $\Omega_Y \rightarrow \mathbb{F}_p[A_n]$ are maps of X -spaces, we see that each α_n lies in $\mathbb{F}_p[A_n]^X$.

Let the integer r be least such that $\alpha_r \neq 0$. Consider $\alpha_r \in \mathbb{F}_p[A_r]^X$; thus $\alpha_r = \sum_{\mathcal{C} \in \mathcal{O}(A_r)} \lambda_{\mathcal{C}} \hat{\mathcal{C}}$ and not all the $\lambda_{\mathcal{C}}$ are zero.

Pick a $\mathcal{C} \in \mathcal{O}(A_r)$ with $\lambda_{\mathcal{C}} \neq 0$. Since π_{r+1} is a map of X -spaces, $\pi_{r+1}^{-1}(\mathcal{C}) = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \dots \cup \mathcal{D}_k$ is a union of X -orbits, with $\pi_{r+1}(\mathcal{D}_j) = \mathcal{C}$ for $j = 1, \dots, k$ and $\pi_{r+1}(\mathcal{D}_j) \cap \mathcal{C} = \emptyset$ for $j > k$, if we let $\mathcal{D}_{k+1}, \dots, \mathcal{D}_m$ denote the remaining elements of $\mathcal{O}(A_{r+1})$.

We claim we can find a \mathcal{D}_j with $1 \leq j \leq k$ such that $|\mathcal{D}_j| = |\mathcal{C}|$.

For, suppose not. Then $|\mathcal{D}_j| > |\mathcal{C}|$ for each $j = 1, \dots, k$. As $\pi_{r+1} : \mathcal{D}_j \rightarrow \mathcal{C}$ is a surjective map of finite transitive X -spaces, and because X is a pro- p group, we deduce that each fibre $(\pi_{r+1}|_{\mathcal{D}_j})^{-1}(s)$ for $s \in \mathcal{C}$ has size a power of p greater than 1. But then, because we are working over \mathbb{F}_p , we must have $\pi_{r+1}(\hat{\mathcal{D}}_j) = 0$, for each $1 \leq j \leq k$.

Now, since $\alpha_{r+1} \in \mathbb{F}_p[A_{r+1}]^X$, we can write $\alpha_{r+1} = \sum_{j=1}^m \mu_j \hat{\mathcal{D}}_j$ for some $\mu_j \in \mathbb{F}_p$. So, $\alpha_r = \pi_{r+1}(\alpha_{r+1}) = \sum_{j=k+1}^m \mu_j \pi_{r+1}(\hat{\mathcal{D}}_j)$. But $\pi_{r+1}(\mathcal{D}_j) \cap \mathcal{C} = \emptyset$ for all $j > k$, contradicting the fact that $\mathcal{C} \subseteq \text{supp}(\alpha_r)$.

Hence, we can find $\mathcal{C}_{r+1} \in \mathcal{O}(A_{r+1})$ with $|\mathcal{C}_{r+1}| = |\mathcal{C}_r|$ and $\pi_{r+1}(\mathcal{C}_{r+1}) = \mathcal{C}_r$, where we set \mathcal{C}_r to be \mathcal{C} . It is clear that we can continue this process of “lifting” the X -orbits, without ever increasing the sizes. Thus, we get a sequence

$$\dots \xrightarrow{\pi_{n+2}} \mathcal{C}_{n+1} \xrightarrow{\pi_{n+1}} \mathcal{C}_n \xrightarrow{\pi_n} \dots \xrightarrow{\pi_{r+1}} \mathcal{C}_r$$

of X -orbits, each having the same size as \mathcal{C}_r .

Now, pick some $s_r \in \mathcal{C}_r$ and inductively choose lifts $s_n \in \mathcal{C}_n$ for each $n \geq r$. Let s be the element of Y determined by these lifts. It is then straightforward to see that the X -orbit of s in Y is finite and that the image of this orbit in A_r equals \mathcal{C} . Let $\mathcal{F}_{\mathcal{C}}$ denote this element of $\mathcal{O}(Y)$.

Finally, we can consider the element $\beta = \sum_{\mathcal{C} \in \mathcal{O}(A_r)} \lambda_{\mathcal{C}} \hat{\mathcal{F}}_{\mathcal{C}}$. Obviously β lies in $\mathbb{F}_p[Y]^X$, and the image of β in $\mathbb{F}_p[A_r]$ coincides with

α_r . Hence, $\alpha - \beta$ has norm strictly smaller than that of α and also lies in Ω_Y^X . Applying the argument above to $\alpha - \beta$ instead of α and iterating, we see that α can be approximated arbitrarily closely by elements of $\mathbb{F}_p[Y]^X$. \square

Next we turn to the analogous proposition over the p -adics. Let

$$\Lambda_Y = \varprojlim \mathbb{Z}_p[A_n].$$

This is naturally a Λ_X -module and there is a natural isomorphism $\Lambda_Y/p\Lambda_Y \cong \Omega_Y$ of Λ_X -modules. Moreover, since $\Lambda_Y \cong \varprojlim (\mathbb{Z}/p^n\mathbb{Z})[A_n]$ is a countably based pro- p group, it is a compact metric space.

PROPOSITION 2.2. *With the notations above, $\Lambda_Y^X = \overline{\mathbb{Z}_p[Y]^X}$.*

Proof. As in the proof of Proposition 2.1, the inclusion $\overline{\mathbb{Z}_p[Y]^X} \subseteq \Lambda_Y^X$ is clear. Let $\bar{\cdot} : \Lambda_Y \rightarrow \Omega_Y$ denote reduction mod p .

Let $\alpha \in \Lambda_Y^X$ so that $\bar{\alpha} \in \Omega_Y^X$. By Proposition 2.1, $\bar{\alpha} = \lim_{n \rightarrow \infty} u_n$ for some $u_n \in \mathbb{F}_p[Y]^X$. Since $\mathbb{F}_p[Y]^X = \mathbb{F}_p[\mathcal{O}(\hat{Y})]$ and $\mathbb{Z}_p[Y]^X = \mathbb{Z}_p[\mathcal{O}(\hat{Y})]$, we can choose $v_n \in \mathbb{Z}_p[Y]^X$ such that $\bar{v}_n = u_n$ for all n . Since Λ_Y is compact, by passing to a convergent subsequence we may assume that v_n converges to $\beta_0 \in \overline{\mathbb{Z}_p[Y]^X}$. Now $\bar{\alpha} = \lim_{n \rightarrow \infty} \bar{v}_n = \bar{\beta}_0$, so $\alpha - \beta_0 \in p\Lambda_Y \cap \Lambda_Y^X = p\Lambda_Y^X$, since Λ_Y is p -torsion free.

Hence we can write $\alpha = \beta_0 + p\alpha_1$ where $\alpha_1 \in \Lambda_Y^X$. Iterating the above argument, we obtain elements $\beta_1, \beta_2, \dots \in \overline{\mathbb{Z}_p[Y]^X}$ and $\alpha_1, \alpha_2, \dots \in \Lambda_Y^X$ such that $\alpha_n = \beta_n + p\alpha_{n+1}$ for all $n \geq 1$. So $\alpha = \sum_{n=0}^{\infty} \beta_n p^n \in \overline{\mathbb{Z}_p[Y]^X}$. \square

3 MAIN RESULTS

We immediately make use of the above Propositions.

Proof of Theorem A. Since G is countably based, we can write G as an inverse limit of the countable system $A_n = G/G_n$, for some suitable open normal subgroups $G_1 \supset G_2 \supset \dots \supset G_n \supset \dots$ of G . Each A_n is a finite G -space, where G acts by conjugation. Now apply Propositions 2.1 and 2.2. \square

Proof of Corollary A. $Z(\mathbb{F}_p[G])$ is spanned over \mathbb{F}_p by all conjugacy class sums $\hat{\mathcal{C}}$, where \mathcal{C} is a *finite* conjugacy class of G . Let \mathcal{C} be such a conjugacy class and let $x \in \mathcal{C}$. Then the centralizer $C_G(x)$ of x in G is a closed subgroup of G of finite index.

Let $y \in G$; then $y^{p^n} \in C_G(x)$ for some n , so $(x^{-1}yx)^{p^n} = y^{p^n}$. Since G is p -valued, the map $g \mapsto g^p$ is injective on G by [5, Chapter III, Proposition 2.1.4], so $x^{-1}yx = y$ and $\mathcal{C} \subseteq Z$.

Hence $Z(\mathbb{F}_p[G]) = \mathbb{F}_p[Z]$, and similarly $Z(\mathbb{Z}_p[G]) = \mathbb{Z}_p[Z]$. The result follows from Theorem A. \square

We remark that Corollary A does not extend to arbitrary torsion free analytic pro- p groups. This can be easily checked for the group given in [5, Chapter III, Example 3.2.5].

We now turn to the proof of Theorem B. Let $G = \varprojlim G/G_n$ be a pro- p group, H a closed subgroup. Let $M = \mathbb{F}_p \otimes_{\Omega_H} \Omega_G$ be the induced module from the trivial module for Ω_H . G acts on the coset space $Y = H \backslash G$ by right translation and we can write $Y = \varprojlim HG_n \backslash G$ as an inverse limit of finite G -spaces. It is easy to see that Ω_Y is then naturally isomorphic to M .

Let R denote the endomorphism ring $\text{End}_{\Omega_G} M$ of M . Each element $f \in R$ gives rise to a trivial Ω_H -submodule of M generated by $f(1 \otimes 1)$, when we view M as an Ω_H -module by restriction. This gives an isomorphism of \mathbb{F}_p -vector spaces

$$R = \text{Hom}_{\Omega_G}(\mathbb{F}_p \otimes_{\Omega_H} \Omega_G, M) \cong \text{Hom}_{\Omega_H}(\mathbb{F}_p, M),$$

expressing the fact “induction is left adjoint to restriction”.

Now $\text{Hom}_{\Omega_H}(\mathbb{F}_p, M)$ can be thought of as the sum of all trivial Ω_H -submodules of M , which is precisely the set $M^H = \Omega_Y^H$, where H acts on Y by right translation. In view of Proposition 2.1, we are interested in the finite H -orbits on Y ; these are given by those double cosets of H in G which are finite unions of left cosets of H . Suppose HxH is such a double coset; then $\text{Stab}_H(Hx) = \{h \in H : Hxh = Hx\} = H \cap H^x$ has finite index in H , so the set $\mathcal{N}_G(H) = \{x \in G : H \cap H^x \leq_o H\}$ is of interest; we observe that it contains the usual normalizer $N_G(H)$ of H in G . This set is sometimes called the *commensurator* of H in G .

Proof of Theorem B. Recall [3, 9.5] that G contains an open normal uniform subgroup K and that the \mathbb{Q}_p -Lie algebra of G can be defined by

$$\mathcal{L}(G) = K \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

where K is viewed as a \mathbb{Z}_p -module of finite rank [3, 4.17]. The conjugation action of G on K is \mathbb{Z}_p -linear and therefore extends to an action of G on $\mathcal{L}(G)$, which is easily checked to be independent of the choice of K . This is just the adjoint action of G on $\mathcal{L}(G)$. Next, we observe that when $x \in G$,

$$\begin{aligned} H \cap H^x \leq_o H &\Leftrightarrow \mathcal{L}(H \cap H^x) = \mathcal{L}(H) \cap \mathcal{L}(H)^x = \mathcal{L}(H) \\ &\Leftrightarrow \mathcal{L}(H)^x = \mathcal{L}(H), \end{aligned}$$

so $N := \mathcal{N}_G(H) = \text{Stab}_G \mathfrak{h}$ is a (closed) subgroup of G .

By [3, Exercise 9.10], we see that the Lie algebra of N is equal to the normalizer $N_{\mathfrak{g}}(\mathfrak{h})$ of \mathfrak{h} in \mathfrak{g} . We remark in passing that this implies that $N_G(H)$ has finite index in N when dealing with analytic pro- p groups; this is not true in general.

Now, by Proposition 2.1 and the above remarks, R is finite dimensional over \mathbb{F}_p if and only if the number of finite H -orbits on $Y = H \backslash G$ is finite.

Clearly $\{Hx : HxH \text{ is a finite } H\text{-orbit}\} = H \backslash N$, so the number of finite H -orbits on Y is finite if and only if H has finite index in N . This happens if and only if $\mathfrak{h} = \mathcal{L}(H) = \mathcal{L}(N) = N_{\mathfrak{g}}(\mathfrak{h})$, as required. \square

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