Time-Like Isothermic Surfaces Associated to Grassmannian Systems

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Abstract. We establish that, as is the case with space-like isothermic surfaces, time-like isothermic surfaces in pseudo-riemannian space $\mathbb{R}^{n-j,j}$ are associated to the $O(n-j+1,j+1)/O(n-j,j)\times O(1,1)$-system.

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1 Introduction

There is no doubt that the recent renaissance in interest about isothermic surfaces is principally due to the fact that they constitute an integrable system, as can be seen in several new works where it is shown, for instance, that the theory of isothermic surfaces in $\mathbb{R}^3$ can be reformulated within the modern theory of soliton theory [4], or can be analyzed as curved flats in the symmetric space $O(4,1)/O(3)\times O(1,1)$ [3]. Additionally, in a recent work of Burstall [1], we find an account of the theory of isothermic surfaces in $\mathbb{R}^n$ from both points of view: of classic surfaces geometry as well as from the perspective of the modern theory of integrable systems and loop groups.
The key point of this class of surfaces, as well as of the classic pseudospherical surfaces and those with constant mean curvature, is that the Gauss-Codazzi equations are soliton equations, they have a zero-curvature formulation, i.e.,

the equations should amount to the flatness of a family of connections depending on an auxiliary parameter. It is well known that this special property allows actions of an infinite dimensional group on the space of solutions, called the "dressing action" in the soliton theory. For instance, the geometric transformations found for the surfaces above such as Backlund, Darboux and Ribaucour, arise as the dressing action of some simple elements.

More recently, in 1997, Terng in [12] defined a new integrable system, the U/K-system (or n-dimensional system associated to U/K), which is very closely related to that of curved flats discovered by Ferus and Pedit [8]. Terng, in [12], showed that the U/K-system admits a Lax connection and initiated the project to study the geometry associated with these systems. In fact, using the existence of this Lax connection, in 2002 Bruck-Du-Park-Terng ([2]) studied the geometry involved in two particular cases of U/K-systems: $O(m+n)/O(m)\times O(n)$ and $O(m+n,1)/O(m)\times O(n,1)$-systems. For these cases, they found that the isothermic surfaces, submanifolds with constant sectional curvatures and submanifolds admitting principal curvature coordinates are associated to them, and, that the dressing actions of simple elements on the space of solutions corresponded to Backlund, Darboux and Ribaucour transformations for submanifolds.

Later, looking for a relation between space-like isothermic surfaces in pseudo-riemannian space and the U/K-systems, the first author found in [6] that the class of space-like isothermic surfaces in pseudo-riemannian space $\mathbb{R}^{n-j,j}$ for any signature j, were associated to the $O(n-j+1,j+1)/O(n-j,j)\times O(1,1)$-system. The principal point in this study was the suitable choice of a one maximal abelian subalgebra, which allows one to obtain elliptic Gauss equations, which are appropriate for space-like surfaces.

The main goal of this note is to show that time-like isothermic surfaces in the pseudo-riemannian space $\mathbb{R}^{n-j,j}$ are also associated to the $O(n-j+1,j+1)/O(n-j,j)\times O(1,1)$-systems, defined by other two maximal abelian subalgebras, that are not conjugate under the $Ad(K)$-action, where $K=O(n-j,j)\times O(1,1)$. We study the class of time-like surfaces both with diagonal and non-diagonal second fundamental form, in the cases when its principal curvatures are real and distinct and when they are complex conjugates. We show that an isothermic pair i.e, two isothermic time-like surfaces which are dual, in the diagonal or non-diagonal case, are associated to our systems. Additionally, in this paper we present a review of the principal results recently obtained in [7], about the geometric transformations associated to the dressing action of certain elements with two simple poles on the space of solutions of the complex $O(n-j+1,j+1)/O(n-j,j)\times O(1,1)$-system, corresponding to the timelike isothermic surfaces whose second fundamental forms are non-diagonal. The geometric transformations associated to real case of timelike isothermic surfaces with second fundamental forms are diagonal, were already studied in [14].

Finally, we note that all time-like surfaces of constant mean curvature, all time-like rotation surfaces and all time-like members of Bonnet families are examples of time-like isothermic surfaces [11].

2 The U/K-systems

In this section, we introduce the definition of U/K-system given by Terng in [12]. Let U be a semi-simple Lie group, σ an involution on U and K the fixed point set of σ. Then U/K is a symmetric space. The Lie algebra $\mathcal{K}$ is the fixed point set of the differential σ_* of σ at the identity, in others words, it is the $+1$ eigenspace of σ_*. Let now $\mathcal{P}$ denote the -1 eigenspace of σ_*. Then we have the Lie algebra of U, $\mathcal{U} = \mathcal{K} \oplus \mathcal{P}$ and

$$[\mathcal{K},\mathcal{K}] \subset \mathcal{K},\ [\mathcal{K},\mathcal{P}] \subset \mathcal{P},\ [\mathcal{P},\mathcal{P}] \subset \mathcal{K}.$$

Let $\mathcal{A}$ be a non-degenerate maximal abelian subalgebra in $\mathcal{P}$, $a_1, a_2, ..., a_n$ a basis for $\mathcal{A}$ and $\mathcal{A}^{\perp}$ the orthogonal complement of $\mathcal{A}$ in the algebra $\mathcal{U}$ with respect to the Killing form $<,>$. Then the U/K-system is the following first order system of non-linear partial differential equations for $v : \mathbb{R}^n \to \mathcal{P} \cap \mathcal{A}^{\perp}$.

$$[a_i, v_{x_j}] - [a_j, v_{x_i}] = [[a_i, v], [a_j, v]], \qquad 1 \leq i \neq j \leq n, \tag{1}$$

where $v_{x_j} = \frac{\partial v}{\partial x_j}$.

The first basic result established in [12] is the existence of one-parameter family of connections whose flatness condition is exactly the U/K-system.

Theorem 2.1. *([12]) The following statements are equivalent for a map $v : \mathbb{R}^n \to \mathcal{P} \cap \mathcal{A}^{\perp}$:*
i) v is solution of the U/K-system (1).
ii)

$$[\frac{\partial}{\partial x_i} + \lambda a_i + [a_i, v], \frac{\partial}{\partial x_j} + \lambda a_j + [a_j, v]] = 0 \ for\ all\ \lambda \in \mathbb{C}, \tag{2}$$

iii) θ_λ is a flat $\mathcal{U}_{\mathbb{C}} = \mathcal{U} \otimes \mathbb{C}$-connection 1-form on $\mathbb{R}^n$ for all $\lambda \in \mathbb{C}$, where

$$\theta_\lambda = \sum (a_i \lambda + [a_i, v]) dx_i. \tag{3}$$

iv) There exists E so that $E^{-1}dE = \theta_\lambda$.

The one-parameter family of flat connections θ_λ given by (3) is called the *Lax connection* of the U/K-system (1).
It is well known that for a flat connection $\theta = \sum_{i=1}^{n} A_i(x) dx_i$, the *trivialization* of θ, is a solution E for the following linear system:

$$E_{x_i} = E A_i. \tag{4}$$

Or equivalently of $E^{-1}dE = \theta$.

3 Main Results

In the next two subsections we establish our results that time-like isothermic surfaces are associated to the Grassmannian system $O(n-j+1,j+1)/O(n-j,j)\times O(1,1)$. In fact, using the existence of another two maximal abelian subalgebras in the subspace $\mathcal{P}$, different from that of the space-like case given in [6] in which the first author obtained elliptic Gauss equations, we associate to each of these maximal abelian subalgebras one $O(n-j+1,j+1)/O(n-j,j)\times O(1,1)$-system. As we will see, these systems are not equivalent and for each of these maximal abelian subalgebra we obtain hyperbolic Gauss equations, which are correct for time-like surfaces.
Let $U/K = O(n-j+1,j+1)/O(n-j,j)\times O(1,1)$, where

$$O(n-j+1,j+1) = \\ \{X \in GL(n+2) | X^t \begin{pmatrix} I_{n-j,j} & 0 \\ 0 & J' \end{pmatrix} X = \begin{pmatrix} I_{n-j,j} & 0 \\ 0 & J' \end{pmatrix}\},$$

$$I_{n-j,j} = \begin{pmatrix} I_{n-j} & 0 \\ 0 & -I_j \end{pmatrix} \quad \text{and} \quad J' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let $\mathcal{U} = o(n-j+1,j+1)$ be the Lie algebra of U and $\sigma : \mathcal{U} \to \mathcal{U}$ be an involution defined by $\sigma(X) = I_{n,2}^{-1} X I_{n,2}$. Denote by $\mathcal{K}, \mathcal{P}$ the +1, -1 eigenspaces of σ respectively, i.e.,

$$\mathcal{K} = \{\begin{pmatrix} Y_1 & 0 \\ 0 & Y_2 \end{pmatrix} \Big| Y_1 \in o(n-j,j), Y_2 \in o(1,1)\} = o(n-j,j)\times o(1,1),$$

and

$$\mathcal{P} = \{\begin{pmatrix} 0 & \xi \\ -J'\xi^t I_{n-j,j} & 0 \end{pmatrix} \Big| \xi \in \mathcal{M}_{n\times 2}\}.$$

3.1 Time-like case with diagonal second fundamental form

Here we assume the elements $a_1, a_2 \in \mathcal{M}_{(n+2)\times(n+2)}$, where

$$a_1 = e_{n,n+1} + e_{n,n+2} + e_{n+1,n} + e_{n+2,n}$$

$$a_2 = -e_{1,n+1} + e_{1,n+2} - e_{n+1,1} + e_{n+2,1},$$

and e_{ij} is the $(n+2)\times(n+2)$ elementary matrix whose only non-zero entry is 1 in the ij^{th} place.
Then it is easy to see that the subalgebra $\mathcal{A} = < a_1, a_2 >$ is maximal abelian in $\mathcal{P}$, that $Tr[a_1^2]Tr[a_2^2] - Tr[a_1a_2]^2 = 16$ with $Tr[a_1^2] = 4$, so the induced metric on $\mathcal{A}$ is positive definite and finally that

$$\mathcal{P}\cap\mathcal{A}^{\perp} = \{\begin{pmatrix} 0 & \xi \\ -J'\xi^t I_{n-j,j} & 0 \end{pmatrix} \Big| \xi \in \mathcal{M}_{n\times 2},\ \xi_{11} = \xi_{12},\ \xi_{n1} = -\xi_{n2}\}.$$

So using this basis $\{a_1, a_2\}$, the U/K-system (1) for this symmetric space is the following PDE for

$$\xi = \begin{pmatrix} \xi_1 & \xi_1 \\ r_{1,1} & r_{1,2} \\ \vdots & \vdots \\ r_{n-2,1} & r_{n-2,2} \\ \xi_2 & -\xi_2 \end{pmatrix} : \mathbb{R}^2 \to \mathcal{M}_{n\times 2},$$

$$\begin{cases} (r_{i,2})_{x_1} - (r_{i,1})_{x_1} = -2(r_{i,1} + r_{i,2})\xi_2, \quad i = 1, ..., n-2 \\ (r_{i,2})_{x_2} + (r_{i,1})_{x_2} = 2(r_{i,2} - r_{i,1})\xi_1, \quad i = 1, ..., n-2 \\ 2((\xi_1)_{x_2} + (\xi_2)_{x_1}) = \sum_{i=1}^{n-2} \sigma_i (r_{i1}^2 - r_{i2}^2) \\ (\xi_2)_{x_2} + (\xi_1)_{x_1} = 0, \end{cases} \tag{5}$$

where $\sigma_i = 1,\ i = 1, ..., n-j-1$ and $\sigma_i = -1,\ i = n-j, ..., n-2$.
We now denote the entries of ξ by:

$$\begin{pmatrix} \xi_1 & \xi_1 \\ \xi_2 & -\xi_2 \end{pmatrix} = F \quad \text{and} \quad \begin{pmatrix} r_{1,1} & r_{1,2} \\ \vdots & \vdots \\ r_{n-2,1} & r_{n-2,2} \end{pmatrix} = G.$$

For convenience, we call the U/K-system (5) the real $O(n-j+1, j+1)/O(n-j, j) \times O(1,1)$-system, because this system will correspond to time-like surfaces in $\mathbb{R}^{n-j,j}$ whose shape operators are diagonalizable.
Continuing with the same notation used in [2], the real $O(n-j+1, j+1)/O(n-j, j) \times O(1,1)$-system II is the PDE for $(F, G, B) : \mathbb{R}^2 \to gl_*(2) \times \mathcal{M}_{(n-2)\times 2} \times O(1,1)$, where $gl_*(2) = \{N \in \mathcal{M}_{2\times 2} | N_{11} = N_{12},\ N_{21} = -N_{22}\}$

$$\begin{cases} (r_{i,2})_{x_1} - (r_{i,1})_{x_1} = -2\xi_2(r_{i,1} + r_{i,2}), \quad i = 1, ..., n-2 \\ (r_{i,1})_{x_2} + (r_{i,2})_{x_2} = 2\xi_1(r_{i,2} - r_{i,1}), \quad i = 1, ..., n-2 \\ 2((\xi_1)_{x_2} + (\xi_2)_{x_1}) = \sum_{i=1}^{n-2} \sigma_i (r_{i1}^2 - r_{i2}^2) \\ (b_{11})_{x_1} - (b_{12})_{x_1} = 2\xi_2(b_{11} + b_{12}) \\ (b_{21})_{x_1} - (b_{22})_{x_1} = 2\xi_2(b_{22} + b_{21}) \\ (b_{11})_{x_2} + (b_{12})_{x_2} = -2\xi_1(b_{11} - b_{12}) \\ (b_{21})_{x_2} + (b_{22})_{x_2} = -2\xi_1(b_{21} - b_{22}) \end{cases} \tag{6}$$

where the matrix $B = (b_{ij}) \in O(1,1)$. Now we recall that if we take $g = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ solution of $g^{-1}dg = \theta_0$ and B being the particular case $B = \begin{pmatrix} b & 0 \\ 0 & \frac{1}{b} \end{pmatrix} = \begin{pmatrix} e^{2u} & 0 \\ 0 & e^{-2u} \end{pmatrix}$, we obtain the relation

$$B^{-1}dB = \begin{pmatrix} -2\xi_1 dx_2 + 2\xi_2 dx_1 & 0 \\ 0 & -2\xi_2 dx_1 + 2\xi_1 dx_2 \end{pmatrix},$$

which implies that $\xi_1 = -u_{x_2}$ and $\xi_2 = u_{x_1}$, hence the matrix ξ becomes:

$$\xi = \begin{pmatrix} -u_{x_2} & -u_{x_2} \\ r_{1,1} & r_{1,2} \\ \vdots & \vdots \\ r_{n-2,1} & r_{n-2,2} \\ u_{x_1} & -u_{x_1} \end{pmatrix}.$$

So the real $O(n-j+1, j+1)/O(n-j, j) \times O(1,1)$-system II is the set of partial differential equations for $(u, r_{1,1}, r_{1,2}, \dots, r_{n-2,1}, r_{n-2,2})$:

$$\begin{cases} (r_{i,2})_{x_1} - (r_{i,1})_{x_1} = -2(r_{i,1} + r_{i,2})u_{x_1}, & i = 1, ..., n-2 \\ (r_{i,1})_{x_2} + (r_{i,2})_{x_2} = -2(r_{i,2} - r_{i,1})u_{x_2}, & i = 1, ..., n-2 \\ 2(u_{x_1x_1} - u_{x_2x_2}) = \sum_{i=1}^{n-2} \sigma_i (r_{i1}^2 - r_{i2}^2) \end{cases} \tag{7}$$

We observe that the next proposition follows from Proposition 2.5 in [2].

Proposition 3.1. *the following statements are equivalent for map* $(F, G, B) : \mathbb{R}^2 \to gl_*(2) \times \mathcal{M}_{(n-2)\times 2} \times O(1,1)$*:*
(1) (F, G, B) *is solution of (6).*
(2) $\theta_\lambda^{II} := g_2 \theta_\lambda g_2^{-1} - dg_2 g_2^{-1}$ *is a flat connection on* $\mathbb{R}^2$ *for all* $\lambda \in \mathbb{C}$*, where* θ_λ *is the Lax connection associated to the solution* ξ *of the system (5) and* $g_2 = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}$ *is the* $O(1,1)$*-part of the trivialization* $g = (g_1, g_2)$ *of* θ_0*.*
(3) $\theta_\lambda^{II} := g_2 \theta_\lambda g_2^{-1} - dg_2 g_2^{-1}$ *is flat for* $\lambda = 1$*, where* g_2 *is the same as in item (2).*

Before showing the relationship between the Grassmannian system and isothermic surfaces we give the definition of a time-like isothermic surface with shape operators diagonalized over $\mathbb{R}$.

Definition 3.1. *(*Real isothermic surface*) Let* $\mathcal{O}$ *be a domain in* $\mathbb{R}^{1,1}$*. An immersion* $X : \mathcal{O} \to \mathbb{R}^{n-j,j}$ *is called a real time-like isothermic surface if it has flat normal bundle and the two fundamental forms are:*

$$I = e^{2v}(-dx_1^2 + dx_2^2), \; II = e^v \sum_{i=2}^{n-1} (g_{i-1,2} dx_2^2 - g_{i-1,1} dx_1^2) e_i,$$

with respect to some parallel normal frame $\{e_i\}$*. Or equivalently* $(x_1, x_2) \in \mathcal{O}$ *is conformal and line of curvature coordinate system for* X*.*

We note that each isothermic surface has a dual surface ([11]) and make the following related definition.

Definition 3.2. *(*Real isothermic time-like dual pair in $\mathbb{R}^{n-j,j}$ of type $O(1,1)$*). Let* $\mathcal{O}$ *be a domain in* $\mathbb{R}^{1,1}$ *and* $X_i : \mathcal{O} \to \mathbb{R}^{n-j,j}$ *an immersion with*

flat and non-degenerate normal bundle for $i = 1, 2$. (X_1, X_2) is called a real isothermic time-like dual pair in $\mathbb{R}^{n-j,j}$ of type $O(1,1)$ if :
(i) The normal plane of $X_1(x)$ is parallel to the normal plane of $X_2(x)$ and $x \in \mathcal{O}$,
(ii) there exists a common parallel normal frame $\{e_2, ..., e_{n-1}\}$, where $\{e_i\}_2^{n-j}$ and $\{e_i\}_{n-j+1}^{n-1}$ are space-like and time-like vectors resp.
(iii) $x \in \mathcal{O}$ is a conformal line of curvature coordinate system with respect to $\{e_2, ..., e_{n-1}\}$ for each X_k such that the fundamental forms of X_k are:

$$\begin{cases} I_1 = b^{-2}(-dx_1^2 + dx_2^2), \\ II_1 = b^{-1}\sum_{i=2}^{n-1}[-(g_{i-1,1} + g_{i-1,2})dx_1^2 + (g_{i-1,2} - g_{i-1,1})dx_2^2]e_i, \\ I_2 = b^2(-dx_1^2 + dx_2^2), \\ II_2 = b\sum_{i=2}^{n-1}[-(g_{i-1,1} + g_{i-1,2})dx_1^2 - (g_{i-1,2} - g_{i-1,1})dx_2^2]e_i, \end{cases} \tag{8}$$

where $B = \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}$ is in $O(1,1)$ and a $\mathcal{M}_{(n-2)\times 2}$-valued map $G = (g_{ij})$.

Our first result, whose proof follows the same lines of Theorem 6.8 or 7.4 in [2], gives us the relationship between the dual pair of real isothermic timelke surfaces in $\mathbb{R}^{n-j,j}$ of type $O(1,1)$ and the solutions of the real $O(n-j+1, j+1)/O(n-j,j) \times O(1,1)$-system II (6):

Theorem 3.1. *Suppose $(u, r_{1,1}, r_{1,2}, \ldots, r_{n-2,1}, r_{n-2,2})$ is solution of (7) and F, B are given by*

$$F = \begin{pmatrix} \xi_1 & \xi_1 \\ \xi_2 & -\xi_2 \end{pmatrix} = \begin{pmatrix} -u_{x_2} & -u_{x_2} \\ u_{x_1} & -u_{x_1} \end{pmatrix}, \quad B = \begin{pmatrix} e^{2u} & 0 \\ 0 & e^{-2u} \end{pmatrix}.$$

Then: (a)
$\omega =$

$$\begin{pmatrix} 0 & \epsilon_1\beta_1 dx_2 & \ldots & \epsilon_{n-2}\beta_{n-2}dx_2 & 2(-\xi_1 dx_1 + \xi_2 dx_2) \\ -\beta_1 dx_2 & 0 & \ldots & 0 & -\eta_1 dx_1 \\ -\beta_2 dx_2 & 0 & \ldots & 0 & -\eta_2 dx_1 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ -\beta_{n-2}dx_2 & 0 & \ldots & 0 & -\eta_{n-2}dx_1 \\ 2(\xi_2 dx_2 - \xi_1 dx_1) & -\epsilon_1\eta_1 dx_1 & \ldots & -\epsilon_{n-2}\eta_{n-2}dx_1 & 0 \end{pmatrix} \tag{9}$$

where $\epsilon_i = 1$ for $i < n-j$ and $\epsilon_i = -1$ for $i \geq n-j$, and where $\beta_i = (r_{i,2} - r_{i,1})$, $\eta_i = (r_{i,1} + r_{i,2})$, $i = 1, ..., n-2$, is a flat $o(n-j,j)$-valued connection 1-form. Hence there exists $A : \mathbb{R}^2 \to O(n-j,j)$ such that

$$A^{-1}dA = \omega, \tag{10}$$

where ω is given by (9).

(*b*)

$$A\begin{pmatrix} -dx_2 & 0 & \dots & 0 & dx_1 \\ dx_2 & 0 & \dots & 0 & dx_1 \end{pmatrix}^t B^{-1}$$

is exact. So there exists a map $X : \mathbb{R}^2 \to \mathcal{M}_{n\times 2}$ *such that*

$$dX = A\begin{pmatrix} -dx_2 & 0 & \dots & 0 & dx_1 \\ dx_2 & 0 & \dots & 0 & dx_1 \end{pmatrix}^t B^{-1} \tag{11}$$

(*c*) *Let* $X_j : \mathbb{R}^2 \to \mathbb{R}^{n-j,j}$ *denote the* j*-th column of* X *(solution of 11) and* e_i *denote the* i*-th column of* A*. Then* (X_1, X_2) *is a dual pair of real isothermic timelike surfaces in* $\mathbb{R}^{n-j,j}$ *of type* $O(1,1)$*. I.e.* (X_1, X_2) *have the following properties:*
(1) e_1, e_n *are resp. space-like and time-like tangent vectors to* X_1 *and* X_2*, i.e, the tangent planes of* X_1, X_2 *are parallel.*
(2) $\{e_2, ..., e_{n-1}\}$ *is a parallel normal frame for* X_1 *and* X_2*, with* $\{e_2, ..., e_{n-j}\}$ *and* $\{e_{n-j+1}, ..., e_{n-1}\}$ *being resp. space-like and time-like vectors.*
(3) the two fundamental forms for the immersion X_k *are:*

$$\begin{cases} I_1 = e^{-4u}(dx_2^2 - dx_1^2) \\ II_1 = e^{-2u}\sum_{i=2}^{n-1}[(r_{i-1,2} - r_{i-1,1})dx_2^2 - (r_{i-1,1} + r_{i-1,2})dx_1^2]e_i \\ I_2 = e^{4u}(dx_2^2 - dx_1^2) \\ II_2 = e^{2u}\sum_{i=2}^{n-1}[-(r_{i-1,2} - r_{i-1,1})dx_2^2 - (r_{i-1,1} + r_{i-1,2})dx_1^2]e_i \end{cases}$$

Remark 3.1. *We observe that we can prove a theorem like Theorem (3.1) for a general solution* (F, G, B) *of system (6) by taking a generic* $F = (f_{ij})$ *and* $B = (b_{ij}) = \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} \in O(1,1)$*, i.e, we conclude that if* (F, G, B) *is a solution of system (6), we obtain a real isothermic timelike dual pair in* $\mathbb{R}^{n-j,j}$ *of type* $O(1,1)$ *with I and II fundamental forms like in (8).*

Now for the converse, we have the following result.

Theorem 3.2. *Let* (X_1, X_2) *be a real isothermic time-like dual pair in* $\mathbb{R}^{n-j,j}$ *of type* $O(1,1)$*,* $\{e_2, ..., e_{n-1}\}$ *a common parallel normal frame and* (x_1, x_2) *a common isothermal line of curvature coordinates for* X_1 *and* X_2*, such that the two fundamental forms* I_k*,* II_k *for* X_k *are given by (8). Set* $f_{11} = -\frac{b_{x_2}}{2b} = f_{12}$*,* $f_{22} = -\frac{b_{x_1}}{2b} = -f_{21}$*, and* $F = (f_{ij})_{2\times 2}$*. Then if all entries of* G *and the* $g_{i-1,1} + g_{i-1,2}$*,* $g_{i-1,2} - g_{i-1,1}$ *are non-zero, then* (F, G, B) *is a solution of (6).*

Proof. From the definition of real isothermic time-like pair in $\mathbb{R}^{n-j,j}$, we have

$$\omega_1^{(1)} = -b^{-1}dx_2, \quad \omega_n^{(1)} = b^{-1}dx_1 \quad \omega_1^{(2)} = bdx_2, \quad \omega_n^{(2)} = bdx_1$$

is a dual 1-frame for X_k and $\omega_{1\alpha}^{(k)} = l_\alpha(g_{\alpha-1,2} - g_{\alpha-1,1})dx_2$, $\omega_{n\alpha}^{(k)} = -l_\alpha(g_{\alpha-1,1} + g_{\alpha-1,2})dx_1$ for each X_k, where $l_\alpha = 1$ if $\alpha = 2, ..., n-j$ and $l_\alpha = -1$ if

$\alpha = n-j+1,...,n-1$. We observe that $\omega_{i\alpha}^{(k)}$, $i = 1, n$, $\alpha = 2, ..., n-1$ are independent of k. We find that the Levi-civita connection 1-form for the metric I_k is:

$$\omega_{1n}^{(k)} = \frac{b_{x_1}}{b}dx_2 + \frac{b_{x_2}}{b}dx_1 = 2(-f_{22}^{(k)}dx_2 - f_{11}^{(k)}dx_1),$$

which are independent from k. Hence $\omega_{1n}^{(k)} = \omega_{1n}^{(1)} = 2(-f_{22}dx_2 - f_{11}dx_1) = 2(\xi_2 dx_2 - \xi_1 dx_1)$. So the structure equations and the Gauss-Codazzi equations for X_1, X_2 imply that (F, G, B) is a solution of system (6). ■

So, from Theorems (3.1), (3.2) and Remark (3.1), it follows that there exists a correspondence between the solutions (F, G, B) of system (6) and a dual pair of real isothermic timelike surfaces in $\mathbb{R}^{n-j,j}$ of type $O(1,1)$.

Theorem 3.3. *The real $O(n-j+1, j+1)/O(n-j, j) \times O(1,1)$-system II (6) is the Gauss-Codazzi equation for a time-like surface in $\mathbb{R}^{n-j,j}$ such that:*

$$\begin{cases} I = e^{4u}(dx_2^2 - dx_1^2) \\ II = e^{2u}\sum_{i=2}^{n-1}[-(r_{i-1,2} - r_{i-1,1})dx_2^2 - (r_{i-1,1} + r_{i-1,2})dx_1^2]e_i \end{cases} \tag{12}$$

Proof. We can read from I and II that: $\omega_1 = e^{2u}dx_2, \omega_n = e^{2u}dx_1$, $\omega_{1,i} = \eta_i(r_{i-1,2} - r_{i-1,1})dx_2$, and $\omega_{n,i} = -\eta_i(r_{i-1,2} + r_{i-1,1})dx_1$, where $\eta_i = 1$ if $i = 2, ..., n-j$ and $\eta_i = -1$ if $i = n-j+1, ..., n-1$. Now use the structure equations: $d\omega_1 = \omega_n \wedge \omega_{1n}$ and $d\omega_n = \omega_1 \wedge \omega_{n1}$, to obtain:

$$\omega_{1n} = 2(u_{x_1}dx_2 + u_{x_2}dx_1).$$

Now from the Gauss equation: $d\omega_{1n} = -\sum_{i=2}^{n-j}\omega_{1,i}\wedge\omega_{n,i} + \sum_{i=n-j+1}^{n-1}\omega_{1,i}\wedge\omega_{n,i}$, we have that

$$u_{x_1x_1} - u_{x_2x_2} = \frac{1}{2}[\sum_{i=1}^{n-2}\sigma_i(r_{i1}^2 - r_{i2}^2)].$$

The Codazzi equations: $d\omega_{1,i} = -\omega_{1n} \wedge \omega_{n,i}$ and $d\omega_{n,i} = -\omega_{n1} \wedge \omega_{1,i}$ for $i = 2, ..., n-1$, yield, for these values of i,

$$\begin{aligned} (r_{i-1,2})_{x_1} - (r_{i-1,1})_{x_1} &= -2(r_{i-1,2} + r_{i-1,1})u_{x_1} \\ (r_{i-1,1})_{x_2} + (r_{i-1,2})_{x_2} &= -2(r_{i-1,2} - r_{i-1,1})u_{x_2}. \end{aligned}$$

Collecting our information we see that the Gauss-Codazzi equation is the following system for $(u, r_{1,1}, r_{1,2}, \ldots, r_{n-2,1}, r_{n-2,2})$:

$$\begin{cases} (r_{i-1,2})_{x_1} - (r_{i-1,1})_{x_1} = -2(r_{i-1,2} + r_{i-1,1})u_{x_1}, & i = 2, ..., n-1 \\ (r_{i-1,1})_{x_2} + (r_{i-1,2})_{x_2} = -2(r_{i-1,2} - r_{i-1,1})u_{x_2}, & i = 2, ..., n-1 \\ 2(u_{x_1x_1} - u_{x_2x_2}) = \sum_{i=1}^{n-2}\sigma_i(r_{i1}^2 - r_{i2}^2) \end{cases} \tag{13}$$

Hence if we put

$$B = \begin{pmatrix} e^{2u} & 0 \\ 0 & e^{-2u} \end{pmatrix}, \ F = \begin{pmatrix} -u_{x_2} & -u_{x_2} \\ u_{x_1} & -u_{x_1} \end{pmatrix}, \ G = \begin{pmatrix} r_{1,1} & r_{1,2} \\ \vdots & \vdots \\ r_{n-2,1} & r_{n-2,2} \end{pmatrix}, \tag{14}$$

we see that (F, G, B) is solution of the real $O(n-j+1, j+1)/O(n-j, j)\times O(1,1)$-system II. Conversely, if (F, G, B) is solution of the real $O(n-j+1, j+1)/O(n-j, j)\times O(1,1)$-system II (6), and we assume B being as in (14), then the fourth and sixth equation of system (6), imply that

$$\xi_2 = u_{x_1}, \;\; \xi_1 = -u_{x_2}$$

ie, (F, G, B) is the form (14). Finally writing the real $O(n-j+1, j+1)/O(n-j, j)\times O(1,1)$-system II for this (F, G, B) in terms of u and r_{ij} we get equation (13). ■

The next result follows from Theorem (3.1) and Theorem (3.3).

Theorem 3.4. *Let $\mathcal{O}$ be a domain of $\mathbb{R}^{1,1}$, and $X_2 : \mathcal{O} \to \mathbb{R}^{n-j,j}$ an immersion with flat normal bundle and $(x_1, x_2) \in \mathcal{O}$ an isothermal line of curvature coordinate system with respect to a parallel normal frame $\{e_2, ..., e_{n-1}\}$, such that I and II fundamental forms are given by (12). Then there exists an immersion X_1, unique up to translation, such that (X_1, X_2) is a real isothermic timelike dual pair in $\mathbb{R}^{n-j,j}$ of type $O(1,1)$. Moreover, the fundamental forms of X_1, X_2 are respectively:*

$$\begin{cases} I_1 = e^{-4u}(dx_2^2 - dx_1^2) \\ II_1 = e^{-2u}\sum_{i=2}^{n-1}[(r_{i-1,2} - r_{i-1,1})dx_2^2 - (r_{i-1,1} + r_{i-1,2})dx_1^2]e_i \\ I_2 = e^{4u}(dx_2^2 - dx_1^2) \\ II_2 = e^{2u}\sum_{i=2}^{n-1}[-(r_{i-1,2} - r_{i-1,1})dx_2^2 - (r_{i-1,1} + r_{i-1,2})dx_1^2]e_i \end{cases} \tag{15}$$

It follows from Gauss equation that the Gaussian curvatures of X_1 and X_2 of the real isothermic timelike dual pair (15), denoted by $K_G^{(1)}$, $K_G^{(2)}$, and the mean curvatures, denoted by $\eta^{(1)}$ and $\eta^{(2)}$, are given by

$$K_G^{(1)} = -e^{4u}\sum_{i=1}^{n-2}\sigma_i(r_{i,1}^2 - r_{i,2}^2), \qquad K_G^{(2)} = e^{-4u}\sum_{i=1}^{n-2}\sigma_i(r_{i,1}^2 - r_{i,2}^2),$$

$$\eta^{(1)} = e^{2u}\sum_{i=1}^{n-2} r_{i,2}e_{i+1}, \qquad \eta^{(2)} = e^{-2u}\sum_{i=1}^{n-2} r_{i,1}e_{i+1},$$

where $\sigma_i = 1, \; i = 1, ..., n-j-1$ and $\sigma_i = -1, \; i = n-j, ..., n-2$.

3.2 Timelike case with non-diagonal second fundamental form

We continue with the same notational convention used in the subsection above. For this new case, we take the elements $a_1, a_2 \in \mathcal{M}_{(n+2)\times(n+2)}$, to be

$$a_1 = e_{1,n+1} + e_{n,n+2} + e_{n+1,n} - e_{n+2,1}$$

$$a_2 = -e_{1,n+2} + e_{n,n+1} + e_{n+1,1} + e_{n+2,n}.$$

We note that $Tr[a_1^2]Tr[a_2^2] - Tr[a_1a_2]^2 = -16$ and $Tr[a_1^2] = 0$, so that the induced metric on $\mathcal{A}$ is time-like.
One can see easily that the space $\mathcal{A}$ spanned by a_1 and a_2 is a maximal abelian subalgebra contain in $\mathcal{P}$, and that

$$\mathcal{A}^\perp \cap \mathcal{P} = \{\begin{pmatrix} 0 & \xi \\ -J'\xi^t I_{n-j,j} & 0 \end{pmatrix} | \xi \in \mathcal{M}_{n\times 2}, \xi_{11} = -\xi_{n2}, \xi_{12} = \xi_{n1}\}.$$

So the matrix $v \in \mathcal{A}^\perp \cap \mathcal{P}$ if and only if

$$v = \begin{pmatrix} 0 & \dots & \dots & & 0 & \xi_1 & \xi_2 \\ 0 & \dots & \dots & & 0 & r_{1,1} & r_{1,2} \\ \vdots & \dots & \dots & & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & & 0 & r_{n-2,1} & r_{n-2,2} \\ 0 & \dots & \dots & & 0 & \xi_2 & -\xi_1 \\ -\xi_2 & -r_{1,2} & \dots & r_{n-2,2} & -\xi_1 & 0 & 0 \\ -\xi_1 & -r_{1,1} & \dots & r_{n-2,1} & \xi_2 & 0 & 0 \end{pmatrix}.$$

Then using this basis $\{a_1, a_2\}$, the U/K-system (1) for this symmetric space is the following PDE for

$$\xi = \begin{pmatrix} \xi_1 & \xi_2 \\ r_{1,1} & r_{1,2} \\ \vdots & \vdots \\ r_{n-2,1} & r_{n-2,2} \\ \xi_2 & -\xi_1 \end{pmatrix} : \mathbb{R}^2 \to \mathcal{M}_{n\times 2},$$

$$\begin{cases} -r_{i,2_{x_2}} - r_{i,1_{x_1}} & = 2(r_{i,2}\xi_1 - r_{i,1}\xi_2), \quad i = 1, ..., n-2 \\ -r_{i,1_{x_2}} + r_{i,2_{x_1}} & = -2(r_{i,1}\xi_1 + r_{i,2}\xi_2), \quad i = 1, ..., n-2 \\ (-2\xi_1)_{x_2} + (2\xi_2)_{x_1} & = \sum_{i=1}^{n-2} \sigma_i(r_{i,1}^2 + r_{i,2}^2) \\ (2\xi_2)_{x_2} - (2\xi_1)_{x_1} & = 0. \end{cases} \tag{16}$$

We now denote the entries of ξ by:

$$\begin{pmatrix} \xi_1 & \xi_2 \\ \xi_2 & -\xi_1 \end{pmatrix} = F \quad \text{and} \quad \begin{pmatrix} r_{1,1} & r_{1,2} \\ \vdots & \vdots \\ r_{n-2,1} & r_{n-2,2} \end{pmatrix} = G.$$

For convenience, we call the U/K-system (16) the complex $O(n-j+1, j+1)/O(n-j, j) \times O(1,1)$-system, because this system will correspond to time-like surfaces in $\mathbb{R}^{n-j,j}$ whose shape operators have complex eigenvalues.
Now, the complex $O(n-j+1, j+1)/O(n-j, j) \times O(1,1)$-system II is the following PDE for $(F, G, B) : \mathbb{R}^2 \to gl_*(2) \times \mathcal{M}_{(n-2)\times 2} \times O(1,1)$, where $gl_*(2)$

is the set of matrices 2×2 such that $-f_{11} = f_{22},\ f_{21} = f_{12}$,

$$\begin{cases} -r_{i,2_{x_2}} - r_{i,1_{x_1}} & = 2(r_{i,2}\xi_1 - r_{i,1}\xi_2), \\ -r_{i,1_{x_2}} + r_{i,2_{x_1}} & = -2(r_{i,1}\xi_1 + r_{i,2}\xi_2), \\ (-2\xi_1)_{x_2} + (2\xi_2)_{x_1} & = \sum_{i=1}^{n-2} \sigma_i(r_{i,1}^2 + r_{i,2}^2) \\ b_{22_{x_2}} + b_{21_{x_1}} & = -2b_{22}\xi_1 + 2b_{21}\xi_2, \\ b_{12_{x_2}} + b_{11_{x_1}} & = -2b_{12}\xi_1 + 2b_{11}\xi_2, \\ b_{21_{x_2}} - b_{22_{x_1}} & = 2b_{21}\xi_1 + 2b_{22}\xi_2, \\ b_{11_{x_2}} - b_{12_{x_1}} & = 2b_{11}\xi_1 + 2b_{12}\xi_2, \end{cases} \tag{17}$$

where the matrix $B = (b_{ij}) \in O(1,1)$ and $1 \le i \le n-2$. Now taking $B = \begin{pmatrix} e^{2u} & 0 \\ 0 & e^{-2u} \end{pmatrix}$, and using the fact that

$$B^{-1}dB = \begin{pmatrix} 2\xi_2 dx_1 + 2\xi_1 dx_2 & 0 \\ 0 & -2\xi_2 dx_1 - 2\xi_1 dx_2 \end{pmatrix},$$

we have

$$\xi = \begin{pmatrix} u_{x_2} & u_{x_1} \\ r_{1,1} & r_{1,2} \\ \vdots & \vdots \\ r_{n-2,1} & r_{n-2,2} \\ u_{x_1} & -u_{x_2} \end{pmatrix}.$$

So the complex $O(n-j+1, j+1)/O(n-j,j) \times O(1,1)$-system II is the PDE for $(u, r_{1,1}, r_{1,2}, \dots, r_{n-2,1}, r_{n-2,2})$:

$$\begin{cases} -r_{i,2_{x_2}} - r_{i,1_{x_1}} & = 2(r_{i,2}\xi_1 - r_{i,1}\xi_2), \\ -r_{i,1_{x_2}} + r_{i,2_{x_1}} & = -2(r_{i,1}\xi_1 + r_{i,2}\xi_2), \\ -2u_{x_2x_2} + 2u_{x_1x_1} & = \sum_{i=1}^{n-2} \sigma_i(r_{i,1}^2 + r_{i,2}^2). \end{cases} \tag{18}$$

Remark 3.2. *We recall that the complex $O(n-j+1, j+1)/O(n-j,j) \times O(1,1)$-system II is the flatness condition for the family:*

$$\theta_\lambda^{II} = \begin{pmatrix} \omega & MB^{-1} \\ BN & 0 \end{pmatrix}$$

where $B = (b_{ij}) \in O(1,1)$ and the matrices $\omega \in \mathcal{M}_{n\times n}, M \in \mathcal{M}_{n\times 2}, N \in \mathcal{M}_{2\times n}$ are given by:

$$\omega = \begin{pmatrix} 0 & \vec{a} & \vec{b} & c \\ -\vec{a}^{\,t} & 0 & 0 & \vec{d}^{\,t} \\ \vec{b}^{\,t} & 0 & 0 & \vec{e}^{\,t} \\ c & \vec{d} & -\vec{e} & 0 \end{pmatrix}, \quad M = \lambda \begin{pmatrix} dx_1 & -dx_2 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ dx_2 & dx_1 \end{pmatrix} \tag{19}$$

$$N = \lambda \begin{pmatrix} dx_2 & 0 & \dots & 0 & dx_1 \\ -dx_1 & 0 & \dots & 0 & dx_2 \end{pmatrix}$$

where

$$\begin{aligned}
\vec{a} &= (a_1, \ldots, a_{n-j-1}) \text{ and } a_k = r_{k,1}dx_2 - r_{k,2}dx_1, \text{ for } 1 \le k \le n-j-1,\\
\vec{b} &= (b_{n-j}, \ldots, b_{n-2}) \text{ and } b_q = -r_{q,1}dx_2 + r_{q,2}dx_1, \text{ for } n-j \le q \le n-2,\\
c &= -2\xi_1 dx_1 - 2\xi_2 dx_2\\
\vec{d} &= (d_1, \ldots, d_{n-j-1}) \text{ and } d_k = -r_{k,1}dx_1 - r_{k,2}dx_2, \text{ for } 1 \le k \le n-j-1,\\
\vec{e} &= (e_q, \ldots, e_{n-2}) \text{ and } e_q = -r_{q,1}dx_1 - r_{q,2}dx_2, \text{ for } n-j \le q \le n-2.
\end{aligned}$$

We note that a proposition similar to Proposition (3.1), can be proven in this new case.
At this point we need the appropriate definition of a complex isothermic surface, i.e., one that has an isothermal coordinate system with respect to which all the shape operators are diagonalized over $\mathbb{C}$.

Definition 3.3. (Complex isothermic surface) *Let $\mathcal{O}$ be a domain in $\mathbb{R}^{1,1}$. An immersion $X : \mathcal{O} \to \mathbb{R}^{n-j,j}$ is called a complex time-like isothermic surface if it has flat normal bundle and the two fundamental forms are:*

$$I = \pm e^{2v}(-dx_1^2 + dx_2^2), \; II = \sum_{i=2}^{n-1} e^v (g_{i1}(dx_2^2 - dx_1^2) - 2g_{i2}dx_1dx_2)e_i,$$

with respect to some parallel normal frame $\{e_i\}$.

Remark 3.3. *We note that given any complex isothermic surface there is a dual isothermic surface with parallel normal space ([11]). The U/K system generates this pair of dual surfaces, making it clear that they should be considered essentially as a single unit.*

Definition 3.4. (Complex isothermic time-like dual pair in $\mathbb{R}^{n-j,j}$ of type $O(1,1)$). *Let $\mathcal{O}$ be a domain in $\mathbb{R}^{1,1}$ and $X_i : \mathcal{O} \to \mathbb{R}^{n-j,j}$ an immersion with flat and non-degenerate normal bundle for $i = 1, 2$. (X_1, X_2) is called a complex isothermic timelike dual pair in $\mathbb{R}^{n-j,j}$ of type $O(1,1)$ if :*
(i) The normal plane of $X_1(x)$ is parallel to the normal plane of $X_2(x)$ and $x \in \mathcal{O}$,
(ii) there exists a common parallel normal frame $\{e_2, ..., e_{n-1}\}$, where $\{e_i\}_2^{n-j}$ and $\{e_i\}_{n-j+1}^{n-1}$ are space-like and time-like vectors resp.
(iii) $x \in \mathcal{O}$ is a isothermal coordinate system with respect to $\{e_2, ..., e_{n-1}\}$, for each X_k, such that the fundamental forms of X_k are diagonalizable over $\mathbb{C}$. Namely,

$$\begin{cases} I_1 = b^{-2}(dx_1^2 - dx_2^2),\\ II_1 = -b^{-1}\sum_{i=1}^{n-2}[g_{i,2}(dx_2^2 - dx_1^2) + 2g_{i,1}dx_1dx_2]e_{i+1},\\ I_2 = b^2(-dx_1^2 + dx_2^2),\\ II_2 = b\sum_{i=1}^{n-2}[g_{i,1}(dx_2^2 - dx_1^2) - 2g_{i,2}dx_1dx_2]e_{i+1}, \end{cases} \tag{20}$$

where $B = \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}$ *is in* $O(1,1)$ *and a* $\mathcal{M}_{(n-2)\times 2}$*-valued map* $G = (g_{ij})$.

Theorem 3.5. *Suppose* $(u, r_{1,1}, r_{1,2}, \ldots, r_{n-2,1}, r_{n-2,2})$ *is solution of (18) and* F, B *are given by*

$$F = \begin{pmatrix} u_{x_2} & u_{x_1} \\ u_{x_1} & -u_{x_2} \end{pmatrix}, B = \begin{pmatrix} e^{2u} & 0 \\ 0 & e^{-2u} \end{pmatrix}.$$

Then: (a) *The* ω *defined by (19) is a flat* $o(n-j,j)$*-valued connection 1-form. Hence there exists* $A : \mathbb{R}^2 \to O(n-j,j)$ *such that*

$$A^{-1} dA = \omega. \tag{21}$$

(b)

$$A \begin{pmatrix} dx_1 & 0 & \ldots & 0 & dx_2 \\ -dx_2 & 0 & \ldots & 0 & dx_1 \end{pmatrix}^t B^{-1}$$

is exact. So there exists a map $X : \mathbb{R}^2 \to \mathcal{M}_{n\times 2}$ *such that*

$$dX = A \begin{pmatrix} dx_1 & 0 & \ldots & 0 & dx_2 \\ -dx_2 & 0 & \ldots & 0 & dx_1 \end{pmatrix}^t B^{-1}. \tag{22}$$

(c) *Let* $X_i : \mathbb{R}^2 \to \mathbb{R}^{n-j,j}$ *denote the i-th column of* X *(solution of (22)) and* e_i *denote the i-th column of* A. *Then* X_1 *and* X_2 *are a dual pair of isothermic time-like surfaces in* $\mathbb{R}^{n-j,j}$ *with common isothermal coordinates and second fundamental forms diagonalized over* $\mathbb{C}$, *so that:*

(1) e_1, e_n *are space-like and time-like tangent vectors to* X_1 *and* X_2, *i.e, the tangent planes of* X_1, X_2 *are parallel.*

(2) $\{e_2, \ldots, e_{n-1}\}$ *form a parallel normal frame for* X_1 *and* X_2 *of signature* $\{n-j-1, j-1\}$.

(3) the two fundamental forms for the immersion X_i *are:*

$$\begin{cases} I_1 = e^{-4u}(dx_1^2 - dx_2^2) \\ II_1 = -e^{-2u}\sum_{i=1}^{n-2}[r_{i,2}(dx_2^2 - dx_1^2) + 2r_{i,1}dx_1dx_2]e_{i+1} \\ I_2 = e^{4u}(dx_2^2 - dx_1^2) \\ II_2 = e^{2u}\sum_{i=1}^{n-2}[r_{i,1}(dx_2^2 - dx_1^2) - 2r_{i,2}dx_1dx_2]e_{i+1}. \end{cases} \tag{23}$$

Remark 3.4. *We observe that we can prove a theorem like Theorem (3.5) for a general solution* (F, G, B) *of system (17) by taking a generic* $F = (f_{ij})$ *and* $B = (b_{ij}) = \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} \in O(1,1)$, *i.e, we conclude that if* (F, G, B) *is a solution of system (17), we obtain a complex isothermic timelike dual pair in* $\mathbb{R}^{n-j,j}$ *of type* $O(1,1)$ *with I and II fundamental forms like in (20).*

Now for the converse, we have the following result.

Theorem 3.6. *Let (X_1, X_2) be a complex isothermic timelike dual pair in $\mathbb{R}^{n-j,j}$ of type $O(1,1)$, $\{e_2, ..., e_{n-1}\}$ a common parallel normal frame and (x_1, x_2) a common isothermal coordinates for X_1 and X_2, such that the two fundamental forms I_k, II_k for X_k are given by (20). Set $f_{11} = \frac{b_{x_2}}{2b} = -f_{22}$, $f_{12} = \frac{b_{x_1}}{2b} = f_{21}$, and $F = (f_{ij})_{2\times 2}$. Then if all entries of G are non-zero, then (F, G, B) is a solution of (17).*

Proof. From the definition of complex isothermic timelike dual pair in $\mathbb{R}^{n-j,j}$, we have

$$\omega_1^{(1)} = b^{-1}dx_1, \quad \omega_n^{(1)} = b^{-1}dx_2, \quad \omega_1^{(2)} = -bdx_2, \quad \omega_n^{(2)} = bdx_1$$

is a dual 1-frame for X_k and $\omega_{1\alpha}^{(k)} = l_\alpha(-g_{\alpha-1,2}dx_1 + g_{\alpha-1,1}dx_2)$, $\omega_{n\alpha}^{(k)} = -l_\alpha(g_{\alpha-1,1}dx_1 + g_{\alpha-1,2}dx_2)$ for each X_k, where $l_\alpha = 1$ if $\alpha = 2, ..., n-j$ and $l_\alpha = -1$ if $\alpha = n-j+1, ..., n-1$. We observe that $\omega_{i\alpha}^{(k)}$, $i = 1, n$, $\alpha = 2, ..., n-1$ are independent of k. We find that the Levi-civita connection 1-form for the metric I_k is:

$$\omega_{1n}^{(k)} = -\frac{b_{x_2}}{b}dx_1 - \frac{b_{x_1}}{b}dx_2,$$

which are independent from k. Hence $\omega_{1n}^{(k)} = \omega_{1n}^{(1)} = 2(f_{22}dx_1 - f_{12}dx_2) = -2(\xi_1 dx_1 + \xi_2 dx_2)$. So the structure equations and the Gauss-Codazzi equations for X_1, X_2 imply that (F, G, B) is a solution of system (17). ■

So, from Theorems (3.5), (3.6) and Remark (3.4), follows that exists a correspondence between the solutions (F, G, B) of system (17) and a dual pair of complex isothermic timelike surfaces in $\mathbb{R}^{n-j,j}$ of type $O(1,1)$.

Theorem 3.7. *The complex $O(n-j+1, j+1)/O(n-j, j) \times O(1,1)$-system II (17) is the Gauss-Codazzi equation for a time-like surface in $\mathbb{R}^{n-j,j}$ such that:*

$$\begin{cases} I = e^{4u}(dx_2^2 - dx_1^2) \\ II = e^{2u}\sum_{i=1}^{n-2}[r_{i,1}(dx_2^2 - dx_1^2) - 2r_{i,2}dx_1dx_2]e_{i+1}. \end{cases} \tag{24}$$

Proof. For this surface we can read off from the fundamental forms I and II that

$$\begin{cases} \omega_1 = -e^{2u}dx_2 \\ \omega_n = e^{2u}dx_1 \\ \omega_{1\,i} = \sigma_i(r_{i-1,1}dx_2 - r_{i-1,2}dx_1) \quad \text{for} \quad 2 \le i \le n-1 \\ \omega_{n\,i} = -\sigma_i(r_{i-1,1}dx_1 + r_{i-1,2}dx_2) \quad \text{for} \quad 2 \le i \le n-1. \end{cases}$$

Using the structure equations, we can see that

$$\omega_{1n} = -2u_{x_2}dx_1 - 2u_{x_1}dx_2,$$

and that the Gauss and Codazzi equations are the same as (18), since we have

$$u_{x_1} = \xi_2, \qquad u_{x_2} = \xi_1.$$

Hence if we put

$$B = \begin{pmatrix} e^{2u} & 0 \\ 0 & e^{-2u} \end{pmatrix}, \quad F = \begin{pmatrix} u_{x_2} & u_{x_1} \\ u_{x_1} & -u_{x_2} \end{pmatrix}, \tag{25}$$
$$G = \begin{pmatrix} r_{1,1} & r_{1,2} \\ \vdots & \vdots \\ r_{n-2,1} & r_{n-2,2} \end{pmatrix}$$

we have that (F, G, B) is solution of the complex $O(n-j+1, j+1)/O(n-j, j) \times O(1,1)$-system II (17). Conversely, if (F, G, B) is solution of the complex $O(n-j+1, j+1)/O(n-j, j) \times O(1,1)$-system II (17), and B is as in (25), then the fourth and sixth equation from system (17), imply that

$$\xi_2 = u_{x_1}, \;\; \xi_1 = u_{x_2}$$

i.e., (F, G, B) has the form (25). Finally writing the $O(n-j+1, j+1)/O(n-j, j) \times O(1,1)$-system II (17) for this (F, G, B), in terms of u and r_i we get equation (18). ■

The next result follows from Theorem (3.5) and Theorem (3.7).

Theorem 3.8. *Let $\mathcal{O}$ be a domain of $\mathbb{R}^{1,1}$, and $X_2 : \mathcal{O} \to \mathbb{R}^{n-j,j}$ an immersion with flat normal bundle and $(x_1, x_2) \in \mathcal{O}$ a isothermal coordinates system with respect to a parallel normal frame $\{e_2, ..., e_{n-1}\}$, such that I and II fundamental forms are given by (24). Then there exists an immersion X_1, unique up to translation, such that (X_1, X_2) is a complex isothermic timelike dual pair in $\mathbb{R}^{n-j,j}$ of type $O(1,1)$. Moreover, the fundamental forms of X_1, X_2 are respectively:*

$$\begin{cases} I_1 = e^{-4u}(dx_1^2 - dx_2^2) \\ II_1 = -e^{-2u} \sum_{i=1}^{n-2} [r_{i,2}(dx_2^2 - dx_1^2) + 2r_{i,1}dx_1dx_2]e_{i+1} \\ I_2 = e^{4u}(dx_2^2 - dx_1^2) \\ II_2 = e^{2u} \sum_{i=1}^{n-2} [r_{i,1}(dx_2^2 - dx_1^2) - 2r_{i,2}dx_1dx_2]e_{i+1}. \end{cases} \tag{26}$$

Finally, it follows from Gauss equation that the Gaussian curvatures of X_1 and X_2 of a complex isothermic timelike dual pair (26), denoted by $K_G^{(1)}$, $K_G^{(2)}$, are given by

$$K_G^{(1)} = e^{4u} \sum_{i=1}^{n-2} \sigma_i (r_{i,1}^2 + r_{i,2}^2), \qquad K_G^{(2)} = e^{-4u} \sum_{i=1}^{n-2} \sigma_i (r_{i,1}^2 + r_{i,2}^2),$$

where $\sigma_i = 1, \; i = 1, ..., n-j-1$ and $\sigma_i = -1, \; i = n-j, ..., n-2$.

Example: Next we give an explicit example of a dual pair of complex timelike isothermic surfaces in $\mathbb{R}^{2,1}$ and the associated solution to the complex $O(3,2)/O(2,1) \times O(1,1)$-system II.

We consider first the Lorentzian helicoid

$$X(x_1, x_2) = (x_2, \sinh(x_1)\sinh(x_2), \cosh(x_2)\sinh(x_1))$$

with normal vector:

$$N(x_1, x_2) = \frac{1}{\cosh(x_1)}(-\sinh(x_1), \cosh(x_2), \sinh(x_2)).$$

The dual surface to this surface is:

$$\widehat{X}(x_1, x_2) = \frac{1}{\cosh(x_1)}(\sinh(x_1), -\cosh(x_2), -\sinh(x_2)),$$

which is a parametrization of part of the standard immersion of the Lorenztian sphere (see[11]). They constitute a dual pair of complex timelike isothermic surfaces in $\mathbb{R}^{2,1}$, with first and second fundamental forms given resp. by

$$I^1 = \cosh^2(x_1)[-dx_1^2 + dx_2^2], \quad II^1 = 2dx_1dx_2,$$

$$I^2 = (1/\cosh^2(x_1))[dx_1^2 - dx_2^2], \quad II^2 = (1/\cosh^2(x_1))[dx_2^2 - dx_1^2].$$

Here

$$B = \begin{pmatrix} \cosh x_1 & 0 \\ 0 & \cosh^{-1} x_1 \end{pmatrix}, \; F = \begin{pmatrix} 0 & \frac{\tanh x_1}{2} \\ \frac{\tanh x_1}{2} & 0 \end{pmatrix},$$

and

$$G = \begin{pmatrix} 0, & -\cosh^{-1} x_1 \end{pmatrix},$$

are a solution of the complex $O(3,2)/O(2,1) \times O(1,1)$-system II. More specifically, taking $e^{2u} = \cosh x_1$, we have $(u, 0, -\cosh^{-1} x_1)$ is a solution of the complex $O(3,2)/O(2,1) \times O(1,1)$-system II.

4 Appendix: Associated Geometric transformations

The first part of this appendix concerns the geometric transformations on surfaces in the pseudo-euclidean space $\mathbb{R}^{n-j,j}$ corresponding to the action of an element with two simple poles on the space of local solutions of our complex $O(n-j+1, j+1)/O(n-j, j) \times O(1,1)$-system II (17). In particular, the results which will be established here were proved by the authors in [7], hence we invite the reader to see in [7] the proof's details. In addition, the reader will find in [7], an explicit example of an isothermic timelike dual pair in $\mathbb{R}^{2,1}$ of type $O(1,1)$ constructed by applying the Darboux transformation to the trivial solution of complex system II (18). We note that the study of the geometric transformations associated to the real case, was already considered in [14].

In the second part of this appendix, we establish the moving frame formulas for timelike surfaces in $\mathbb{R}^{n-j,j}$.

Initially in [7], we made a natural extension of the Ribaucour transformation definition given in [5], and of the definition of Darboux transformation for surfaces in $\mathbb{R}^m$ for our case of complex timelike surfaces. Later, we found the rational element $g_{s,\pi}$ whose action corresponds to the Ribaucour and Darboux transformations just as we defined them. We now review the principal results of [7].

We start by defining Ribaucour and Darboux transformations for timelike surfaces in $\mathbb{R}^{n-j,j}$ whose shape operators have conjugate eigenvalues as follows:
For $x \in \mathbb{R}^{n-j,j}$ and $v \in (T\mathbb{R}^{n-j,j})_x$, where let $\gamma_{x,v}(t) = x + tv$ denote the geodesic starting at x in the direction of v.

Definition 4.1. *Let M^m and $\widetilde{M}^m$ be Lorentzian submanifolds whose shape operators are all diagonalizable over $\mathbb{R}$ or $\mathbb{C}$ immersed in the pseudo-riemannian space $\mathbb{R}^{n-j,j}$, $0 \leq j < n$. A sphere congruence is a vector bundle isomorphism $P : \mathcal{V}(M) \to \mathcal{V}(\widetilde{M})$ that covers a diffeomorphism $\phi : M \to \widetilde{M}$ with the following conditions:*
(1) If ξ is a parallel normal vector field of M, then $P \circ \xi \circ \phi^{-1}$ is a parallel normal field of $\widetilde{M}$.
(2) For any nonzero vector $\xi \in \mathcal{V}_x(M)$, the geodesics $\gamma_{x,\xi}$ and $\gamma_{\phi(x),P(\xi)}$ intersect at a point that is the same parameter value t away from x and $\phi(x)$.

For the following definition we assume that each shape operator is diagonalized over the real or complex numbers. We note that there are submanifolds for which this does not hold.

Definition 4.2. *A sphere congruence $P : \mathcal{V}(M) \to \mathcal{V}(\widetilde{M})$ that covers a diffeomorphism $\phi : M \to \widetilde{M}$ is called a Ribaucour transformation if it satisfies the following additional properties:*
(1) If e is an eigenvector of the shape operator A_ξ of M, corresponding to a real eigenvalue then $\phi_*(e)$ is an eigenvector of the shape operator $A_{P(\xi)}$ of $\widetilde{M}$ corresponding to a real eigenvalue.
If $e_1 + ie_2$ is an eigenvector of A_ξ on $(TM)^{\mathbb{C}}$ corresponding to the complex eigenvalue $a + ib$ (so that $e_1 - ie_2$ corresponds to the eigenvalue $a - ib$), then $\phi_*(e_1) + i\phi_*(e_2)$ is an eigenvector corresponding to a complex eigenvalue for $A_{P(\xi)}$.
(2) The geodesics $\gamma_{x,e}$ and $\gamma_{\phi(x),\phi_*(e)}$ intersect at a point that is equidistant to x and $\phi(x)$ for real eigenvectors e, and γ_{x,e_j} and $\gamma_{\phi(x),\phi_*(e_j)}$ meet for the real and imaginary parts of complex eigenvectors $e_1 + ie_2$, i.e., for $j = 1, 2$.

Definition 4.3. *Let $M, \widetilde{M}$ be two timelike surfaces in $\mathbb{R}^{n-j,j}$ with flat and non-degenerate normal bundle, shape operators that are diagonalizable over $\mathbb{C}$ and $P : \mathcal{V}(M) \to \mathcal{V}(\widetilde{M})$ a Ribaucour transformation that covers the map $\phi : M \to \widetilde{M}$. If, in addition, ϕ is a sign-reversing conformal diffeomorphism then P is called a Darboux transformation.*

In definition (4.3), by a sign-reversing conformal diffeomorphism we mean that the time-like and space like vectors are interchanged and the conformal coordinates remain conformal.
Next we define the rational element

$$g_{s,\pi}(\lambda) = (\pi + \frac{\lambda - is}{\lambda + is}(I - \pi))(\overline{\pi} + \frac{\lambda + is}{\lambda - is}(I - \overline{\pi})) \tag{27}$$

where $0 \neq s \in \mathbb{R}$, π is the orthogonal projection of $\mathbb{C}^{n+2}$ onto the span of $\begin{pmatrix} W \\ iZ \end{pmatrix}$ with respect to the bi-linear form $\langle , \rangle_2$ given by

$$\langle U, V \rangle_2 = \overline{u}_1 v_1 + ... + \overline{u}_{n-j} v_{n-j} - \overline{u}_{n-j+1} v_{n-j+1} - ... - \overline{u}_n v_n + \overline{u}_{n+1} v_{n+2} + \overline{u}_{n+2} v_{n+1},$$

for $W \in \mathbb{R}^{n-j,j}$, $Z \in \mathbb{R}^{1,1}$ unit vectors.
It is easy to see that $g_{s,\pi}$ belongs to the group:

$$G_- = \{g : S^2 \to U_{\mathbb{C}} \mid g \text{ is meromorphic, } g(\infty) = I \text{ and satisfies the reality conditions}\},$$

where $U_{\mathbb{C}} = O(n - j + 1, j + 1; \mathbb{C})$ and the reality conditions are the following, for a map $g : \mathbb{C} \to U_{\mathbb{C}}$:

$$\begin{cases} \overline{g(\overline{\lambda})} = g(\lambda) \\ I_{n,2}\ g(-\lambda)\ I_{n,2} = g(\lambda) \\ g(\lambda)^t \begin{pmatrix} I_{n-j,j} & 0 \\ 0 & J' \end{pmatrix} g(\lambda) = \begin{pmatrix} I_{n-j,j} & 0 \\ 0 & J' \end{pmatrix}. \end{cases} \tag{28}$$

With this, we have:

Theorem 4.1. *Let (X_1, X_2) be a complex isothermic timelike dual pair in $\mathbb{R}^{n-j,j}$ of type $O(1,1)$ corresponding to the solution (u, G) of the system (18), and let $\xi = \begin{pmatrix} F \\ G \end{pmatrix}$ the corresponding solution of the system (16), where*

$$F = \begin{pmatrix} u_{x_2} & u_{x_1} \\ u_{x_1} & -u_{x_2} \end{pmatrix}, B = \begin{pmatrix} e^{2u} & 0 \\ 0 & e^{-2u} \end{pmatrix}.$$

Let $g_{s,\pi}$ defined in (27), and $\widehat{W}$, $\widehat{Z}$ as in Main Lemma 4.1 (see below), for the solution ξ of the system (16). Let $(\widetilde{E^{\sharp}}^{II}, \widetilde{A}^{\sharp}, \widetilde{B}^{\sharp}) = g_{s,\pi}.(E^{II}, A, B)$ the action of $g_{s,\pi}$ over (E^{II}, A, B) where $A, B, \widetilde{A}^{\sharp}, \widetilde{B}^{\sharp}$ are the entries of

$$E(x, 0) = \begin{pmatrix} A(x) & 0 \\ 0 & B(x) \end{pmatrix}, \quad \widetilde{E^{\sharp}}(x, 0) = \begin{pmatrix} \widetilde{A}^{\sharp}(x) & 0 \\ 0 & \widetilde{B}^{\sharp}(x) \end{pmatrix}$$

and E^{II} is the frame corresponding to the solution (F,G,B) of the complex system II (18). Write $A=(e_1,...,e_n)$ and $\widetilde{A}^\sharp=(\widetilde{e}_1,...,\widetilde{e}_n)$. Set

$$\begin{cases}\widetilde{X}_1 = X_1 + \frac{2}{s}\widehat{z}_2 e^{-2u}\sum_{i=1}^n \widehat{w}_i e_i,\\ \widetilde{X}_2 = X_2 + \frac{2}{s}\widehat{z}_1 e^{2u}\sum_{i=1}^n \widehat{w}_i e_i,\end{cases} \tag{29}$$

Then
(i) $(\widetilde{u},\widetilde{G})$ is the solution of system (18), corresponding to $\widetilde{X}=(\widetilde{X}_1,\widetilde{X}_2)$, where $e^{4\widetilde{u}}=\frac{4\widehat{z}_2^4}{e^{4u}}$ and $\widetilde{G}=(\widetilde{r}_{ij})$ is defined by Main Lemma 4.1, for the new solution $\widetilde{\xi}$ of the system (16).
(ii) The fundamental forms of pair $(\widetilde{X}_1,\widetilde{X}_2)$ are respectively

$$\begin{cases}\widetilde{I}_1 = e^{4\widetilde{u}}(-dx_1^2+dx_2^2)\\ \widetilde{II}_1 = e^{2\widetilde{u}}\sum_{i=1}^{n-2}[\widetilde{r}_{i,1}(dx_2^2-dx_1^2)-2\widetilde{r}_{i,2}dx_1dx_2]\widetilde{e}_{i+1}.\\ \widetilde{I}_2 = e^{-4\widetilde{u}}(dx_1^2-dx_2^2)\\ \widetilde{II}_2 = -e^{-2\widetilde{u}}\sum_{i=1}^{n-2}\widetilde{r}_{i,2}(dx_2^2-dx_1^2)+2\widetilde{r}_{i,1}dx_1dx_2]\widetilde{e}_{i+1}.\end{cases}$$

(iii) The bundle morphism $P(e_k(x))=\widetilde{e}_k(x)$, $k=2,...,n-1$ covering the map $X_i\to\widetilde{X}_i$ is a Darboux transformation for each $i=1,2$.

Proof. For (i) and (ii) we just observe that

$$d\widetilde{X}=\widetilde{A}^\sharp\begin{pmatrix}dx_1 & 0 & \dots & 0 & dx_2\\ -dx_2 & 0 & \dots & 0 & dx_1\end{pmatrix}^t \widetilde{B^\sharp}^{-1},$$

and calculate.
For (iii) we observe that the map $\phi: X_i\to\widetilde{X}_i$ is sign-reversing conformal because the coordinates (x_1,x_2) are isothermic for X_i and $\widetilde{X}_i$ but timelike and spacelike vectors are interchanged. The rest of the properties of Darboux transformation follows from Lemma 4.2 below. ■

Lemma 4.1. *(Main Lemma) Let $\xi=\begin{pmatrix}F\\G\end{pmatrix}$ be a solution of the system (16), and $E(x,\lambda)$ a frame of ξ such that $E(x,\lambda)$ is holomorphic for $\lambda\in\mathbb{C}$. Let $g_{s,\pi}$ the map defined by (27) and $\widetilde{\pi}(x)$ the orthogonal projection onto $\mathbb{C}\begin{pmatrix}\widetilde{W}\\ i\widetilde{Z}\end{pmatrix}(x)$ with respect to $\langle,\rangle_2$, where*

$$\begin{pmatrix}\widetilde{W}\\ i\widetilde{Z}\end{pmatrix}(x)=E(x,-is)^{-1}\begin{pmatrix}W\\ iZ\end{pmatrix}. \tag{30}$$

Let $\widehat{W}=\frac{\widetilde{W}}{\|\widetilde{W}\|_{n-j,j}}$ and $\widehat{Z}=\frac{\widetilde{Z}}{\|\widetilde{Z}\|_{1,1}}$, $\widetilde{E}(x,\lambda)=g_{s,\pi}(\lambda)E(x,\lambda)g_{s,\widetilde{\pi}(x)}(\lambda)^{-1}$,

$$\widetilde{\xi}=\xi-2s(\widehat{W}\widehat{Z}^tJ')_*, \tag{31}$$

where (ϑ_*) is the projection onto the span of $\{a_1,a_2\}^\perp$. Then $\widetilde{\xi}$ is a solution of system (16), $\widetilde{E}$ is a frame for $\widetilde{\xi}$ and $\widetilde{E}(x,\lambda)$ is holomorphic in $\lambda\in\mathbb{C}$.

For the Proof of the Main Lemma see ([7]).

Writing the new solution given by Main Lemma 4.1 as $\widetilde{\xi} = \begin{pmatrix} \widetilde{F} \\ \widetilde{G} \end{pmatrix}$, one sees the components of $\widetilde{\xi}$ are:

$$\begin{cases} \widetilde{f}_{11} = -\widetilde{f}_{22} = f_{11} - s(\widehat{w}_1 \widehat{z}_2 - \widehat{w}_n \widehat{z}_1), \\ \widetilde{f}_{12} = \widetilde{f}_{21} = f_{12} - s(\widehat{w}_1 \widehat{z}_1 + \widehat{w}_n \widehat{z}_2), \\ \widetilde{r}_{i1} = r_{i1} - 2s\widehat{w}_{1+i}\widehat{z}_2 \\ \widetilde{r}_{i2} = r_{i2} - 2s\widehat{w}_{1+i}\widehat{z}_1, \end{cases} \tag{32}$$

for $F = (f_{ij})_{2\times 2},\ G = (r_{ij})_{(n-2)\times 2},\ \widetilde{F} = (\widetilde{f}_{ij})_{2\times 2},\ \widetilde{G} = (\widetilde{r}_{ij})_{(n-2)\times 2}$.

Lemma 4.2. *Let* $\xi = \begin{pmatrix} F \\ G \end{pmatrix}$ *solution of (16),* E *frame of* ξ*,* $E(x,0) = \begin{pmatrix} A(x) & 0 \\ 0 & B(x) \end{pmatrix}$*,* (F,G,B) *a solution corresponding to complex* $O(n-j+1, j+1)/O(n-j,j) \times O(1,1)$*-system II, and*

$$(\widetilde{F}, \widetilde{G}, \widetilde{B}^\sharp, \widetilde{E^\sharp}^{II}) = g_{s,\pi}.(F,G,B,E^{II}), \quad \widetilde{A}^\sharp = g_{s,\pi}.A$$

the action of $g_{s,\pi}$ *over the solution* (F,G,B) *and the matrix* A*, resp.. Let* $e_i, \widetilde{e}_i$ *denote the i-th column of* A *and* $\widetilde{A}^\sharp$ *resp. Then we have*
(i) $X = (X_1, X_2)$ *and* $\widetilde{X} = (\widetilde{X}_1, \widetilde{X}_2)$ *are complex isothermic timelike dual pairs in* $\mathbb{R}^{n-j,j}$ *of type* $O(1,1)$ *such that* $\{e_2, ... e_{n-1}\}$ *and* $\{\widetilde{e}_2, ..., \widetilde{e}_{n-1}\}$ *are parallel normal frames for* X_j *and* $\widetilde{X}_j$ *respectively for* $j = 1, 2$*, where* $\{e_\alpha\}_{\alpha=2}^{n-j}$ *and* $\{e_\alpha\}_{\alpha=n-j+1}^{n-1}$ *are spacelike and timelike vectors resp.*
(ii) The solutions of the complex $O(n-j+1, j+1)/O(n-j,j) \times O(1,1)$*-system II corresponding to* X *and* $\widetilde{X}$ *are* (F,G,B) *and* $(\widetilde{F}, \widetilde{G}, \widetilde{B}^\sharp)$ *resp.*
(iii) The bundle morphism $P(e_k(x)) = \widetilde{e}_k(x)$ $k = 2, ..., n-1$*, is a Ribaucour Transformation covering the map* $X_j(x) \mapsto \widetilde{X}_j(x)$ *for each* $j = 1, 2$*.*
(iv) There exist smooth functions ψ_{ik} *such that* $X_i + \psi_{ik} e_k = \widetilde{X}_i + \psi_{ik} \widetilde{e}_k$ *for* $1 \le i \le 2$ *and* $1 \le k \le n$*.*

For the proof of Lemma 4.2, see ([7]).

Now we begin the second part of this appendix, where we review the method of moving frames for time-like surfaces in the Lorentz space $\mathbb{R}^{n-j,j}$. Set

$$e_A \cdot e_B = \sigma_{AB} = I_{n-j,j} = \begin{pmatrix} I_{n-j} & 0 \\ 0 & -I_j \end{pmatrix}.$$

We also let $\sigma_i := \sigma_{ii}$.

For the time-like immersion X set $dX = \omega_1 e_1 + \omega_n e_n$, so that a space-like unit tangent vector to the surface is e_1, a time-like unit vector to the surface is e_n and the normal space is spanned by e_α, for $2 \leq \alpha \leq n-1$. Define

$$de_B = \sum_A \omega_{AB} e_A. \tag{33}$$

This gives $\omega_{AB} = \sigma_A e_A \cdot de_B$ and

$$\omega_{AB}\sigma_A + \omega_{BA}\sigma_B = 0. \tag{34}$$

From $d(dX) = 0$ we get:

$$d\omega_1 = \omega_n \wedge \omega_{1n} \tag{35}$$

$$d\omega_n = \omega_1 \wedge \omega_{n1} \tag{36}$$

$$\omega_1 \wedge \omega_{\alpha 1} + \omega_n \wedge \omega_{\alpha n} = 0, \tag{37}$$

for α as above.
In addition, by Cartan's Lemma we have:

$$\omega_{1\alpha} = h^\alpha_{11}\omega_1 + h^\alpha_{1n}\omega_n, \quad \omega_{n\alpha} = h^\alpha_{n1}\omega_1 + h^\alpha_{nn}\omega_n.$$

This makes the first fundamental form:

$$I : \omega_1^2 - \omega_n^2 \tag{38}$$

and the second fundamental form is:

$$II : \quad -\sum_{k=1,n}\sum_\alpha \omega_{k\alpha}\sigma_k\,\omega_k\,\sigma_\alpha\,e_\alpha = \tag{39}$$
$$-\sum_\alpha (h^\alpha_{11}\omega_1 + h^\alpha_{1n}\omega_n)\,\omega_1\,\sigma_\alpha\,e_\alpha + \sum_\alpha (h^\alpha_{n1}\omega_1 + h^\alpha_{nn}\omega_n)\,\omega_n\,\sigma_\alpha\,e_\alpha.$$

We also have: $d\omega_{CA} = -\sum_B \omega_{CB} \wedge \omega_{BA}$, which yield the Gauss and Codazzi equations. The Gauss equation comes from examining $d\omega_{1n}$, while the Codazzi equations are from $d\omega_{1\alpha}$ and $d\omega_{n\alpha}$.

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