Characterising Weak-Operator Continuous Linear Functionals on $\mathcal{B}(H)$ constructively

Douglas S. Bridges

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Abstract. Let $\mathcal{B}(H)$ be the space of bounded operators on a not-necessarily-separable Hilbert space H. Working within Bishop-style constructive analysis, we prove that certain weak-operator continuous linear functionals on $\mathcal{B}(H)$ are finite sums of functionals of the form $T \rightsquigarrow \langle Tx, y\rangle$. We also prove that the identification of weak- and strong-operator continuous linear functionals on $\mathcal{B}(H)$ cannot be established constructively.

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1 Introduction

Let H be a complex Hilbert space that is nontrivial (that is, contains a unit vector), $\mathcal{B}(H)$ the space of all bounded operators on H, and $\mathcal{B}_1(H)$ the unit ball of $\mathcal{B}(H)$. In this paper we carry out, within Bishop-style constructive mathematics (BISH),[1] an investigation of weak-operator continuous linear functionals on $\mathcal{B}(H)$.

Depending on the context, we use, for example, $\mathbf{x}$ to represent either the element $(x_1, \ldots, x_N)$ of the finite direct sum $H_N \equiv \bigoplus_{n=1}^{N} H$ of N copies of H or else the element $(x_n)_{n\geq 1}$ of the direct sum $H_\infty \equiv \bigoplus_{n\geq 1} H$ of a sequence of copies of H. We use I to denote the identity projection on H.

The following are the topologies of interest to us here.

[1]That is, mathematics that uses only intuitionistic logic and is based on a suitable set- or type-theoretic foundation [1, 2, 12]. For more on BISH see [3, 4, 8].

- ▷ The WEAK OPERATOR TOPOLOGY: the weakest topology on $\mathcal{B}(H)$ with respect to which the mapping $T \rightsquigarrow \langle Tx, y\rangle$ is continuous for all $x, y \in H$; sets of the form
$$\{T \in \mathcal{B}(H) : |\langle Tx, y\rangle| < \varepsilon\},$$
with $x, y \in H$ and $\varepsilon > 0$, form a sub-base of weak-operator neighbourhoods of 0 in $\mathcal{B}(H)$.

- ▷ The ULTRAWEAK OPERATOR TOPOLOGY: the weakest topology on $\mathcal{B}(H)$ with respect to which the mapping $T \rightsquigarrow \sum_{n=1}^{\infty} \langle Tx_n, y_n\rangle$ is continuous for all $\mathbf{x}, \mathbf{y} \in H_\infty$; sets of the form
$$\left\{T \in \mathcal{B}(H) : \left|\sum_{n=1}^{\infty} \langle Tx_n, y_n\rangle\right| < \varepsilon\right\},$$
with $\mathbf{x}, \mathbf{y} \in H_\infty$ and $\varepsilon > 0$, form a sub-base of ultraweak-operator neighbourhoods of 0 in $\mathcal{B}(H)$.

These topologies are induced, respectively, by the seminorms of the form $T \rightsquigarrow |\langle Tx, y\rangle|$ with $x, y \in H$, and those of the form $T \rightsquigarrow |\sum_{n=1}^{\infty} \langle Tx_n, y_n\rangle|$ with $\mathbf{x}, \mathbf{y} \in H_\infty$.
An important theorem in classical operator algebra theory states that the weak-operator continuous linear functionals on (any linear subspace of) $\mathcal{B}(H)$ all have the form $T \rightsquigarrow \sum_{n=1}^{N} \langle Tx_n, y_n\rangle$ with $\mathbf{x}, \mathbf{y} \in H_N$ for some N; and the ultraweak-operator continuous linear functionals on $\mathcal{B}(H)$ have the form $T \rightsquigarrow \sum_{n=1}^{\infty} \langle Tx_n, y_n\rangle$, where $\mathbf{x}, \mathbf{y} \in H_\infty$. However, the classical proofs, such as those found in [10, 11, 14], depend on applications of nonconstructive versions of the Hahn-Banach theorem, the Riesz representation theorem, and polar decomposition.
The foregoing characterisation of ultraweak-operator continuous functionals was derived constructively, when H is separable, in [9].[2] A variant of it was derived in [8] (Proposition 5.4.16) without the requirement of separability, and using not the standard ultraweak operator topology but one that is classically, though not constructively, equivalent to it. Our aim in the present work is to provide a constructive proof of the standard classical characterisation of weak-operator continuous linear functionals (Theorem 10) on $\mathcal{B}(H)$, without the requirement of separability but with one hypothesis in addition to the classical ones. In presenting this work, we emphasise that, in contrast to their classical counterparts, our proofs contain extractable, implementable algorithms for the desired representation of weak-operator continuous linear functionals; moreover, the constructive proofs themselves verify that those algorithms meet their specifications.

[2]The characterisation was derived by Spitters in the case where H is separable and the subspace is $\mathcal{B}(H)$ itself ([15],Theorem 5); but his proof uses Brouwer's continuity principle and so is intuitionistic, rather than in the style of Bishop.

2 PRELIMINARY LEMMAS

The proof of our main theorem depends on a sequence of (at-times-complicated) lemmas. For the first one, we remind the reader of two elementary definitions in constructive analysis: we say that an inhabited set S—that is, one in which we can construct an element—is FINITELY ENUMERABLE if there exist a positive integer N and a mapping of $\{1, \ldots, N\}$ onto S; if that mapping is one-one, then S is called FINITE.

LEMMA 1 *If u is a weak-operator continuous linear functional on $\mathcal{B}(H)$, then there exist a finitely enumerable subset F of $H \times H$ and a positive number C such that $|u(T)| \leq C \sum_{(x,y)\in F} |\langle Tx, y\rangle|$ for all $T \in \mathcal{B}(H)$.*

PROOF. This is an immediate consequence of Proposition 5.4.1 in [8]. ∎

We shall need some information about locally convex spaces. Let $(p_j)_{j\in J}$ be a family of seminorms defining the topology on a locally convex linear space V, and let A be a subset of V. A subset S of A is said to be LOCATED (in A) if

$$\inf\left\{\sum_{j\in F} p_j(x-s) : s \in S\right\}$$

exists for each $x \in A$ and each finitely enumerable subset F of J. We say that A is TOTALLY BOUNDED if for each finitely enumerable subset F of J and each $\varepsilon > 0$, there exists a finitely enumerable subset S of A—called an ε-APPROXIMATION TO S RELATIVE TO $(p_j)_{j\in F}$—such that for each $x \in A$ there exists $s \in S$ with $\sum_{j\in F} p_j(x-s) < \varepsilon$.
The unit ball $\mathcal{B}_1(H)$ is weak-operator totally bounded ([8], Proposition 5.4.15); but, in contrast to the classical situation, it cannot be proved constructively that $\mathcal{B}_1(H)$ is weak-operator complete [5].
A mapping f between locally convex spaces $\left(X, (p_j)_{j\in J}\right)$ and $\left(Y, (q_k)_{k\in K}\right)$ is UNIFORMLY CONTINUOUS on a subset S of X if for each $\varepsilon > 0$ and each finitely enumerable subset G of K, there exist $\delta > 0$ and a finitely enumerable subset F of J such that if $x, x' \in S$ and $\sum_{j\in F} p_j(x-x') < \delta$, then $\sum_{k\in G} q_k\left(f(x) - f(x')\right) < \varepsilon$.
We recall four facts about total boundedness, locatedness, and uniform continuity in a locally convex space V. The proofs are found on pages 129–130 of [8].

- ▷ If f is a uniformly continuous mapping of a totally bounded subset A of V into a locally convex space, then $f(A)$ is totally bounded.
- ▷ If f is a uniformly continuous, real-valued mapping on a totally bounded subset A of V, then $\sup_{x\in A} f(x)$ and $\inf_{x\in A} f(x)$ exist.
- ▷ A totally bounded subset of V is located in V.

▷ If $A \subset V$ is totally bounded and $S \subset A$ is located in A, then S is totally bounded.

We remind the reader that a bounded linear mapping $T : X \to Y$ between normed linear spaces is NORMED if its NORM,

$$\|T\| \equiv \sup\{\|Tx\| : x \in X, \|x\| \leq 1\},$$

exists. If X is finite-dimensional, then $\|T\|$ exists; but the statement 'Every bounded linear functional on an infinite-dimensional Hilbert space is normed'[3] is essentially nonconstructive.

Lemma 2 *Every weak-operator continuous linear functional on $\mathcal{B}(H)$ is normed.*

Proof. This follows from observations made above, since, in view of Lemma 1, the linear functional is weak-operator uniformly continuous on the weak-operator totally bounded set $\mathcal{B}_1(H)$. ■

We note the following stronger form of Lemma 1.

Lemma 3 *Let u be a weak-operator continuous linear functional on $\mathcal{B}(H)$. Then there exist $\delta > 0$, and finitely many nonzero*[4] *elements $\xi_1, \ldots, \xi_N$ and $\zeta_1, \ldots, \zeta_N$ of H with $\sum_{n=1}^{N} \|\xi_n\|^2 = \sum_{n=1}^{N} \|\zeta_n\|^2 = 1$, such that $|u(T)| \leq \delta \sum_{n=1}^{N} |\langle T\xi_n, \zeta_n\rangle|$ for each $T \in \mathcal{B}(H)$.*

Proof. By Lemma 1, there exist a positive integer ν,$C > 0$, and vectors $\mathbf{x}, \mathbf{y} \in H_\nu$ such that $|u(T)| \leq C \sum_{n=1}^{\nu} |\langle Tx_n, y_n\rangle|$ for all $T \in \mathcal{B}(H)$.[5] For each $n \leq \nu$, construct nonzero vectors x_n', y_n' such that $x_n' \neq x_n$ and $y_n' \neq y_n$. The desired result follows from the inequality

$$\begin{aligned}\sum_{n=1}^{\nu} |\langle Tx_n, y_n\rangle| \leq \sum_{n=1}^{\nu} |\langle T(x_n - x_n'), y_n - y_n'\rangle| + \sum_{n=1}^{\nu} |\langle Tx_n', y_n - y_n'\rangle| \\ + \sum_{n=1}^{\nu} |\langle T(x_n - x_n'), y_n'\rangle| + \sum_{n=1}^{\nu} |\langle Tx_n', y_n'\rangle|\end{aligned}$$

the fact that each of the vectors $x_n', x_n - x_n', y_n'$, and $y_n - y_n'$ is nonzero, and scaling to get the desired norm sums equal to 1 and then the positive δ. ■

The next lemma will be used in an application of the separation theorem in the proof of Lemma 6.

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[3]In fact, a nonzero linear functional on a normed space is normed if and only its kernel is located ([8], Proposition 2.3.6).

[4]A vector in a locally convex space is NONZERO if it is mapped to a positive number by at least one seminorm.

[5]At this stage, it is trivial to prove Lemma 3 classically by simply deleting terms $\langle Tx_n, y_n\rangle$ when either x_n or y_n is 0. With intuitionistic logic we need to work a little harder, because we cannot generally decide whether a given vector in H is, or is not, equal to 0.

Lemma 4 *Let $\zeta_1, \ldots, \zeta_N$ be elements of H with $\sum_{n=1}^{N} \|\zeta_n\|^2 = 1$. Let K be a finite-dimensional subspace of H_N, and let $\| \|^*$ be the standard norm on the dual space K^* of K:*

$$\|f\|^* = \sup \{|f(\mathbf{x})| : \mathbf{x} \in K, \|\mathbf{x}\| \leq 1\} \quad (f \in K^*) .$$

Define a mapping F of $\mathcal{B}(H)$ into $\left(K^*, \| \|^*\right)$ by

$$F(T)(\mathbf{x}) \equiv \sum_{n=1}^{N} \langle T x_n, \zeta_n \rangle \quad (\mathbf{x} \in K) .$$

Then F is weak-operator uniformly continuous on $\mathcal{B}_1(H)$.

Proof. Given $\varepsilon > 0$, let $\{\mathbf{x}_1, \ldots, \mathbf{x}_m\}$ be an ε-approximation to the (compact) unit ball of K. Writing $\mathbf{x}_i = (x_{i,1}, \ldots, x_{i,N})$, consider $S, T \in \mathcal{B}_1(H)$ with

$$\sum_{i=1}^{m} \sum_{n=1}^{N} |\langle (S-T) x_{i,n}, \zeta_n \rangle| < \varepsilon.$$

For each $\mathbf{x}$ in the unit ball of K, there exists i such that $\|\mathbf{x} - \mathbf{x}_i\| < \varepsilon$. We compute

$$\begin{aligned} |F(S)(\mathbf{x}) - F(T)(\mathbf{x})| &\leq |F(S)(\mathbf{x}) - F(S)(\mathbf{x}_i)| + |F(S)(\mathbf{x}_i) - F(T)(\mathbf{x}_i)| \\ &\quad + |F(T)(\mathbf{x}) - F(T)(\mathbf{x}_i)| \\ &\leq \sum_{n=1}^{N} |\langle S(x_n - x_{i,n}), \zeta_n \rangle| + \sum_{n=1}^{N} |\langle (S-T) x_{i,n}, \zeta_n \rangle| \\ &\quad + \sum_{n=1}^{N} |\langle T(x_n - x_{i,n}), \zeta_n \rangle| \\ &\leq 2 \sum_{n=1}^{N} \|x_n - x_{i,n}\| \|\zeta_n\| + \varepsilon \\ &\leq 2 \|\mathbf{x} - \mathbf{x}_i\| \|\zeta\| + \varepsilon < 3\varepsilon. \end{aligned}$$

Hence $\|F(S) - F(T)\|^* \leq 3\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we conclude that F is uniformly continuous on $\mathcal{B}_1(H)$. ∎

In order to ensure that the unit kernel $\mathcal{B}_1(H) \cap \ker u$ of a weak-operator continuous linear functional u on $\mathcal{B}(H)$ is weak-operator totally bounded, and hence weak-operator located, we derive a generalisation of Lemma 5.4.9 of [8].

Lemma 5 *Let $\left(V, (p_j)_{j \in J}\right)$ be a locally convex space. Let V_1 be a balanced, convex, and totally bounded subset of V. Let u be a linear functional on V that, on V_1, is both uniformly continuous and nonzero. Then $V_1 \cap \ker u$ is totally bounded.*

Proof. Since u is nonzero and uniformly continuous on the totally bounded set V_1,

$$C = \sup\{|u(y)| : y \in V_1\}$$

exists and is positive. Choose y_1 in V_1 such that $u(y_1) > C/2$. Then

$$y_0 \equiv \frac{C}{2u(y_1)} y_1$$

belongs to the balanced set V_1, and $u(y_0) = C/2$. Let $\varepsilon > 0$, and let F be a finitely enumerable subset of J. Since each p_j is uniformly continuous on V, it maps the totally bounded set V_1 onto a totally bounded subset of $\mathbf{R}$.[6] Hence there exists $b > 0$ such that $p_j(x) \leq b$ for each $j \in F$ and each $x \in V_1$. Using Theorem 5.4.6 of [8], compute t with

$$0 < t < \frac{C\varepsilon}{C + 4b}$$

such that

$$S_t = \{y \in V_1 : |u(y)| \leq t\}$$

is totally bounded. Pick a t-approximation $\{s_1, \ldots, s_n\}$ of S_t relative to $(p_j)_{j\in F}$, and set

$$y_k = \frac{C}{C + 2t} s_k - \frac{2}{C + 2t} u(s_k) y_0 \quad (1 \leq k \leq n).$$

Then $y_k \in \ker(u)$. Since $|u(s_k)| \leq t$ and V_1 is balanced,

$$\frac{-u(s_k)}{t} y_0 \in V_1.$$

Thus

$$y_k = \frac{C}{C + 2t} s_k + \left(1 - \frac{C}{C + 2t}\right) \left(\frac{-u(s_k)}{t} y_0\right) \in V_1.$$

[6] We use $\mathbf{R}$ and $\mathbf{C}$ for the sets of real and complex numbers, respectively.

Now consider any element y of $V_1 \cap \ker(u)$. Since $y \in S_t$, there exists k such that $\sum_{j\in F} p_j(y-s_k) < t$ and therefore

$$\begin{aligned}\sum_{j\in F} p_j(y-y_k) &\leq \sum_{j\in F} p_j(y-s_k) + \sum_{j\in F} p_j(s_k-y_k) \\ &< t + \frac{2}{C+2t}\sum_{j\in F} p_j(ts_k + u(s_k)y_0) \\ &\leq t + \frac{2}{C+2t}\sum_{j\in F} (tp_j(s_k) + u(s_k)p_j(y_0)) \\ &\leq t + \frac{2t}{C}\sum_{j\in F} (p_j(s_k) + p_j(y_0)) \\ &\leq t\left(1+\frac{4b}{C}\right) < \varepsilon.\end{aligned}$$

Thus $\{y_1,\ldots,y_n\}$ is a finitely enumerable ε-approximation to $V_1 \cap \ker(u)$ relative to the family $(p_j)_{j\in F}$ of seminorms. ∎

The next lemma, the most complicated in the paper, extracts much of the sting from the proof of our main theorem by showing how to find finitely many mappings of the form $T \rightsquigarrow \langle Tx,\zeta\rangle$ whose sum is small on the unit kernel of u.

Lemma 6 *Let u be a nonzero weak-operator continuous linear functional on $\mathcal{B}(H)$. Let δ be a positive number, and $\xi_1,\ldots,\xi_N$ and $\zeta_1,\ldots,\zeta_N$ nonzero elements of H, such that*[7]

$$\sum_{n=1}^{N} \|\xi_n\|^2 = \sum_{n=1}^{N} \|\zeta_n\|^2 = 1,$$

and

$$|u(T)| \leq \delta \sum_{n=1}^{N} |\langle T\xi_n,\zeta_n\rangle| \quad (T \in \mathcal{B}(H)). \tag{1}$$

Then for each $\varepsilon > 0$, there exists a unit vector $\mathbf{x}$ in the subspace

$$K \equiv \mathbf{C}\xi_1 \times \mathbf{C}\xi_2 \times \cdots \times \mathbf{C}\xi_N$$

of H_N, such that $x_n \neq 0$ for $1 \leq n \leq N$ and $\left|\sum_{n=1}^{N} \langle Tx_n,\zeta_n\rangle\right| < \varepsilon$ for all $T \in \mathcal{B}_1(H) \cap \ker u$.

Proof. First note that since each ξ_n is nonzero, K is an N-dimensional subspace of H_N. Now, an application of Lemma 5 tells us that the unit kernel

[7] Such ξ_k, ζ_k, and δ exist, by Lemma 3.

$\mathcal{B}_1(H) \cap \ker u$ of u is weak-operator totally bounded. For each $\mathbf{x} \in H_N$, since the mapping $T \rightsquigarrow \sum_{n=1}^{N} \langle Tx_n, \zeta_n \rangle$ is weak-operator uniformly continuous on the unit kernel, we see that

$$\|\mathbf{x}\|_0 = \sup \left\{ \left| \sum_{n=1}^{N} \langle Tx_n, \zeta_n \rangle \right| : T \in \mathcal{B}_1(H) \cap \ker u \right\}$$

exists. The mapping $\mathbf{x} \rightsquigarrow \|\mathbf{x}\|_0$ is a seminorm on H_N satisfying $\|\mathbf{x}\|_0 \leq \|\zeta\| \, \|\mathbf{x}\| = \|\mathbf{x}\|$; whence the identity mapping from $(H_N, \| \, \|)$ to $(H_N, \| \, \|_0)$ is uniformly continuous. Since the subset

$$\{\mathbf{x} \in K : \|\mathbf{x}\| = 1\}$$

of the finite-dimensional Banach space $(K, \| \, \|)$ is totally bounded, it follows that

$$\beta \equiv \inf \left\{ \|\mathbf{x}\|_0 : \mathbf{x} \in K, \|\mathbf{x}\| = 1 \right\},$$

exists. It will suffice to prove that $\beta = 0$. For then, given ε with $0 < \varepsilon < 1$, we can construct a unit vector $\mathbf{x}' \in K$ such that $\left| \sum_{n=1}^{N} \langle Tx'_n, \zeta_n \rangle \right| < \varepsilon/2$ for all $T \in \mathcal{B}_1(H) \cap \ker u$. Picking nonzero vectors $y_n \in \mathbf{C}\xi_n$ such that $\left(\sum_{n=1}^{N} \|x'_n - y_n\|^2 \right)^{1/2} < \varepsilon/8$, we have

$$\left| 1 - \left(\sum_{n=1}^{N} \|y_n\|^2 \right)^{1/2} \right| < \frac{\varepsilon}{8},$$

so

$$\mathbf{x} \equiv \left(\sum_{n=1}^{N} \|y_n\|^2 \right)^{-1/2} \mathbf{y}$$

is a unit vector in $\mathbf{C}\xi_1 \times \cdots \times \mathbf{C}\xi_N$ with each $x_n \neq 0$. Moreover,

$$\begin{aligned} \|\mathbf{x} - \mathbf{y}\|^2 &= \sum_{k=1}^{N} \left| \left(\sum_{n=1}^{N} \|y_n\|^2 \right)^{-1/2} - 1 \right|^2 \|y_k\|^2 \\ &\leq \left(\frac{\varepsilon}{8} \right)^2 \sum_{k=1}^{N} \|y_k\|^2 \\ &\leq \frac{\varepsilon^2}{64} \left(1 + \frac{\varepsilon}{8} \right)^2 < \frac{\varepsilon^2}{16}, \end{aligned}$$

so for each $T \in \mathcal{B}_1(H) \cap \ker u$,

$$\begin{aligned}\left|\sum_{n=1}^{N} \langle Tx_n, \zeta_n \rangle\right| &\leq \left|\sum_{n=1}^{N} \langle Tx'_n, \zeta_n \rangle\right| + \sum_{n=1}^{N} \left|\langle T\left(x_n - x'_n\right), \zeta_n \rangle\right| \\ &\leq \frac{\varepsilon}{2} + \sum_{n=1}^{N} \|x_n - x'_n\| \, \|\zeta_n\| \\ &\leq \frac{\varepsilon}{2} + \|\mathbf{x} - \mathbf{x}'\| \, \|\zeta\| \\ &\leq \frac{\varepsilon}{2} + \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y} - \mathbf{x}'\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{8} < \varepsilon.\end{aligned}$$

To prove that $\beta = 0$, we suppose that $\beta > 0$. Then $\| \, \|_0$ is a norm equivalent to the original norm on K, so $(K, \| \, \|_0)$ is an N-dimensional Banach space. Define norms $\| \, \|^*$ and $\| \, \|_0^*$ on the dual K^* of K by

$$\begin{aligned}\|f\|^* &\equiv \sup\left\{|f(\mathbf{x})| : \mathbf{x} \in K, \|\mathbf{x}\| \leq 1\right\}, \\ \|f\|_0^* &\equiv \sup\left\{|f(\mathbf{x})| : \mathbf{x} \in K, \|\mathbf{x}\|_0 \leq 1\right\}.\end{aligned}$$

For each $T \in \mathcal{B}(H)$ and each $\mathbf{x} \in K$ let

$$F(T)\,(\mathbf{x}) \equiv \sum_{n=1}^{N} \langle Tx_n, \zeta_n \rangle .$$

Then, by Lemma 4, F is weak-operator uniformly continuous as a mapping of $\mathcal{B}_1(H)$ into $\left(K^*, \| \, \|^*\right)$; since the norms $\| \, \|^*$ and $\| \, \|_0^*$ are equivalent on the finite-dimensional dual space K^*, F is therefore weak-operator uniformly continuous as a mapping of $\mathcal{B}_1(H)$ into $\left(K^*, \, \| \, \|_0^*\right)$. Hence

$$D = F(\mathcal{B}_1(H) \cap \ker u)$$

is a totally bounded, and therefore located, subset of $\left(K^*, \| \, \|_0^*\right)$. Moreover, for each $T \in \mathcal{B}_1(H) \cap \ker u$ and each $\mathbf{x} \in K$, $|F(T)\,(\mathbf{x})| \leq \|\mathbf{x}\|_0$; so D is a subset of the unit ball S_0^* of $\left(K^*, \, \| \, \|_0^*\right)$. We shall use the separation theorem from functional analysis to prove that D is $\| \, \|_0^*$-dense in S_0^*. Consider any ϕ in S_0^*, and suppose that

$$0 < d = \inf\left\{\|\phi - F(T)\|_0^* : T \in \mathcal{B}_1(H) \cap \ker u\right\}.$$

Now, D is bounded, convex, balanced, and located; so, by Corollary 5.2.10 of [8], there exists a linear functional v on $\left(K^*, \| \, \|_0^*\right)$ with norm 1 such that

$$v(\phi) > |v(F(T))| + \frac{d}{2} \qquad (T \in \mathcal{B}_1(H) \cap \ker u).$$

It is a simple exercise[8] to show that since $\left(K^{*}, \| \|_0^*\right)$ is N-dimensional, there exists $\mathbf{y} \in K$ such that $\|\mathbf{y}\|_0 = 1$ and $v(f) = f(\mathbf{y})$ for each $f \in K^*$. Hence

$$\begin{aligned}\phi(\mathbf{y}) &\geq \sup\{|F(T)(\mathbf{y})| : T \in \mathcal{B}_1(H) \cap \ker u\} + \frac{d}{2} \\ &> \sup\left\{\left|\sum_{n=1}^{N} \langle T y_n, \zeta_n\rangle\right| : T \in \mathcal{B}_1(H) \cap \ker u\right\} = \|\mathbf{y}\|_0,\end{aligned}$$

which contradicts the fact that $\phi \in S_0^*$. We conclude that $d = 0$ and therefore that D is $\| \|_0^*$-dense in S_0^*.
Continuing our proof that $\beta = 0$, pick $T_0 \in \mathcal{B}_1(H)$ with $u(T_0) > 0$. Replacing u by $u(T_0)^{-1}u$ if necessary, we may assume that $u(T_0) = 1$. Define a linear functional Ψ on $(K, \| \|_0)$ by setting

$$\Psi(\mathbf{x}) = \beta \sum_{n=1}^{N} \langle T_0 x_n, \zeta_n\rangle \quad (\mathbf{x} \in K)\ .$$

Note that for $\mathbf{x} \in K$ we have

$$|\Psi(\mathbf{x})| \leq \beta \sum_{n=1}^{N} \|x_n\| \, \|\zeta_n\| \leq \beta \, \|\mathbf{x}\| \, \|\zeta\| \leq \|\mathbf{x}\|_0.$$

Hence $\Psi \in S_0^*$. By the work of the previous paragraph, we can find $T \in \mathcal{B}_1(H) \cap \ker u$ such that $\|\Psi - F(T)\|_0^* < \beta/2\delta$. In particular, since $\|\xi\|_0 \leq \|\xi\| = 1$,

$$\left|\sum_{n=1}^{N} \langle(\beta T_0 - T)\, \xi_n, \zeta_n\rangle\right| < \frac{\beta}{2\delta}. \tag{2}$$

In order to apply the defining property of δ and thereby obtain a contradiction, we need to estimate not the sum on the left hand side of (2), but $\sum_{n=1}^{N} |\langle(\beta T_0 - T)\, \xi_n, \zeta_n\rangle|$. To do so, we write

$$\{n : 1 \leq n \leq N\} = P \cup Q,$$

where P, Q are disjoint sets,

$$n \in P \Rightarrow \langle(\beta T_0 - T)\, \xi_n, \zeta_n\rangle \neq 0, \text{ and}$$

$$n \in Q \Rightarrow |\langle(\beta T_0 - T)\, \xi_n, \zeta_n\rangle| < \frac{\beta}{2\delta N}.$$

If $n \in P$, we set

$$\lambda_n = \frac{1}{\langle(\beta T_0 - T)\, \xi_n, \zeta_n\rangle} \, |\langle(\beta T_0 - T)\, \xi_n, \zeta_n\rangle|,$$

[8] Alternatively, we can refer to [3] (page 287, Theorem 10) or [8] (Theorem 5.4.14).

and if $n \in Q$, we set $\lambda_n = 0$; in each case, we define $\gamma_n \equiv \lambda_n \xi_n$. Then $\gamma \equiv (\gamma_1, \ldots, \gamma_N) \in K$ and

$$\|\gamma\|_0^2 \leq \|\gamma\|^2 = \sum_{n=1}^N |\lambda_n|^2 \|\xi_n\|^2 \leq \|\xi\|^2 = 1.$$

Hence

$$\left| \sum_{n=1}^N \langle (\beta T_0 - T) \gamma_n, \zeta_n \rangle \right| \leq \|\Psi - F(T)\|_0^* < \frac{\beta}{2\delta}.$$

Moreover,

$$\begin{aligned} \left| \sum_{n=1}^N \langle (\beta T_0 - T) \gamma_n, \zeta_n \rangle \right| &= \left| \sum_{n \in P} \langle (\beta T_0 - T) \lambda_n \xi_n, \zeta_n \rangle \right| \\ &= \sum_{n \in P} |\langle (\beta T_0 - T) \xi_n, \zeta_n \rangle| , \end{aligned}$$

so

$$\begin{aligned} &\sum_{n=1}^N |\langle (\beta T_0 - T) \xi_n, \zeta_n \rangle| \\ &= \sum_{n \in P} |\langle (\beta T_0 - T) \xi_n, \zeta_n \rangle| + \sum_{n \in Q} |\langle (\beta T_0 - T) \xi_n, \zeta_n \rangle| \\ &\leq \left| \sum_{n=1}^N \langle (\beta T_0 - T) \gamma_n, \zeta_n \rangle \right| + N \left(\frac{\beta}{2\delta N} \right) < \frac{\beta}{\delta} \end{aligned}$$

and therefore $u(\beta T_0 - T) < \beta$. But $u(\beta T_0 - T) = \beta u(T_0) - u(T) = \beta$, a contradiction which ensures that β actually equals 0. ∎

We shall apply Lemma 6 shortly; but its application requires another construction.

Lemma 7 *Let N be a positive integer, let $\xi_1, \ldots, \xi_N$ be linearly independent vectors in H, and let $\zeta_1, \ldots, \zeta_N$ be nonzero elements of H, such that $\sum_{n=1}^N \|\xi_n\|^2 = \sum_{n=1}^N \|\zeta_n\|^2 = 1$. Then there exists a positive number c with the following property: for each unit vector $\mathbf{z}$ in the subspace*

$$K \equiv \mathbf{C}\xi_1 \times \cdots \times \mathbf{C}\xi_N,$$

there exists $T \in \mathcal{B}_1(H)$ such that $\sum_{n=1}^N \langle T z_n, \zeta_n \rangle > c$.

Proof. Let

$$m \equiv \inf \left\{ \|\zeta_n\|^2 : 1 \leq n \leq N \right\} > 0.$$

Define a norm on the N-dimensional span V of $\{\xi_1, \dots, \xi_N\}$ by

$$\left\| \sum_{n=1}^{N} \alpha_n \xi_n \right\|_1 \equiv \max_{1 \leq n \leq N} |\alpha_n| .$$

Since V is finite-dimensional, there exists $b > 0$ such that $\|\mathbf{x}\|_1 \leq b \|\mathbf{x}\|$ for each $\mathbf{x} \in V$. Let $\mathbf{z} \equiv (\lambda_1 \xi_1, \dots, \lambda_N \xi_N)$ in H_N satisfy $\|\mathbf{z}\| = 1$. If $|\lambda_n| < 1/\sqrt{N}$ for each n, then

$$1 = \sum_{n=1}^{N} \|\lambda_n \xi_n\|^2 = \sum_{n=1}^{N} |\lambda_n|^2 \|\xi_n\|^2 < \sum_{n=1}^{N} \left(\frac{1}{\sqrt{N}} \right)^2 = 1,$$

which is absurd. Hence we can pick ν such that $|\lambda_\nu| > 1/\sqrt{2N}$. Define a linear mapping T on H such that

$$T\xi_\nu = \frac{\lambda_\nu^*}{b |\lambda_\nu|} \zeta_\nu, \quad T\xi_n = 0 \ \ (n \neq \nu),$$

and $Tx = 0$ whenever x is orthogonal to V. Then

$$\left\| T \left(\sum_{n=1}^{N} \alpha_n \xi_n \right) \right\| = \frac{|\lambda_\nu^*|}{b |\lambda_\nu|} |\alpha_\nu| \leq \frac{1}{b} \left\| \sum_{n=1}^{N} \alpha_n \xi_n \right\|_1 \leq \left\| \sum_{n=1}^{N} \alpha_n \xi_n \right\|,$$

so $T \in \mathcal{B}_1(H)$. Moreover,

$$\langle Tz_n, \zeta_n \rangle = \begin{cases} 0 & \text{if } n \neq \nu \\ \frac{1}{b} |\lambda_\nu| \|\zeta_\nu\|^2 & \text{if } n = \nu, \end{cases}$$

so

$$\sum_{n=1}^{N} \langle Tz_n, \zeta_n \rangle = \frac{1}{b} |\lambda_\nu| \|\zeta_\nu\|^2 > \frac{m}{b\sqrt{2N}}.$$

It remains to take $c \equiv m/b\sqrt{2N}$. ∎

The next lemma takes the information arising from the preceding two, and shows that when the vectors ξ_n in (1) are linearly independent, we can approximate u by a finite sum of mappings of the form $T \rightsquigarrow \langle Tx, y \rangle$, not just on its unit kernel but on the entire unit ball of $\mathcal{B}(H)$. At the same time, we produce a priori bounds on the sums of squares of the norms of the components of the vectors x, y that appear in the terms $\langle Tx, y \rangle$ whose sum approximates $u(T)$. Those bounds will be needed in the proof of our characterisation theorem.

Lemma 8 *Let H be a Hilbert space, and u a nonzero weak-operator continuous linear functional on $\mathcal{B}(H)$. Let δ be a positive number, $\xi_1, \dots, \xi_N$ linearly independent vectors in H, and $\zeta_1, \cdots, \zeta_N$ nonzero vectors in H, such that*

$$\sum_{n=1}^{N} \|\xi_n\|^2 = \sum_{n=1}^{N} \|\zeta_n\|^2 = 1$$

and (1) *holds. Let* $c > 0$ *be as in* Lemma 7.*Then for each* $\varepsilon > 0$, *there exists* $\mathbf{x} \in \mathbf{C}\xi_1 \times \cdots \times \mathbf{C}\xi_N$ *such that* $x_n \neq 0$ *for each* n,

$$\|\mathbf{x}\| < \frac{2\,\|u\|}{c},$$

and

$$\left| u(T) - \sum_{n=1}^{N} \langle Tx_n, \zeta_n \rangle \right| < \varepsilon$$

for all $T \in \mathcal{B}_1(H)$.

Proof. Pick $T_0 \in \mathcal{B}_1(H)$ with $u(T_0) > 0$. To begin with, take the case where $u(T_0) = 1$ and therefore $\|u\| \geq 1$. Given $\varepsilon > 0$, set

$$\alpha \equiv \frac{\min\{\varepsilon, 1\}}{2\,\|u\|\,(1 + \|u\|)}.$$

Applying Lemma 6, we obtain nonzero vectors $z_n \in \mathbf{C}\xi_n$ $(1 \leq n \leq N)$ such that $\sum_{n=1}^{N} \|z_n\|^2 = 1$ and

$$\left| \sum_{n=1}^{N} \langle Tz_n, \zeta_n \rangle \right| < c\alpha \quad (T \in \mathcal{B}_1(H) \cap \ker u)\,.$$

For each $T \in \mathcal{B}_1(H)$, since

$$(1 + \|u\|)^{-1}\,(T - u(T)T_0) \in \mathcal{B}_1(H) \cap \ker u,$$

we have

$$\left| \sum_{n=1}^{N} \langle (T - u(T)T_0)\, z_n, \zeta_n \rangle \right| < (1 + \|u\|)\, c\alpha.$$

By Lemma 7, there exists $T_1 \in \mathcal{B}_1(H)$ such that $\sum_{n=1}^{N} \langle T_1 z_n, \zeta_n \rangle > c$. We compute

$$\begin{aligned} c &< \sum_{n=1}^{N} \langle T_1 z_n, \zeta_n \rangle \\ &\leq \left| \sum_{n=1}^{N} \langle (T_1 - u(T_1)T_0)\, z_n, \zeta_n \rangle \right| \quad + |u(T_1)| \left| \sum_{n=1}^{N} \langle T_0 z_n, \zeta_n \rangle \right| \\ &\leq (1 + \|u\|)c\alpha + \|u\| \left| \sum_{n=1}^{N} \langle T_0 z_n, \zeta_n \rangle \right|. \end{aligned}$$

Hence

$$\begin{aligned} \left| \sum_{n=1}^{N} \langle T_0 z_n, \zeta_n \rangle \right| &> \frac{c}{\|u\|}\,(1 - (1 + \|u\|)\,\alpha) \\ &\geq \frac{c}{\|u\|}\left(1 - \frac{1}{2\,\|u\|}\right) > \frac{c}{2\,\|u\|}, \end{aligned}$$

since $\|u\| \geq 1$. Setting

$$\mathbf{x} \equiv \left(\sum_{n=1}^{N} \langle T_0 z_n, \zeta_n \rangle \right)^{-1} \mathbf{z},$$

we have $0 \neq x_n \in \mathbf{C}\xi_n$ for each n, and

$$\|\mathbf{x}\| = \left| \sum_{n=1}^{N} \langle T_0 z_n, \zeta_n \rangle \right|^{-1} \|\mathbf{z}\| < \frac{2\|u\|}{c}.$$

Moreover, for each $T \in \mathcal{B}_1(H)$,

$$\begin{aligned}\left| u(T) - \sum_{n=1}^{N} \langle T x_n, \zeta_n \rangle \right| &= \left| \sum_{n=1}^{N} \langle T_0 z_n, \zeta_n \rangle \right|^{-1} \left| \sum_{n=1}^{N} \langle (u(T)T_0 - T) z_n, \zeta_n \rangle \right| \\ &< \frac{2\|u\|}{c}(1 + \|u\|)c\alpha \leq \varepsilon.\end{aligned}$$

We now remove the restriction that $u(T_0) = 1$. Applying the first part of the theorem to $v \equiv u(T_0)^{-1}u$, we construct $\mathbf{y} \in K$ such that each component $y_n \neq 0$, $\|\mathbf{y}\| \leq 2\|v\|/c$, and

$$\left| v(T) - \sum_{n=1}^{N} \langle T y_n, \zeta_n \rangle \right| < u(T_0)^{-1}\varepsilon,$$

and we obtain the desired conclusion by taking $\mathbf{x} \equiv u(T_0)\mathbf{y}$. ∎

Lemma 9 *Under the hypotheses of* Lemma 8, but without the assumption that u is nonzero, *for all* $\varepsilon, \varepsilon' > 0$, *there exists* $\mathbf{x} \in \mathbf{C}\xi_1 \times \cdots \times \mathbf{C}\xi_N$ *such that* $x_n \neq 0$ *for each* n,

$$\|\mathbf{x}\| < \frac{2(\|u\| + \varepsilon')}{c},$$

and

$$\left| u(T) - \sum_{n=1}^{N} \langle T x_n, \zeta_n \rangle \right| < \varepsilon$$

for all $T \in \mathcal{B}_1(H)$.

Proof. Either $\|u\| > 0$ and we can apply Lemma 8, or else $\|u\| < \varepsilon/2$. In the latter event, pick $\mathbf{x}$ in $\mathbf{C}\xi_1 \times \cdots \times \mathbf{C}\xi_N$ such that $x_n \neq 0$ for each n and

$$\|\mathbf{x}\| < \min\left\{ \frac{\varepsilon}{2}, \frac{2(\|u\| + \varepsilon')}{c} \right\}.$$

Then for each $T \in \mathcal{B}_1(H)$ we have

$$\left| \sum_{n=1}^{N} \langle T x_n, \zeta_n \rangle \right| \leq \sum_{n=1}^{N} \|x_n\| \|\zeta_n\| \leq \|\mathbf{x}\| \|\zeta\| < \frac{\varepsilon}{2}$$

and therefore

$$\left|u(T) - \sum_{n=1}^{N} \langle Tx_n, \zeta_n \rangle\right| \leq \|u\| + \left|\sum_{n=1}^{N} \langle Tx_n, \zeta_n \rangle\right| < \varepsilon.$$

■

3 The Characterisation Theorem

We are finally able to prove our main result, by inductively applying Lemma 9.

Theorem 10 *Let H be a nontrivial Hilbert space, and u a nonzero weak-operator continuous linear functional on $\mathcal{B}(H)$. Let δ be a positive number, $\xi_1, \ldots, \xi_N$ linearly independent vectors in H,*[9] *and $\zeta_1, \cdots, \zeta_N$ nonzero vectors in H, such that $|u(T)| \leq \delta \sum_{n=1}^{N} |\langle T\xi_n, \zeta_n \rangle|$ for all $T \in \mathcal{B}(H)$. Then there exists $\mathbf{x} \in \mathbf{C}\xi_1 \times \cdots \times \mathbf{C}\xi_N$ such that*

$$u(T) = \sum_{n=1}^{N} \langle Tx_n, \zeta_n \rangle \tag{3}$$

for all $T \in \mathcal{B}(H)$.

Proof. Re-scaling if necessary, we may assume that $\|u\| < 2^{-3}$. In the notation of, and using, Lemma 9, compute $\mathbf{x}^{(1)}$ in $K \equiv \mathbf{C}\xi_1 \times \cdots \times \mathbf{C}\xi_N$ such that[10]

$$\left\|\mathbf{x}^{(1)}\right\| \leq \frac{2}{c}\left(\|u\| + 2^{-3}\right) < \frac{1}{2c}$$

and

$$\left|u(T) - \sum_{n=1}^{N} \left\langle Tx_n^{(1)}, \zeta_n \right\rangle\right| < 2^{-4} \quad (T \in \mathcal{B}_1(H)).$$

Suppose that for some positive integer k we have constructed vectors $\mathbf{x}^{(i)} \in K$ $(1 \leq i \leq k)$ such that

$$\left\|\mathbf{x}^{(k)}\right\| < \frac{1}{2^k c}, \tag{4}$$

[9]The requirement that the vectors ξ_n be linearly independent is the one place where we have a stronger hypothesis than is needed in the classical theorem. It is worth noting here that *if* $u(T)$ has the desired form $\sum_{n=1}^{N} \langle T\xi_n, \zeta_n \rangle$, then *classically* we can find a set F of indices $n \leq N$ such that (i) the set S of those ξ_n with $n \in F$ is linearly independent and (ii) if $\xi_k \notin S$, then ξ_k is linearly dependent on S. We can then write

$$u(T) = \sum_{n \in F} \langle T\xi_n, \lambda_n \zeta_n \rangle,$$

with each $\lambda_n \in \mathbf{C}$. Constructively, this is not possible, since we cannot necessarily determine whether or not ξ_n is linearly dependent on the vectors $\xi_1, \ldots, \xi_{n-1}$.

[10]In this proof we do not need the fact that, according to Lemma 9, we can arrange for the components of the vector $\mathbf{x}^{(1)}$, and of the subsequently constructed vectors $\mathbf{x}^{(k)}$, to be nonzero.

and

$$\left|u(T)-\sum_{n=1}^{N}\left\langle T\left(x_n^{(1)}+\cdots+x_n^{(k)}\right),\zeta_n\right\rangle\right|<2^{-k-3}\quad (T\in\mathcal{B}_1(H)). \tag{5}$$

Consider the weak-operator continuous linear functional

$$v: T \rightsquigarrow u(T)-\sum_{n=1}^{N}\left\langle T\left(x_n^{(1)}+\cdots+x_n^{(k)}\right),\zeta_n\right\rangle$$

on $\mathcal{B}(H)$. Writing

$$x_n^{(1)}+\cdots+x_n^{(k)}=\lambda_n\xi_n$$

and

$$\gamma\equiv\max\{|\lambda_1|,\ldots,|\lambda_n|\},$$

for each $T\in\mathcal{B}(H)$ we have

$$\begin{aligned}|v(T)| &\leq |u(T)|+\sum_{n=1}^{N}\left|\left\langle T\left(x_n^{(1)}+\cdots+x_n^{(k)}\right),\zeta_n\right\rangle\right| \\ &\leq \delta\sum_{n=1}^{N}|\langle T\xi_n,\zeta_n\rangle|+\sum_{n=1}^{N}|\lambda_n|\,|\langle T\xi_n,\zeta_n\rangle| \\ &\leq (\delta+\gamma)\sum_{n=1}^{N}|\langle T\xi_n,\zeta_n\rangle|.\end{aligned}$$

We can now apply Lemma 9, to obtain

$$\mathbf{x}^{(k+1)}=\left(x_1^{(k+1)},\ldots,x_N^{(k+1)}\right)\in K$$

such that

$$\left\|\mathbf{x}^{(k+1)}\right\|<\frac{2}{c}\left(\|\nu\|+2^{-k-3}\right)<\frac{1}{2^{k+1}c}$$

and

$$\begin{aligned}&\left|u(T)-\sum_{n=1}^{N}\left\langle T\left(x_n^{(1)}+\cdots+x_n^{(k)}+x_n^{(k+1)}\right),\zeta_n\right\rangle\right| \\ &=\left|v(T)-\sum_{n=1}^{N}\left\langle Tx_n^{(k+1)},\zeta_n\right\rangle\right|<2^{-k-4}\end{aligned}$$

for all $T\in\mathcal{B}_1(H)$. This completes the inductive construction of a sequence $\left(\mathbf{x}^{(k)}\right)_{k\geq1}$ in K such that (4) and (5) hold for each k. The series $\sum_{k=1}^{\infty}\mathbf{x}^{(k)}$ converges to a sum $\mathbf{x}$ in the finite-dimensional Banach space K, by comparison with $\sum_{k=1}^{\infty}2^{-k}c^{-1}$. Letting $k\to\infty$ in (5), we obtain (3) for all $T\in\mathcal{B}_1(H)$ and hence for all $T\in\mathcal{B}(H)$. ∎

For nonzero u, the proof of our theorem can be simplified at each stage of the induction, since we can use Lemma 8 directly. If H has dimension $> N$, we can then construct the classical representation of u in the general case as follows. Either $\|u\| > 0$ and there is nothing to prove, or else $\|u\| < \delta$ (the same δ as in the statement of the theorem). In the latter case, we construct a unit vector ξ_{N+1} orthogonal to each of the vectors ξ_n $(1 \leq n \leq N)$, set $\zeta_{N+1} = \xi_{N+1}$, and consider the weak-operator continuous linear functional

$$v : T \rightsquigarrow u(T) + \delta \left\langle T\xi_{N+1}, \zeta_{N+1} \right\rangle .$$

We have

$$|v(T)| \leq |u(T)| + \delta \left|\left\langle T\xi_{N+1}, \zeta_{N+1} \right\rangle\right| \leq \delta \sum_{n=1}^{N+1} \left|\left\langle T\xi_n, \zeta_n \right\rangle\right| .$$

Moreover,

$$|v(I)| \geq \delta \left\|\xi_{N+1}\right\|^2 - |u(I)| \geq \delta - \|u\| > 0,$$

where I is the identity operator on H; so v is nonzero. We can therefore apply the nonzero case to v, to produce a vector $\mathbf{y} \in \mathbf{C}\xi_1 \times \cdots \times \mathbf{C}\xi_{N+1}$ such that

$$v(T) = \sum_{n=1}^{N+1} \left\langle Ty_n, \zeta_n \right\rangle \qquad (T \in \mathcal{B}(H)) .$$

Setting $x_n = y_n$ $(1 \leq n \leq N)$ and $x_{n+1} = y_{N+1} - \delta\xi_{N+1}$, we obtain

$$u(T) = \sum_{n=1}^{N+1} \left\langle Tx_n, \zeta_n \right\rangle$$

for each $T \in \mathcal{B}(H)$. Note, however, that this proof gives $\mathbf{x}$ in $\mathbf{C}\xi_1 \times \cdots \times \mathbf{C}\xi_N \times \mathbf{C}\xi_{N+1}$, not, as in Theorem 10, in $\mathbf{C}\xi_1 \times \cdots \times \mathbf{C}\xi_N$.

As an immediate consequence of Theorem 10, the functional u therein is a linear combination of the functionals $T \rightsquigarrow \left\langle T\xi_n, \zeta_n \right\rangle$ associated with the seminorms that describe the boundedness of u:

Corollary 11 *Under the hypotheses of* Theorem 10*, there exist complex numbers* $\alpha_1, \ldots, \alpha_N$ *such that*

$$u(T) = \sum_{n=1}^{N} \alpha_n \left\langle T\xi_n, \zeta_n \right\rangle$$

for each $T \in \mathcal{B}(H)$.

4 Strong-operator Continuous Functionals

Next we turn briefly to the strong operator topology on $\mathcal{B}(H)$: the locally convex topology generated by the seminorms $T \rightsquigarrow \|Tx\|$ with $x \in H$. (That is, the weakest topology with respect to which the mapping $T \rightsquigarrow Tx$ is continuous for each $x \in H$.) Clearly, a weak-operator continuous linear functional on $\mathcal{B}(H)$ is strong-operator continuous. The converse holds classically, but, as we now show by a Brouwerian example, is essentially nonconstructive.

Let $(e_n)_{n \geq 1}$ be an orthonormal basis of unit vectors in an infinite-dimensional Hilbert space, and let $(a_n)_{n \geq 1}$ be a binary sequence with at most one term equal to 1. Then for $k \geq j$ we have

$$\begin{aligned} \sum_{n=j}^{k} |a_n \langle Te_1, e_n \rangle| &\leq \left(\sum_{n=j}^{k} a_n^2 \right)^{1/2} \left(\sum_{n=j}^{k} |\langle Te_1, e_n \rangle|^2 \right)^{1/2} \\ &\leq \left(\sum_{n=j}^{k} |\langle Te_1, e_n \rangle|^2 \right)^{1/2}. \end{aligned}$$

Since $\sum_{n=1}^{\infty} |\langle Te_1, e_n \rangle|^2$ converges to $\|Te_1\|$, we see that $\sum_{n=j}^{k} |a_n \langle Te_1, e_n \rangle| \to 0$ as $j, k \to \infty$. Hence

$$u(T) \equiv \sum_{n=1}^{\infty} a_n \langle Te_1, e_n \rangle$$

defines a linear functional u on $\mathcal{B}(H)$; moreover, $|u(T)| \leq \|Te_1\|$ for each T, so (by Proposition 5.4.1 of [8]) u is strong-operator continuous. Suppose it is also weak-operator continuous. Then, by Lemma 2, it is normed. Either $\|u\| < 1$ or $\|u\| > 0$. In the first case, if there exists (a unique) ν with $a_\nu = 1$, then $u(T) = \langle Te_1, e_\nu \rangle$ for each $T \in \mathcal{B}(H)$. Defining T such that $Te_1 = e_\nu$ and $Te_n = 0$ for all $n \neq \nu$, we see that $T \in \mathcal{B}_1(H)$ and $u(T) = 1$; whence $\|u\| = 1$, a contradiction. Thus in this case, $a_n = 0$ for all n. On the other hand, in the case $\|u\| > 0$ we can find T such that $u(T) > 0$; whence there exists n such that $a_n = 1$. It now follows that the statement

> If H is an infinite-dimensional Hilbert space, then every strong-operator continuous linear functional on $\mathcal{B}(H)$ is weak-operator continuous

implies the essentially nonconstructive principle

> LPO: For each binary sequence $(a_n)_{n \geq 1}$, either $a_n = 0$ for all n or else there exists n such that $a_n = 1$

and so is itself essentially nonconstructive.

5 Concluding Observations

The ideal constructive form of Theorem 10 would have two improvements over the current one. First, the requirement that the vectors ξ_n be linearly independent would be relaxed to have them only nonzero in Lemma 8, Lemma 9, and Theorem 10. Second, $\mathcal{B}(H)$ would be replaced by a suitable linear subspace $\mathcal{R}$ of itself, and our theorem would apply to linear functionals that are weak-operator continuous on $\mathcal{R}$, where "suitable" probably means "having weak-operator totally bounded unit ball $\mathcal{R}_1 \equiv \mathcal{R} \cap \mathcal{B}_1(H)$". With that notion of suitability and with minor adaptations, Lemma 6 holds and the proof of Lemma 8 goes through as far as the construction of the vector $\mathbf{z} \in K$. In fact, Theorem 10 goes through with $\mathcal{B}(H)$ replaced by any linear subspace $\mathcal{R}$ of $\mathcal{B}(H)$ that has weak-operator totally bounded unit ball and satisfies the following condition (cf. Lemma 7):

> (*) Let N be a positive integer, let $\xi_1, \ldots, \xi_N$ be linearly independent vectors in H, and let $\zeta_1, \ldots, \zeta_N$ be nonzero elements of H, such that $\sum_{n=1}^N \|\xi_n\|^2 = \sum_{n=1}^N \|\zeta_n\|^2 = 1$. Then there exists a positive number c with the following property: for each unit vector $\mathbf{z}$ in the subspace
>
> $$K \equiv \mathbf{C}\xi_1 \times \cdots \times \mathbf{C}\xi_N$$
>
> there exists $T \in \mathcal{R}_1$ such that $\sum_{n=1}^N \langle Tz_n, \zeta_n \rangle > c$.

This condition holds in the special case where $N = 1$, in which case, if also $\mathcal{R}_1$ is weak-operator totally bounded, we obtain Theorem 1 of [6].[11] However, there seems to be no means of establishing (*) for $N > 1$ and a general $\mathcal{R}$. So the ideal form of our theorem remains an ideal and a challenge.

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[11] But the proof of the theorem in [6] is simpler and more direct than the case $N = 1$ of the proof of our Theorem 10 above.

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Douglas S. Bridges
Department of Mathematics
& Statistics
University of Canterbury
Private Bag 4800
Christchurch 8140
New Zealand
d.bridges@math.canterbury.ac.nz