A Simple Criterion for Extending Natural Transformations to Higher K-Theory

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Abstract. In this article we introduce a very simple an widely applicable criterion for extending natural transformations to higher K-theory. More precisely, we prove that every natural transformation defined on the Grothendieck group and with values in an additive theory admits a unique extension to higher K-theory. As an application, the higher trace maps and the higher Chern characters originally constructed by Dennis and Karoubi, respectively, can be obtained in an elegant, unified, and conceptual way from our general results.

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Introduction

In his foundational work, Grothendieck [10] introduced a very simple and elegant construction K_0, the *Grothendieck group*, in order to formulate a far-reaching generalization of the Riemann-Roch theorem. Since then, this versatile construction spawned well-beyond the realm of algebraic geometry to become one of the most important (working) tools in mathematics.
Latter, through revolutionary topological techniques, Quillen [23] extended the Grothendieck group to a whole family of higher K-theory groups $K_n, n \geq 0$. However, in contrast with K_0, these higher K-theory groups are rather misterious and their computation is often out of reach. In order to capture some of its flavour, Connes, Dennis, Karoubi, and others, constructed natural transformations towards simpler theories E making use of a variety of highly involved techniques; see [6, 7, 15]. Typically, the construction of a natural transformation $K_0 \Rightarrow E_0$ is very simple, while its extension $K_n \Rightarrow E_n$ to higher K-theory is a real "tour-de-force". For example, the trace map $K_0 \Rightarrow HH_0$ consists simply in taking the trace of an idempotent, while its extension

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$K_n \Rightarrow HH_n$ makes use of an array of tools (Hurewicz maps, group homology, assembly maps, etc) coming from topology, algebra, representation theory, etc. These phenomena motivate the following general questions:

Questions: *Given a natural transformation $K_0 \Rightarrow E_0$, is it possible to extend it to higher K-theory $K_n \Rightarrow E_n$? If so, is such an extension unique ?*

In this article we prove that if E verifies three very simple conditions, not only such an extension exists, but it is moreover unique. The precisely formulation of our results makes use of the language of Grothendieck derivators, a formalism which allows us to state and prove precise universal properties; see Appendix A.

1. Statement of results

A *differential graded (=dg) category*, over a fixed commutative base ring k, is a category enriched over cochain complexes of k-modules (morphisms sets are such complexes) in such a way that composition fulfills the Leibniz rule: $d(f \circ g) = (df) \circ g + (-1)^{\deg(f)} f \circ (dg)$. Dg categories extend the classical notion of (dg) k-algebra and solve many of the technical problems inherent to triangulated categories; see Keller's ICM address [16]. In non-commutative algebraic geometry in the sense of Bondal, Drinfeld, Kaledin, Kontsevich, Van den Bergh, and others, they are considered as differential graded enhancements of (bounded) derived categories of quasi-coherent sheaves on a hypothetic non-commutative space; see [1, 8, 9, 14, 17, 18].

Let $E : \mathsf{dgcat} \to \mathsf{Spt}$ be a functor, defined on the category of dg categories, and with values in the category of spectra [2]. We say that E is an *additive functor* if it verifies the following three conditions:

- (i) filtered colimits of dg categories are mapped to filtered colimits of spectra;
- (ii) *derived Morita equivalences* (*i.e.* dg functors which induce an equivalence on the associated derived categories; see [16, §4.6]) are mapped to weak equivalences of spectra;
- (iii) *split exact sequences* (*i.e.* sequences of dg categories which become split exact after passage to the associated derived categories; see [24, §13]) are mapped to direct sums

$$0 \longrightarrow \mathcal{A} \rightleftarrows \mathcal{B} \rightleftarrows \mathcal{C} \longrightarrow 0 \quad \mapsto \quad E(\mathcal{A}) \oplus E(\mathcal{C}) \simeq E(\mathcal{B})$$

in the homotopy category of spectra.

Examples of additive functors include Hochschild homology (HH), cyclic homology (HC), and algebraic K-theory (K); see [16, §5]. Recall from [25] that the category dgcat carries a Quillen model structure whose weak equivalences are the derived Morita equivalences. Given an additive functor E, we obtain then an induced morphism of derivators $\mathbb{E} : \mathsf{HO}(\mathsf{dgcat}) \to \mathsf{HO}(\mathsf{Spt})$. Associated to E, we have also the composed functors

$$E_n : \mathsf{dgcat} \xrightarrow{E} \mathsf{Spt} \xrightarrow{\pi_n^s} \mathsf{Ab} \qquad n \geq 0\,,$$

where π_n^s denotes the n^{th} stable homotopy group functor and Ab the category of abelian groups. Our answer to the questions stated in the Introduction is:

THEOREM 1.1. *For any additive functor E, the natural map*

$$\mathrm{Nat}(\mathbb{K}, \mathbb{E}) \xrightarrow{\sim} \mathrm{Nat}(K_0, E_0) \tag{1.2}$$

is bijective. In particular, every natural transformation $\phi : K_0 \Rightarrow E_0$ admits a canonical extension $\phi_n : K_n \Rightarrow E_n$ to all higher K-theory groups.

Intuitively speaking, Theorem 1.1 show us that all the information concerning a natural transformation is encoded on the Grothendieck group. Its proof relies in an essential way on the theory of *non-commutative motives*, a subject envisioned by Kontsevich [17, 19] and whose development was initiated in [3, 4, 24, 25, 27, 28]. In the next section we illustrate the potential of this general result by explaining how the highly involved constructions of Dennis and Karoubi can be obtained as simple instantiations of the above theorem. Due to its generality and simplicity, we believe that Theorem 1.1 will soon be part of the toolkit of any mathematician whose research comes across the above conditions (i)-(iii).

2. APPLICATIONS

2.1. HIGHER TRACE MAPS. Recall from [16, §5.3] the construction of the Hochschild homology complex $HH(\mathcal{A})$ of a dg category $\mathcal{A}$. This construction is functorial in $\mathcal{A}$ and so by promoting it to spectra we obtain a well-defined functor

$$HH : \mathsf{dgcat} \longrightarrow \mathsf{Spt}\,. \tag{2.1}$$

As explained in *loc. cit.*, this functor verifies conditions (i)-(iii) and hence it is additive. Now, given a k-algebra A, recall from [20, Example 8.3.6] the construction of the classical trace map

$$K_0(A) \to HH_0(A) = A/[A, A]\,.$$

Roughly, it is the map induced by sending an idempotent matrix to the image of its trace (*i.e.* the sum of the diagonal entries) in the quocient $A/[A, A]$. This construction extends naturally from k-algebras to dg categories (see [26]) giving rise to a natural transformation

$$K_0 \Rightarrow HH_0\,. \tag{2.2}$$

PROPOSITION 2.3. *In Theorem 1.1 let E be the additive functor (2.1) and let ϕ be the natural transformation (2.2). Then, for every k-algebra A, the canonical extension $\phi_n : K_n(A) \to HH_n(A)$ of ϕ agrees with the n^{th} trace map constructed originally by Dennis (see [20, §8.4 and §11.4]).*

2.2. Higher Chern characters. Recall also from [16, §5.3] the construction of the cyclic homology complex $HC(\mathcal{A})$ of a dg category $\mathcal{A}$. By promoting this construction to spectra we obtain a functor

$$HC : \mathsf{dgcat} \longrightarrow \mathsf{Spt} \tag{2.4}$$

which verifies conditions (i)-(iii). Given a k-algebra A, recall from [20, Theorem 8.3.4] the construction of the Chern characters

$$ch_{0,i} : K_0(A) \longrightarrow HC_{2i}(A) \qquad i \geq 0\,.$$

Morally, these are the non-commutative analogues of the classical Chern character with values in even dimensional de Rham cohomology. As shown in [26] this construction extends naturally from k-algebras to dg categories giving rise to natural transformations

$$K_0 \Rightarrow HC_{2i} \qquad i \geq 0\,.$$

Proposition 2.5. *In Theorem 1.1 let E be the additive functor $\Omega^{2i}HC$ (obtained by composing (2.4) with the $(2i)^{\mathrm{th}}$-fold looping functor on Spt) and let ϕ be the natural transformation $K_0 \Rightarrow (\Omega^{2i}HC)_0 = HC_{2i}$. Then, for every k-algebra A, the canonical extension $\phi_n : K_n(A) \to (\Omega^{2i}HC)_n(A) = HC_{n+2i}(A)$ of ϕ agrees with the higher Chern character $ch_{n,i}$ constructed originally by Karoubi (see [15, §2.27-2.36]).*

3. Proof of Theorem 1.1

We start by describing the natural map (1.2). As mentioned in §1, the category dgcat carries a Quillen model structure whose weak equivalences are the derived Morita equivalences; see [25, Theorem 5.3]. Let us write Hmo for the associated homotopy category and $l : \mathsf{dgcat} \to \mathsf{Hmo}$ for the localization functor. According to our notation the map (1.2) sends a natural transformation $\Phi \in \mathrm{Nat}(\mathbb{K}, \mathbb{E})$ to the natural transformation $\pi_0^s \circ \Phi(e) \circ l \in \mathrm{Nat}(K_0, E_0)$. Pictorially, we have:

$$\mathsf{dgcat} \xrightarrow{\;l\;} \mathsf{Hmo} \underset{\mathbb{E}(e)}{\overset{\mathbb{K}(e)}{\underset{\Downarrow\,\Phi(e)}{\rightrightarrows}}} \mathsf{Ho}(\mathsf{Spt}) \xrightarrow{\;\pi_0^s\;} \mathsf{Ab}\,. \tag{3.1}$$

The functors $K, E : \mathsf{dgcat} \to \mathsf{Spt}$ are additive and so the following diagrams

$$\begin{array}{ccc} \mathsf{dgcat} & \xrightarrow{K} & \mathsf{Spt} \\ {\scriptstyle l}\downarrow & & \downarrow \\ \mathsf{Hmo} & \xrightarrow[\mathbb{K}(e)]{} & \mathsf{Ho}(\mathsf{Spt}) \end{array} \qquad\qquad \begin{array}{ccc} \mathsf{dgcat} & \xrightarrow{E} & \mathsf{Spt} \\ {\scriptstyle l}\downarrow & & \downarrow \\ \mathsf{Hmo} & \xrightarrow[\mathbb{E}(e)]{} & \mathsf{Ho}(\mathsf{Spt}) \end{array}$$

are commutative. Moreover, the 0^{th} stable homotopy group functor π_0^s descends to the homotopy category $\mathsf{Ho}(\mathsf{Spt})$. These facts show us that the composed horizontal functors in the above diagram (3.1) are in fact K_0 and E_0.
We now study the set $\mathrm{Nat}(K_0, E_0)$. Recall from [16, §5.1] the notion of *additive invariant*. Intutively, it consists of a functor defined on dgcat and with values

in an additive category which verifies conditions similar to (ii)-(iii). Since by hypothesis E is additive, the composed functor

$$E_0 : \mathsf{dgcat} \xrightarrow{l} \mathsf{Hmo} \xrightarrow{\mathbb{E}(e)} \mathsf{Ho}(\mathsf{Spt}) \xrightarrow{\pi_0^s} \mathsf{Ab}$$

is an additive invariant. Hence, as proved in [26, Proposition 4.1], we have the following natural bijection

$$\text{(3.2)} \qquad \mathrm{Nat}(K_0, E_0) \xrightarrow{\sim} E_0(\underline{k}) \qquad \eta \mapsto \eta(\underline{k})([k]) \, .$$

Some explanations are in order: $\underline{k}$ denotes the dg category naturally associated to the base ring k, *i.e.* the dg category with only one object and with k as the dg algebra of endomorphisms (concentrated in degree zero); the symbol $[k]$ stands for the class of k (as a module over itself) in the Grothendieck group $K_0(\underline{k}) = K_0(k)$.

Let us now turn our attention to $\mathrm{Nat}(\mathbb{K}, \mathbb{E})$. Recall from [24, §15] the notion of *additive invariant of dg categories*. Roughly speaking, it consists of a morphism of derivators defined on $\mathsf{HO}(\mathsf{dgcat})$ and with values in a triangulated derivator which verifies conditions analogous to (i)-(iii). Since the functor E is additive, the induced morphism of derivators

$$\mathbb{E} : \mathsf{HO}(\mathsf{dgcat}) \longrightarrow \mathsf{HO}(\mathsf{Spt})$$

is an additive invariant of dg categories. Following [3, Theorem 8.1] we have then a natural bijection[2]

$$\text{(3.3)} \qquad \mathrm{Nat}(\mathbb{K}, \mathbb{E}) \xrightarrow{\sim} \pi_0^s \, \mathbb{E}(\underline{k}) = E_0(\underline{k}) \, .$$

A careful inspection of the proof of [3, Theorem 8.1] show us that (3.3) sends a natural transformation $\Phi \in \mathrm{Nat}(\mathbb{K}, \mathbb{E})$ to the element $\pi_0^s(\Phi(e)(\underline{k}))([k])$ of the abelian group $E_0(\underline{k})$. Note that this element is simply the image of $[k]$ by the abelian group homomorphism

$$K_0(k) = \pi_0^s(\mathbb{K}(e)(\underline{k})) \xrightarrow{\pi_0^s(\Phi(e)(\underline{k}))} \pi_0^s(\mathbb{E}(e)(\underline{k})) = E_0(\underline{k}) \, .$$

We now prove that the following diagram

$$\text{(3.4)} \qquad \begin{array}{ccc} \mathrm{Nat}(\mathbb{K}, \mathbb{E}) & \xrightarrow{(1.2)} & \mathrm{Nat}(K_0, E_0) \\ & \searrow^{(3.3)} & \downarrow{\scriptstyle (3.2)} \\ & & E_0(\underline{k}) \end{array}$$

commutes. Let $\Phi \in \mathrm{Nat}(\mathbb{K}, \mathbb{E})$. On the one hand, we observe that the composed map (3.2) $\circ$ (1.2) sends Φ to the element $(\pi_0^s \circ \Phi(e) \circ l)(\underline{k})([k])$ of the abelian group $E_0(\underline{k})$. On the other hand, the following equalities hold:

$$(\Phi(e) \circ l)(\underline{k}) = \Phi(e)(\underline{k}) \qquad (\pi_0^s \circ \Phi(e))(\underline{k}) = \pi_0^s(\Phi(e)(\underline{k})) \, .$$

[2] In [3] this bijection was established for a localizing invariant $\mathbb{E}$. However, the arguments in the additive case are completely similar.

Therefore, we have

$$(\pi_0^s \circ \Phi(e) \circ l)(\underline{k})([k]) = \pi_0^s(\Phi(e)(\underline{k}))([k])\,.$$

Finally, since the right-hand side in this latter equality coincides with the image of Φ by the map (3.3), we conclude that (3.3) = (3.2) $\circ$ (1.2).
Theorem 1.1 now follows from diagram (3.4) and the fact that both maps (3.2) and (3.3) are bijective. The canonical extension $\phi_n : K_n \Rightarrow E_n$ of $\phi : K_0 \Rightarrow E_0$ is then the composition $\pi_n^s \circ \Phi(e) \circ l$, where Φ is the unique natural transformation associated to ϕ under the bijection (1.2).

4. Proof of Proposition 2.3

The essence of the proof consists in describing the unique natural transformation $\Phi \in \mathrm{Nat}(\mathbb{K}, \mathbb{HH})$ which corresponds to (2.2) under the bijection (1.2). Recall from [24, §15] the construction of the universal additive invariant of dg categories

$$\mathcal{U}_{\mathsf{A}} : \mathsf{HO}(\mathsf{dgcat}) \longrightarrow \mathrm{Mot}_{\mathsf{A}}\ .$$

Given any Quillen model category $\mathcal{M}$ we have an induced equivalence of categories

$$(\mathcal{U}_{\mathsf{A}})^* : \ \underline{\mathsf{Hom}}_{\,!}(\mathrm{Mot}_{\mathsf{A}}, \mathsf{HO}(\mathcal{M})) \xrightarrow{\sim} \underline{\mathsf{Hom}}_{\,\mathsf{A}}(\mathsf{HO}(\mathsf{dgcat}), \mathsf{HO}(\mathcal{M}))\,, \tag{4.1}$$

where the left-hand side denotes the category of homotopy colimit preserving morphisms of derivators and the right-hand side the category of additive invariants of dg categories. The algebraic K-theory functor K is additive and so the induced morphism $\mathbb{K}$ is an additive invariant of dg categories. Thanks to equivalence (4.1), it factors then uniquely through $\mathcal{U}_{\mathsf{A}}$. Recall from [24, Theorem 15.10] that for every dg category $\mathcal{A}$ we have a weak equivalence of spectra

$$\mathbb{R}\mathsf{Hom}(\mathcal{U}_{\mathsf{A}}(\underline{k}), \mathcal{U}_{\mathsf{A}}(\mathcal{A})) \simeq K(\mathcal{A})\,,$$

where $\mathbb{R}\mathsf{Hom}(-,-)$ denotes the spectral enrichment of $\mathrm{Mot}_{\mathsf{A}}$ (see [3, §A.3]). Therefore, we conclude that $\mathbb{K}$ can be expressed as the following composition

$$\mathsf{HO}(\mathsf{dgcat}) \xrightarrow{\mathcal{U}_{\mathsf{A}}} \mathrm{Mot}_{\mathsf{A}} \xrightarrow{\mathbb{R}\mathsf{Hom}(\mathcal{U}_{\mathsf{A}}(\underline{k}),-)} \mathsf{HO}(\mathsf{Spt})\,. \tag{4.2}$$

The Hochschild homology functor, with values in the projective Quillen model category $\mathcal{C}(k)$ of cochain complexes of k-modules (see [12, Theorem 2.3.11]), verifies conditions (i)-(iii). Hence, it gives rise to an additive additive invariant of dg categories which we denote by

$$\mathbb{H}H : \mathsf{HO}(\mathsf{dgcat}) \longrightarrow \mathsf{HO}(\mathcal{C}(k))\,.$$

Note that, according to our notation, $\mathbb{HH}$ can be expressed as the following composition

$$\mathsf{HO}(\mathsf{dgcat}) \xrightarrow{\mathbb{H}H} \mathsf{HO}(\mathcal{C}(k)) \xrightarrow{\mathbb{R}\mathsf{Hom}(k,-)} \mathsf{HO}(\mathsf{Spt})\,. \tag{4.3}$$

Equivalence (4.1) provide us then the following commutative diagram

$$\begin{array}{ccc} \mathsf{HO}(\mathsf{dgcat}) & \xrightarrow{\mathbb{H}H} & \mathsf{HO}(\mathcal{C}(k)) \\ {\scriptstyle \mathcal{U}_{\mathsf{A}}}\downarrow & \nearrow_{\overline{\mathbb{H}H}} & \\ \mathrm{Mot}_{\mathsf{A}} & & \end{array} .$$

By construction, the morphism $\overline{\mathbb{H}H}$ maps $\mathcal{U}_{\mathsf{A}}(\underline{k})$ to $\mathbb{H}H(\underline{k}) = k$. Hence, by making use of the above factorizations (4.2) and (4.3), we conclude that it induces a natural transformation $\Phi \in \mathrm{Nat}(\mathbb{K}, \mathbb{HH})$. We now show that the image of this natural transformation Φ by the map (1.2) is the natural transformation (2.2). By taking $E = HH$ in bijection (3.2) we obtain:

$$\mathrm{Nat}(K_0, HH_0) \simeq HH_0(\underline{k}) \simeq k \qquad \eta \mapsto \eta(\underline{k})([k]) . \tag{4.4}$$

Under this bijection, the natural transformation (2.2) corresponds to the unit of the base ring k; see [26, Theorem 1.3]. Hence, it suffices to show that the same holds for the natural transformation $\pi_0^s \circ \Phi(e) \circ l$ associated to Φ. The class $[k]$ of k (as a module over itself) in the Grothendieck group $K_0(k)$ corresponds to the identity morphism in

$$\mathsf{Hom}_{\mathrm{Mot}_{\mathsf{A}}(e)}(\mathcal{U}_{\mathsf{A}}(\underline{k}), \mathcal{U}_{\mathsf{A}}(\underline{k})) \simeq K_0(\underline{k}) \simeq K_0(k) .$$

By functoriality, $\overline{\mathbb{H}H}(e)$ maps this identity morphism to the identity morphism in $\mathsf{Hom}_{\mathcal{D}(k)}(\mathbb{H}H(\underline{k}), \mathbb{H}H(\underline{k}))$. Under the natural isomorphisms

$$\mathsf{Hom}_{\mathcal{D}(k)}(\mathbb{H}H(\underline{k}), \mathbb{H}H(\underline{k})) \simeq \mathsf{Hom}_{\mathcal{D}(k)}(k, k) \simeq HH_0(k) \simeq k$$

the identity morphism corresponds to the unit of the base ring k and so we conclude that $\pi_0^s \circ \Phi(e) \circ l$ agrees with (2.2). This implies that Φ is in fact the unique natural transformation which corresponds to (2.2) under the bijection (1.2).

Finally, let A be a k-algebra. As proved in [27, Theorem 2.8], the canonical extension $\phi_n : K_n(A) \to HH_n(A)$ of ϕ (*i.e.* the abelian group homomorphism $(\pi_n^s \circ \Phi(e) \circ l)(A)$) agrees with the n^{th} trace map constructed by Dennis and so the proof is finished.

5. Proof of Proposition 2.5

We prove first the particular case ($i = 0$). Let us start by describing the unique natural transformation $\Phi \in \mathrm{Nat}(\mathbb{K}, \mathbb{HC})$ which corresponds to $\phi : K_0 \Rightarrow HC_0$ under the bijection (1.2). Observe that $\mathbb{HC}$ can be expressed as the following composition

$$\mathsf{HO}(\mathsf{dgcat}) \xrightarrow{\mathbb{M}} \mathsf{HO}(\mathcal{C}(\Lambda)) \xrightarrow{\mathbb{P}} \mathsf{HO}(k[u]\text{-Comod}) \xrightarrow{\mathbb{U}} \mathsf{HO}(\mathcal{C}(k)) \xrightarrow{\mathbb{R}\mathsf{Hom}(k,-)} \mathsf{HO}(\mathsf{Spt}) .$$

Some explanations are in order: $\mathcal{C}(\Lambda)$ is the projective Quillen model category of mixed complexes and $\mathbb{M}$ the morphism induced by the mixed complex construction[3] (see [4, Example 7.10]); $k[u]$-Comod is the Quillen model category of $k[u]$-comodules (where $k[u]$ is the Hopf algebra of polynomials in one variable

[3]Denoted by C in [4, Example 7.10].

of degree 2) and $\mathbb{P}$ the morphism induced by the perioditization construction (see [4, Example 7.11]); $\mathbb{U}$ is the morphism induced by the natural forgetful construction. Moreover, as explained in [4, Examples 8.10 and 8.11], negative cyclic homology and periodic cyclic homology admit the following factorizations:

$$\mathbb{HC}^- : \mathsf{HO}(\mathsf{dgcat}) \xrightarrow{\mathbb{M}} \mathsf{HO}(\mathcal{C}(\Lambda)) \xrightarrow{\mathbb{R}\mathsf{Hom}(k,-)} \mathsf{HO}(\mathsf{Spt}) \tag{5.1}$$

$$\mathbb{HP} : \mathsf{HO}(\mathsf{dgcat}) \xrightarrow{(\mathbb{P}\circ\mathbb{M})} \mathsf{HO}(k[u]\text{-Comod}) \xrightarrow{\mathbb{R}\mathsf{Hom}(k[u],-)} \mathsf{HO}(\mathsf{Spt})\,.$$

Therefore, since $\mathbb{P}$ maps k to $k[u]$ and $\mathbb{U}$ maps $k[u]$ to k, we obtain the classical natural transformations

$$\mathbb{HC}^- \Rightarrow \mathbb{HP} \Rightarrow \mathbb{HC} \tag{5.2}$$

between the cyclic homology variants; see [20, §5.1]. The mixed complex morphism $\mathbb{M}$ is an additive invariant of dg categories and so by equivalence (4.1) it factors uniquely through $\mathcal{U}_{\mathsf{A}}$. We obtain then a commutative diagram

$$\begin{array}{ccc} \mathsf{HO}(\mathsf{dgcat}) & \xrightarrow{\mathbb{M}} & \mathsf{HO}(\mathcal{C}(\Lambda)) \\ {\scriptstyle \mathcal{U}_{\mathsf{A}}}\downarrow & \nearrow_{\overline{\mathbb{M}}} & \\ \mathrm{Mot}_{\mathsf{A}} & & \end{array} .$$

By construction, the morphism $\overline{\mathbb{M}}$ maps $\mathcal{U}_{\mathsf{A}}(\underline{k})$ to $\mathbb{M}(\underline{k}) = k$. Therefore, making use of the factorizations (4.2) and (5.1), we conclude that $\overline{\mathbb{M}}$ induces a natural transformation $\Phi_1 : \mathbb{K} \Rightarrow \mathbb{HC}^-$. Its composition with (5.2) gives rise to a natural transformation which we denote by $\Phi \in \mathrm{Nat}(\mathbb{K}, \mathbb{HC})$. We now show that the image of Φ by the map (1.2) is the natural transformation $\phi : K_0 \Rightarrow HC_0$. Recall from [26, Theorem 1.7(ii)] that ϕ admits the following factorization

$$K_0 \overset{ch_0^-}{\Rightarrow} HC_0^- \Rightarrow HP_0 \Rightarrow HC_0\,,$$

where ch_0^- is the negative Chern character and the other natural transformations are the ones associated to (5.2). Hence, it suffices to show that the natural transformation $\pi_0^s \circ \Phi_1(e) \circ l$, associated to $\Phi_1 : \mathbb{K} \Rightarrow \mathbb{HC}^-$, agrees with ch_0^-. This fact is proved in [28, Proposition 4.2] and so we conclude that Φ is the unique natural transformation which corresponds to ϕ under the bijection (1.2). Now, let A be a k-algebra. As explained in [20, §11.4.3], Karoubi's Chern character $ch_{n,0}(A)$ can be expressed as the following composition

$$K_n(A) \xrightarrow{ch_n^-(A)} HC_n^-(A) \longrightarrow HP_n(A) \longrightarrow HC_n(A)\,.$$

Note that the right-hand side maps coincide the ones associated to (5.2). Therefore, it suffices to show that the abelian group homomorphism

$$(\pi_n^s \circ \Phi_1(e) \circ l)(A) : K_n(A) \longrightarrow HC_n^-(A)$$

agrees with $ch_n^-(A)$. This latter fact is proved in [27, Theorem 2.8] and so the proof of the particular case $(i = 0)$ is finished.

We now prove the case ($i > 0$). Recall from [13, §1] that for any dg category $\mathcal{A}$ we have a natural periodicity map $S : \Omega^2\mathbb{M}(\mathcal{A}) \to \mathbb{M}(\mathcal{A})$ in the category $\mathcal{C}(\Lambda)$ of mixed complexes. This construction is natural in $\mathcal{A}$ and so by iterating it we obtain an infinite sequence of maps

$$\cdots \longrightarrow \Omega^{2i}\mathbb{M}(\mathcal{A}) \longrightarrow \cdots \longrightarrow \Omega^2\mathbb{M}(\mathcal{A}) \longrightarrow \mathbb{M}(\mathcal{A})\,. \tag{5.3}$$

Under the natural equivalences

$$\begin{array}{rcl} \mathbb{R}\mathsf{Hom}(k, \Omega^{2i}\mathbb{M}(-)) & \simeq & \Omega^{2i}\mathbb{HC}^- \\ \mathbb{R}\mathsf{Hom}(k[u], \mathbb{P}(\Omega^{2i}\mathbb{M}(-))) & \simeq & \Omega^{2i}\mathbb{HP} \\ \mathbb{R}\mathsf{Hom}(k, \mathbb{U}(\mathbb{P}(\Omega^{2i}\mathbb{M}(-)))) & \simeq & \Omega^{2i}\mathbb{HC}\,, \end{array}$$

the above sequence of maps (5.3) gives rise to the following commutative diagram of natural transformations

$$\begin{array}{ccccccc} \mathbb{K} & \overset{\Phi_1}{\Longrightarrow} & \mathbb{HC}^- & \Longrightarrow & \mathbb{HP} & \Longrightarrow & \mathbb{HC} \\ & & \Uparrow & & \simeq\Uparrow & & \Uparrow \\ & & \Omega^2\mathbb{HC}^- & \Longrightarrow & \Omega^2\mathbb{HP} & \Longrightarrow & \Omega^2\mathbb{HC} \\ & & \Uparrow & & \simeq\Uparrow & & \Uparrow \\ & & \vdots & & \vdots & & \vdots \\ & & \Uparrow & & \simeq\Uparrow & & \Uparrow \\ & & \Omega^{2i}\mathbb{HC}^- & \Longrightarrow & \Omega^{2i}\mathbb{HP} & \Longrightarrow & \Omega^{2i}\mathbb{HC} \\ & & \Uparrow & & \simeq\Uparrow & & \Uparrow \\ & & \vdots & & \vdots & & \vdots \end{array} \tag{5.4}$$

The periodicity map S becomes invertible in periodic cyclic homology and so the middle column in (5.4) consists of natural isomorphisms. Hence, we obtain the classical sequence of natural transformations

$$\mathbb{HP} \Rightarrow \cdots \Rightarrow \Omega^{2i}\mathbb{HC} \Rightarrow \cdots \Rightarrow \Omega^2\mathbb{HC} \Rightarrow \mathbb{HC}$$

which relates periodic cyclic homology with the even dimensional loopings of cyclic homology; see [20, §5.1.8]. Let us then take for Φ the composed natural transformation

$$\mathbb{K} \overset{\Phi_1}{\Rightarrow} \mathbb{HC}^- \Rightarrow \mathbb{HP} \Rightarrow \Omega^{2i}\mathbb{HC}\,.$$

The fact that its image by the map (1.2) is the natural transformation $\phi : K_0 \Rightarrow HC_{2i}$ is now an immediate consequence of the following factorization

$$\phi : K_0 \overset{ch_0^-}{\Rightarrow} HC_0^- \Rightarrow HP_0 \Rightarrow HC_{2i}\,,$$

see [26, Theorem 1.7(ii)], and the particular case ($i = 0$). Similarly, the fact that the canonical extension $\phi_n : K_n(A) \to HC_{n+2i}(A)$ agrees with Karoubi's higher Chern character $ch_{n,i}(A)$ follows from the following factorization

$$ch_{n,i}(A) : K_n(A) \overset{ch_n^-(A)}{\longrightarrow} HC_n^-(A) \longrightarrow HP_n(A) \longrightarrow HC_{n+2i}(A)\,,$$

see [20, §11.4.3], and the particular case ($i = 0$). This achieves the proof.

Appendix A. Grothendieck Derivators

In order to make this article more self-contained we give a brief introduction to Grothendieck's theory of derivators [11]; this language can easily be acquired by skimming through [21], [5, §1] or [3, 4, Appendix A].
Derivators originate in the problem of higher homotopies in derived categories. Given a triangulated category $\mathcal{T}$ and a small category I, it essentially never happens that the diagram category $\mathrm{Fun}(I^{\mathsf{op}}, \mathcal{T})$ remains triangulated; this already fails for the category of arrows in $\mathcal{T}$. However, our triangulated category $\mathcal{T}$ often appears as the homotopy category $\mathcal{T} = \mathsf{Ho}(\mathcal{M})$ of some Quillen model category $\mathcal{M}$ (see [22]). In this case we can consider the category $\mathrm{Fun}(I^{\mathsf{op}}, \mathcal{M})$ of diagrams in $\mathcal{M}$ whose homotopy category $\mathsf{Ho}(\mathrm{Fun}(I^{\mathsf{op}}, \mathcal{M}))$ is triangulated and provides a reasonable approximation to $\mathrm{Fun}(I^{\mathsf{op}}, \mathcal{T})$. More importantly, one can let I vary. This "nebula" of categories $\mathsf{Ho}(\mathrm{Fun}(I^{\mathsf{op}}, \mathcal{M}))$, indexed by small categories I, and the various (adjoint) functors between them is what Grothendieck formalized into the concept of a *derivator*.
A derivator consists of a strict contravariant 2-functor, from the 2-category of small categories to the 2-category of all categories, subject to five natural conditions. We shall not list these conditions here for it would be too long; see [5, §1]. The essential example to keep in mind is the (triangulated) derivator $\mathsf{HO}(\mathcal{M})$ associated to a (stable) Quillen model category $\mathcal{M}$ and defined for every small category I by

$$\mathsf{HO}(\mathcal{M})(I) := \mathsf{Ho}(\mathrm{Fun}(I^{\mathsf{op}}, \mathcal{M}))\,.$$

We will write e for the 1-point category with one object and one identity morphism. Note that $\mathsf{HO}(\mathcal{M})(e)$ is the homotopy category $\mathsf{Ho}(\mathcal{M})$. Given Quillen model categories $\mathcal{M}_1$ and $\mathcal{M}_2$ and weak equivalence preserving functors $E, F : \mathcal{M}_1 \to \mathcal{M}_2$, we will denote by $\mathbb{E}, \mathbb{F} : \mathsf{HO}(\mathcal{M}_1) \longrightarrow \mathsf{HO}(\mathcal{M}_2)$ the induced morphisms of derivators and by $\mathrm{Nat}(\mathbb{E}, \mathbb{F})$ the set of natural transformations from $\mathbb{E}$ to $\mathbb{F}$; see [5, §5]. Note that given $\Phi \in \mathrm{Nat}(\mathbb{E}, \mathbb{F})$, $\Phi(e)$ is a natural transformation between the induced functors $\mathbb{E}(e), \mathbb{F}(e) : \mathsf{Ho}(\mathcal{M}_1) \to \mathsf{Ho}(\mathcal{M}_2)$.

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