Ordinarity of Configuration Spaces and of Wonderful Compactifications

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Received: June 7, 2010
Revised: July 10, 2011

Communicated by Thomas Peternell

Abstract. We prove the following: (1) if X is ordinary, the Fulton-MacPherson configuration space $X[n]$ is ordinary for all n; (2) the moduli of stable n-pointed curves of genus zero is ordinary. (3) More generally we show that a wonderful compactification $X_{\mathcal{G}}$ is ordinary if and only if $(X, \mathcal{G})$ is an ordinary building set. This implies the ordinarity of many other well-known configuration spaces (under suitable assumptions).

2010 Mathematics Subject Classification: 14G17, 14J99
Keywords and Phrases: Ordinary varieties, ordinarity, configuration spaces, wonderful compactification, moduli of n-pointed, stable curves of genus zero.

O Marvelous! what new
configuration will come next?
I am bewildered with multiplicity.

William Carlos Williams

1. Introduction

In the past few years a number of configuration spaces have been studied (see [FM94],[DCP95], [Uly02], [Hu03],[Li09], [CGK06],[KS08]). This class of schemes also includes the moduli of n-pointed stable curves of genus zero, denoted here by $\overline{\mathcal{M}}_{0,n}$ (for $n \geq 3$). All these configuration spaces typically arise from an *initial datum*, which usually consists of a collection of closed non-singular subschemes of a non-singular, projective variety with certain additional properties–like transversal intersection; as well combinatorial data such as an integer or a graph. Given an initial datum, the configuration space associated to it is typically constructed as a sequence of blowups using the subschemes provided in the initial datum. Many configuration schemes constructed in the above references can also be constructed as *wonderful compactifications* of suitable open varieties constructed from the initial datum (see [DCP95, Li09]). Now suppose that k is an algebraically closed field of characteristic $p > 0$. A smooth, projective variety X/k is said to be *ordinary* if $H^i(X, BW\Omega^j_X) = 0$ for all i, j. Here $H^i(X, BW\Omega^j_X)$ are the groups defined in [IR83] using the de Rham-Witt complex. The vanishing of these is equivalent to the vanishing of the Zariski cohomology groups $H^i(X, B\Omega^j_X)$ for all i, j where for any $j \geq 0$, $B\Omega^j_X = d\Omega^{j-1}_X$ is the sheaf of locally exact j-forms (see [IR83]). Ordinarity of a variety is a difficult condition to check in practice as it requires an understanding of crystalline Frobenius. Here are some examples of ordinary varieties: projective spaces, Grassmanians, more generally homogenous spaces G/P for G semisimple, P parabolic subgroup of G; for abelian varieties ordinarity in the above sense is equivalent to ordinarity in the usual sense (invertibility of the Hasse-Witt matrix); that a general abelian variety with a suitable polarization is ordinary is a nontrivial result of Peter Norman and Frans Oort [NO80]; that a general complete intersection in projective space is ordinary is a delicate result of Luc Illusie (see [Ill90]).

Our remark in this note is that *a configuration scheme (of the above type), or more generally a wonderful compactification, arising from an initial datum is ordinary if and only if it arises from an ordinary initial datum* (see Theorem 3.2 and Corollary 3.3). In particular we prove that the following schemes are ordinary: (1) if X is a smooth, ordinary, projective variety and let $X[n]$ be the configuration space of Fulton-MacPherson (see [FM94]) and its generalizations (see [KS08]). The scheme $X[n]$ is a compactification of stable configurations of n-points of X. (2) $\overline{\mathcal{M}}_{0,n}$, the moduli space of n-pointed stable curves of genus zero ([Kee92]). (3) The compactification $X\langle n\rangle$ of Ulyanov [Uly02], (4) the compactification of Kuiperberg-Thurston, [Li09], (5) the spaces $T_{d,n}$ of stable, pointed, rooted trees of [CGK06], (6) the compactification of open varieties due to Yi Hu (see [Hu03]).

The proof is not difficult but as all of these configuration schemes play an important role in many areas of algebraic geometry, so their properties in positive characteristic are not without interest, and hence worth recording.

This note grew out of our attempt to answer a question raised by Indranil Biswas (unfortunately we cannot answer his question–see Remark 3.4 for more on this). It is a pleasure to thank him for many conversations about his question. We thank Ana-Maria Castravet for many conversations about $\overline{\mathcal{M}}_{0,n}$, and especially pointing out the constructions of [Kee92, Kap93]. We thank the referee for comments and suggestions which have improved the readability of this paper.

2. Preliminaries

Let k be a perfect field of characteristic $p > 0$. Let $W\Omega^\bullet_X$ be the de Rham-Witt complex of X. Let $H^i(X, W\Omega^j_X)$ (for $i + j \leq \dim(X)$) be the de Rham-Witt cohomology groups. We say that X is ordinary if $H^i(X, BW\Omega^j_X) = 0$

for $i, j \geq 0$ (as a convention we declare the empty scheme to be ordinary). This is equivalent (see [IR83, Theorem 4.13, Page 209]) to the vanishing of $H^i(X, B\Omega^j_X) = 0$ for $i, j \geq 0$ where $B\Omega^j = d(\Omega^{j-1}_X)$ is the sheaf of locally exact differentials. As we are in characteristic $p > 0$, so $d(f^p\omega) = f^p d\omega$ for any (local) sections f of $\mathcal{O}_X$ and ω of Ω^{j-1}_X, hence $B\Omega^j_X$ is naturally a subsheaf of $F_*(\Omega^i_X)$ and it is in fact locally free as X is smooth of finite type and so at any rate the sheaf $B\Omega^j_X$ carries a natural structure of an $\mathcal{O}_X$-module (see [Ill79, Proposition 2.2.8, Page 520] for more details). The condition of ordinarity is equivalent (see [IR83, Theorem 4.13, Page 209-210] and its proof) to the condition:

$$F : H^i(X, W\Omega^j_X) \to H^i(X, W\Omega^j_X) \tag{2.1}$$

is an isomorphism of W-modules for all $i, j \geq 0$. We say that X is Hodge-Witt if $H^i(X, W\Omega^j_X)$ is finite type over W for all $i, j \geq 0$. By definition any ordinary variety is Hodge-Witt (see [IR83, Defn. 4.12, Page 208]). We will use the following standard results.
The following result is [Eke85, III, Prop 2.1(ii) and Prop 7.2(ii)]

Proposition 2.2 (Ekedahl [Eke85]). *Let X, Y be smooth, projective varieties over k. Then*

1. *If X, Y are ordinary then $X \times_k Y$ is ordinary.*
2. *If $X \times_k Y$ is Hodge-Witt then one of X or Y is ordinary and the other is Hodge-Witt.*

The following result is [Ill90, Prop. 1.4].

Proposition 2.3 (Illusie [Ill90]). *Let X be a smooth, projective variety over a perfect field k. Let V be a vector bundle on X. Let $\mathbb{P}(V) \to X$ be the associated projective bundle. Then X is ordinary if and only if $\mathbb{P}(V)$ is ordinary.*

For a smooth, projective variety X and $Z \subset X$ a smooth, closed subscheme, let $\mathrm{Bl}_Z(X)$ be the blowup of X along Z. For $Y \subset X$ we write $\tilde{Y} \subset \mathrm{Bl}_Z(X)$ for the *dominant transform* of Y in $\mathrm{Bl}_Z(X)$, defined as $\tilde{Y} = \pi^{-1}(Y)$ if $Y \subset Z$ and the strict transform of Y in $\mathrm{Bl}_Z(X)$ otherwise.
We need the following version of [Ill90, Proposition 1.6]:

Proposition 2.4. *Let X be a smooth, projective scheme over an algebraically closed field k. Let $Z \subset X$ be a subscheme of X and let $Y \subset X$ be a smooth, closed subscheme of X. Let $\tilde{Y} \subset \mathrm{Bl}_Z(X)$ be the dominant transform of Y in $\mathrm{Bl}_Z(X)$. Then $\tilde{Y}$ is ordinary if and only if Y and $Y \cap Z$ are ordinary.*

Proof. We write $\pi : \mathrm{Bl}_Z(X) \to X$ for the blowup morphism. Then by [Ill90, Proposition 1.6], $\mathrm{Bl}_Z(X)$ is ordinary if and only if X, Z are ordinary. Next observe that the dominant transform $\tilde{Z}$ of Z is the exceptional divisor and by [Har77, Theorem 8.24(b), page 186], $\tilde{Z} \to Z$ is a projective bundle and so by Proposition 2.3 is $\tilde{Z}$ is ordinary if and only if Z is ordinary.
Now to prove the assertion. Let $Z \subset X$ be a smooth, proper subscheme of a smooth, proper X. Let $Y \subset X$ be a smooth, proper subscheme. Let $\tilde{Y} \subset \mathrm{Bl}_Z(X)$ be the dominant transform of Y in $\mathrm{Bl}_Z(X)$. We consider several

subcases. If Y is a subset of Z, then the dominant transform $\tilde{Y} \to Y$ is a projective bundle over Y and hence $\tilde{Y}$ is ordinary if and only if Y is ordinary (by Proposition 2.3). If $Y = Z$, then the dominant transform $\tilde{Y}$ is the exceptional divisor $E \subset \mathrm{Bl}_Z(X)$. Since E is a projective bundle over Z, we see that $\tilde{Y} = E$ is ordinary if and only if Z is ordinary. If $Y \not\subset Z$ then we proceed as follows. If $Y \cap Z = \emptyset$ then $\tilde{Y} \simeq Y$ and hence is ordinary as Y is ordinary. If $Y \cap Z$ is non-empty and by previous considerations, we may assume that $Y \neq Z$. In this case $\tilde{Y}$ is the blowup of Y along $Y \cap Z$ and so the result follows from [Ill90, Proposition 1.6]. This proves the claim. □

3. Building sets and wonderful compactification

Let X be a smooth, projective scheme over an algebraically closed field k. Let $\mathcal{S}$ be a finite collection of closed, smooth subschemes of X. We say that $\mathcal{S}$ is an *arrangement* if the scheme theoretic intersection of any elements of $\mathcal{S}$ is either empty or an element of $\mathcal{S}$.

Let $\mathcal{S}$ be an arrangement of subschemes of X. We say that $\mathcal{G} \subset \mathcal{S}$ is a *building set* if for all $S \in \mathcal{S} \backslash \mathcal{G}$, the minimal elements in $\{G \in \mathcal{G} : G \supset S\}$ intersect transversally and their intersection is S.

A set of subschemes $\mathcal{G}$ of X is called a building set if the collection of all possible intersections of elements of $\mathcal{G}$ is an arrangement of subschemes of X and $\mathcal{G}$ is a building set of this arrangement.

Let X be a smooth, projective scheme over k and let $\mathcal{G}$ be a building set of X. Let $X_{\mathcal{G}} \subset \prod_{G \in \mathcal{G}} \mathrm{Bl}_G(X)$ be the closure of $X^o = X \backslash \cup_{G \in \mathcal{G}} G$. Then we have the following [Li09, Theorem 1.2]:

Theorem 3.1 ([Li09]). *Let X be a smooth, projective variety over an algebraically closed field k. Let $\mathcal{G}$ be a building set of X. Then $X_{\mathcal{G}}$ is a smooth, projective variety over k.*

The scheme $X_{\mathcal{G}}$ is called the wonderful compactification of $(X, \mathcal{G})$.

We say that a building set $\mathcal{G}$ of X is an *ordinary building set* if X is ordinary and all the scheme theoretic intersections of any members of $\mathcal{G}$ are ordinary (recall that by our convention empty intersections are also ordinary). We say that an *arrangement $\mathcal{S}$ of X is ordinary* if $\mathcal{S}$ arises from an ordinary building set.

Theorem 3.2. *Let X/k be a smooth, projective scheme over a perfect field of characteristic $p > 0$. Let $\mathcal{G}$ be a building set associated to X. Then the wonderful compactification $X_{\mathcal{G}}$ associated to X is ordinary if and only if $\mathcal{G}$ is an ordinary building set.*

Corollary 3.3. *Let X be an smooth, projective variety over k. Assume that X is ordinary. Then the following schemes associated to X are all ordinary:*

1. *the scheme $X[n]$ of Fulton-MacPherson (see* [FM94]*)*
2. *the scheme $X\langle n\rangle$ of Ulyanov (see* [Uly02]*)*
3. *the scheme X^{Γ} of Kuiperberg-Thurston (see* [Li09]*)*

(4) *the generalized Fulton-Macpherson configuration scheme* $X_D^{[n]}, X_D[n]$ *(we assume D is a smooth, ordinary subscheme of X) of* [KS08]*,*
(5) *the moduli,* $\overline{\mathcal{M}}_{0,n}$ *(for* $n \geq 3$*), of n-pointed stable curves of genus zero is ordinary.*
(6) *the scheme of* $T^{d,n}$ *of stable, n-pointed, rooted trees of d-dimensional projective spaces of* [CGK06].

Proof of Theorem 3.2. We recall the details of the construction of $X_{\mathcal{G}}$ from [Li09, Definition 2.12, Proposition 2.13]. The construction is inductively carried out as follows. Let $\mathcal{S}$ be an arrangement of X and $\mathcal{G}$ be a building set of $\mathcal{S}$. Then assume that $\mathcal{G} = \{G_1, \ldots, G_N\}$ is indexed so that $G_i \subset G_j$ if $i \leq j$. We define $(X_k, \mathcal{S}^{(k)}, \mathcal{G}^{(k)})$ as follows. For $k = 0$, set $X_0 = X, \mathcal{S}^{(0)} = \mathcal{S}$, $\mathcal{G}^{(0)} = \mathcal{G}$, $G_i^{(0)} = G_i$ for $1 \leq i \leq N$. Then $(X_0, \mathcal{S}^{(0)}, \mathcal{G}^{(0)})$ is ordinary. Assume by induction that $(X_{k-1}, \mathcal{S}^{(k-1)}, \mathcal{G}^{(k-1)})$ has been constructed so that X_{k-1} is ordinary and $\mathcal{G}^{(k-1)}$ is an ordinary building set for X_{k-1}. Then $\mathcal{S}^{(k-1)}$ consists of ordinary subvarieties of X_{k-1}. Define $X_k = \mathrm{Bl}_{G_k^{(k-1)}}(X_{k-1})$. Then by Proposition 2.4, X_k is ordinary if and only if X_{k-1} and $G_k^{(k-1)}$ are both ordinary. Now define $G^{(k)}$ be the dominant transform of $G^{(k-1)}$ for $G \in \mathcal{G}$. Define $\mathcal{G}^{(k)} = \{G^{(k)} : G \in \mathcal{G}\}$; by Lemma 2.4, $\mathcal{G}^{(k)}$ is ordinary and define $\mathcal{S}^{(k)}$ to be the induced arrangement of $\mathcal{G}^{(k)}$. Since $\mathcal{G}^{(k)}$ is ordinary, we see that $\mathcal{S}^{(k)}$ is ordinary. Finally for $k = N$ we get $X_N = X_{\mathcal{G}}$.
We note that the theorem includes the compactification scheme constructed in [Hu03] as a special case. The fact that this scheme arises from a suitable building set is checked in [Li09]. □

Proof of 3.3. To deduce the Corollary 3.3 from Theorem 3.2 it suffices to produce ordinary building sets to construct $X[n], X\langle n\rangle, X^{\Gamma}$ etc. The building sets for these are constructed in (see the discussion of [Li09, Theorem 1.2] in the paragraph following it). These building sets are building sets of X^n. To prove that they are ordinary building sets if X is ordinary, we note that the building sets for (1)-(3) consists of diagonals or polydiagonals, i.e. self-products of X embedded in X^n by various diagonals. Thus the ordinarity of these building sets follows from Proposition 2.2(i) by ordinarity of self-products of ordinary varieties. Conversely if anyone of these configurations spaces is ordinary then as this space is a blowup, by Proposition 2.4, we see that it must be a blowup of an ordinary variety along an ordinary center, descending down the blowup sequence in this fashion, we deduced that some self-product of X is ordinary, and so by applying Proposition 2.2(ii) (which says for us that self-products of X are Hodge-Witt if and only if X is ordinary) we see that X ordinary. For constructing $X_D^{[n]}$, we use a building set which is constructed by [KS08], from X^n, by blowing up a suitable subschemes which are self products of D, X. By Proposition 2.2 this gives an ordinary building set. The result follows from Theorem 3.2. To construct $X_D[n]$, we start with an ordinary building set in $X_D^{[n]}$, consisting of the proper transform in $X_D^{[n]}$ of the multi-diagonals in X^n. This is again an ordinary building set. Again we can see in both these cases,

by repeating our earlier argument, that if any of $X_D^{[n]}, X_D[n]$ is ordinary then X, D must be ordinary.
(5) This assertion is strictly part of the formalism of wonderful compactification via [Kap93] (see [Li09]) but may be of independent interest and so we give a proof for the sake of completeness using [Kee92] where $\overline{\mathcal{M}}_{0,n}$ is constructed from $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ by a suitable sequence of blowups with smooth, ordinary centers which are related to $\mathcal{M}_{0,j}$ for $j < n$. In [Kee92] provided a construction of $\overline{\mathcal{M}}_{0,n}$ as a sequence of blowups and products. We will prove Theorem 3.3(5) by induction on n. Suppose $n = 3$ then $\overline{\mathcal{M}}_{0,n}$ is a point hence is ordinary. Assume that $n = 4$, then $\overline{\mathcal{M}}_{0,4} = \mathbb{P}^1$ hence is ordinary. Assume that the ordinarity of $\overline{\mathcal{M}}_{0,j}$ has been established for some all $j \leq n$; we will show that $\overline{\mathcal{M}}_{0,n}$ is also ordinary. Recall the construction of [Kee92] (we will notations of that paper for this proof). We let $B_1 = \overline{\mathcal{M}}_{0,n} \times \overline{\mathcal{M}}_{0,4} = \overline{\mathcal{M}}_{0,n} \times \mathbb{P}^1$. Then as $\overline{\mathcal{M}}_{0,n}$ is ordinary by induction and as $\mathbb{P}^1$ is ordinary, so we deduce that B_1 is ordinary (see Proposition 2.2). The construction of $\overline{\mathcal{M}}_{0,n}$ shows that for each subset $T \subset \{1, 2 \ldots, n\}$ with $|T^C| \geq 2$, there exists a collection of smooth subschemes D^T. For each T these subschemes are isomorphic to $\overline{\mathcal{M}}_{0,i} \times \overline{\mathcal{M}}_{0,j}$ for suitable $i, j < n$. Thus by our induction hypothesis and Proposition 2.2 we see that D^T are ordinary and hence, by Proposition 2.3 so is the blowup of B_1 along these D_T for every T. Thus B_2 is ordinary. More generally $B_k \to B_{k-1}$ is the blowup of B_{k-1} along the (disjoint) union of strict transforms of D^T (for $|T^C| = k+1$) under $B_k \to B_1$. Then B_k is ordinary as D^T are isomorphic to $\overline{\mathcal{M}}_{0,i} \times \overline{\mathcal{M}}_{0,j}$ for suitable $i, j < n$. Thus B_k is ordinary and $\overline{\mathcal{M}}_{0,n+1} = B_{n-2}$. This proves the assertion.
For (6) this is not immediate from [Li09] so we recall that $T_{d,n}$ is constructed in [CGK06, Theorem 3.3.1] in a manner similar to the Fulton-MacPherson configuration scheme $X[n]$. The procedure is inductive and starting from $T_{1,3} = \mathbb{P}^1, T_{d,2} = \mathbb{P}^{d-1}$, $T_{1,n} = \overline{\mathcal{M}}_{0,n+1}$ (note that by the previous results these are all ordinary) we construct $T_{d,n}$ as follows: suppose $T_{d,n}$ has been constructed for some d, n. Then $T_{d,n+1}$ is a sequence of blowups of a projective bundle over $T_{d,n}$. Since the later is ordinary by induction, so is the projective bundle over $T_{d,n}$ (by Proposition 2.3). The next blowups are along subschemes of the projective bundle which can be identified with $T_{d,i} \times T_{d,j}$ for $j < n+1$ and so these subschemes are ordinary by Proposition 2.2. This proves the assertion. □

Remark 3.4. Indranil Biswas has asked us the following question: if X is a smooth, projective ordinary surface, then is $\mathrm{Hilb}_n(X)$ ordinary for all $n \geq 1$? We note that it is known that if X is Frobenius split, smooth, projective surface by then [KT01] $\mathrm{Hilb}_n(X)$ is Frobenius split. By [JR03] smooth, proper Frobenius split surfaces are ordinary. However by [JR03] the class of Frobenius split varieties is not a subclass of ordinary varieties in dimensions at least three and we note that the class of ordinary surfaces is much bigger–for instance it includes general type surfaces in $\mathbb{P}^3$ by the result of [Ill90]. In any case Biswas' question presents a natural variant of [KT01].

Unfortunately we do not know how to answer Biswas's question. The methods outlined here are not adequate as they require a far better understanding of the geometry of $\mathrm{Hilb}_n(X)$ than we seem to have at the moment. We note however that we can easily deduce the result for $\mathrm{Hilb}_2(X)$ from our result for the Fulton-MacPherson configuration space $X[2]$.

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