The Singularity Category of an Algebra with Radical Square Zero

Xiao-Wu Chen[1]

Received: April 21, 2011

Communicated by Stefan Schwede

Abstract. To an artin algebra with radical square zero, a regular algebra in the sense of von Neumann and a family of invertible bimodules over the regular algebra are associated. These data describe completely, as a triangulated category, the singularity category of the artin algebra. A criterion on the Hom-finiteness of the singularity category is given in terms of the valued quiver of the artin algebra.

2010 Mathematics Subject Classification: 18E30, 13E10, 16E50
Keywords and Phrases: singularity category, von Neumann regular algebra, invertible bimodule, valued quiver

1. Introduction

Let R be a commutative artinian ring. All algebras, categories and functors are R-linear. We recall that an R-linear category is *Hom-finite* provided that all the Hom sets are finitely generated R-modules.

Let A be an artin R-algebra. Denote by A-mod the category of finitely generated left A-modules, and by $\mathbf{D}^b(A\text{-mod})$ the bounded derived category. Following [14], the *singularity category* $\mathbf{D}_{\rm sg}(A)$ is the quotient triangulated category of $\mathbf{D}^b(A\text{-mod})$ with respect to the full subcategory formed by perfect complexes; see also [3, 12, 10, 15, 2] and [13]. Here, we recall that a complex in $\mathbf{D}^b(A\text{-mod})$ is *perfect* provided that it is isomorphic to a bounded complex consisting of finitely generated projective modules.

The singularity category measures the homological singularity of an algebra in the sense that an algebra A has finite global dimension if and only if its singularity category $\mathbf{D}_{\rm sg}(A)$ vanishes. In the meantime, the singularity category captures the stable homological features of an algebra ([3]). A fundamental result of Buchweitz and Happel states that for a Gorenstein algebra A, the

[1]The author is supported by the Fundamental Research Funds for the Central Universities (WK0010000024), the Special Foundation of President of the Chinese Academy of Sciences (No.1731112304061) and National Natural Science Foundation of China (No.10971206).

singularity category $\mathbf{D}_{\rm sg}(A)$ is triangle equivalent to the stable category of (maximal) Cohen-Macaulay A-modules ([3, 10]). This implies in particular that the singularity category of a Gorenstein algebra is Hom-finite and has Auslander-Reiten triangles. We point out that Buchweitz and Happel's result specializes to Rickard's result ([15]) on self-injective algebras. However, for non-Gorenstein algebras, not much is known about their singularity categories ([4]).

Our aim is to describe the singularity category of an algebra with radical square zero. We point out that such algebras are usually non-Gorenstein ([5]). In what follows, we describe the results in this paper.

We denote by $\mathbf{r}$ the Jacobson radical of A. The algebra A is said to be with *radical square zero* provided that $\mathbf{r}^2 = 0$. In this case, $\mathbf{r}$ has a natural $A/\mathbf{r}$-$A/\mathbf{r}$-bimodule structure. Set $\mathbf{r}^{\otimes 0} = A/\mathbf{r}$ and $\mathbf{r}^{\otimes i+1} = \mathbf{r} \otimes_{A/\mathbf{r}} (\mathbf{r}^{\otimes i})$ for $i \geq 0$. Then there are obvious algebra homomorphisms $\mathrm{End}_{A/\mathbf{r}}(\mathbf{r}^{\otimes i}) \to \mathrm{End}_{A/\mathbf{r}}(\mathbf{r}^{\otimes i+1})$ induced by $\mathbf{r} \otimes_{A/\mathbf{r}} -$. We denote by $\Gamma(A)$ the direct limit of this chain of algebra homomorphisms. It is a regular algebra ([7, 8]) in the sense of von Neumann. We call $\Gamma(A)$ the *associated regular algebra* of A. In most cases, the algebra $\Gamma(A)$ is not semisimple.

For $n \in \mathbb{Z}$ and $i \geq \max\{0, n\}$, $\mathrm{Hom}_{A/\mathbf{r}}(\mathbf{r}^{\otimes i}, \mathbf{r}^{\otimes i-n})$ has a natural $\mathrm{End}_{A/\mathbf{r}}(\mathbf{r}^{\otimes i-n})$-$\mathrm{End}_{A/\mathbf{r}}(\mathbf{r}^{\otimes i})$-bimodule structure. Set $K^n(A)$ to be the direct limit of the chain of maps $\mathrm{Hom}_{A/\mathbf{r}}(\mathbf{r}^{\otimes i}, \mathbf{r}^{\otimes i-n}) \to \mathrm{Hom}_{A/\mathbf{r}}(\mathbf{r}^{\otimes i+1}, \mathbf{r}^{\otimes i+1-n})$, which are induced by $\mathbf{r} \otimes_{A/\mathbf{r}} -$. Then $K^n(A)$ is naturally a $\Gamma(A)$-$\Gamma(A)$-bimodule for each $n \in \mathbb{Z}$. Observe that $K^0(A) = {}_{\Gamma(A)}\Gamma(A)_{\Gamma(A)}$ as bimodules, and that composition of maps induces $\Gamma(A)$-$\Gamma(A)$-bimodule morphisms $\phi^{n,m}\colon K^n(A) \otimes_{\Gamma(A)} K^m(A) \to K^{n+m}(A)$ for all $n, m \in \mathbb{Z}$. These bimodules $K^n(A)$ are called the *associated bimodules* of A.

Recall that for an algebra Γ, a Γ-Γ-bimodule K is *invertible* provided that the functor $K \otimes_\Gamma -$ induces an auto-equivalence on the category of left Γ-modules.

Theorem A. *Let A be an artin algebra with radical square zero. Use the above notation. Then the associated $\Gamma(A)$-$\Gamma(A)$-bimodules $K^n(A)$ are invertible and the maps $\phi^{n,m}$ are isomorphisms of bimodules.*

Since the algebra $\Gamma(A)$ is regular, the category $\mathrm{proj}\ \Gamma(A)$ of finitely generated right projective $\Gamma(A)$-modules is a semisimple abelian category. The invertible bimodule $K^1(A)$ induces an auto-equivalence

$$\Sigma_A = - \otimes_{\Gamma(A)} K^1(A)\colon \mathrm{proj}\ \Gamma(A) \longrightarrow \mathrm{proj}\ \Gamma(A).$$

We observe that the category $\mathrm{proj}\ \Gamma(A)$ has a unique triangulated structure with Σ_A its shift functor; see Lemma 3.4. This unique triangulated category is denoted by $(\mathrm{proj}\ \Gamma(A), \Sigma_A)$.

The following result describes the singularity category of an artin algebra with radical square zero, which is based on a result by Keller and Vossieck ([12]).

Theorem B. *Let A be an artin algebra with radical square zero. Use the above notation. Then we have a triangle equivalence*

$$\mathbf{D}_{\mathrm{sg}}(A) \simeq (\mathrm{proj}\, \Gamma(A), \Sigma_A).$$

We are interested in the Hom-finiteness of singularity categories. For this, we recall the notion of valued quiver of an artin algebra A. Choose a complete set of representatives of pairwise non-isomorphic simple A-modules $\{S_1, S_2, \cdots, S_n\}$. Set $\Delta_i = \mathrm{End}_A(S_i)$; they are division algebras. Observe that $\mathrm{Ext}^1_A(S_i, S_j)$ has a natural Δ_j-Δ_i-bimodule structure. The *valued quiver* Q_A of A is defined as follows: its vertex set is $\{S_1, S_2, \cdots, S_n\}$, here we identify each simple module S_i with its isoclass; there is an arrow from S_i to S_j whenever $\mathrm{Ext}^1_A(S_i, S_j) \neq 0$, in which case the arrow is endowed with a *valuation* $(\dim_{\Delta_j} \mathrm{Ext}^1_A(S_i, S_j), \dim_{\Delta_i{}^{\mathrm{op}}} \mathrm{Ext}^1_A(S_i, S_j))$; here $\Delta_i{}^{\mathrm{op}}$ denotes the opposite algebra of Δ_i. We say that the valuation of Q_A is *trivial* provided that all the valuations are $(1, 1)$. Recall that a vertex in a valued quiver is a source (*resp.* sink) provided that there is no arrows ending (*resp.* starting) at it. For a valued quiver, to adjoin a (new) source (*resp.* sink) is to add a vertex together with some valued arrows starting (*resp.* ending) at this vertex. For details, we refer to [1, III.1].

The following result characterizes when the singularity category is Hom-finite, using the valued quivers.

Theorem C. *Let A be an artin algebra with radical square zero. Then the following statements are equivalent:*

(1) *the singularity category $\mathbf{D}_{\mathrm{sg}}(A)$ is Hom-finite;*
(2) *the associated regular algebra $\Gamma(A)$ is semisimple;*
(3) *the valued quiver Q_A is obtained from a disjoint union of oriented cycles with the trivial valuation by repeatedly adjoining sources or sinks.*

The paper is structured as follows. In Section 2, we collect some facts on singularity categories and recall a basic result of Keller and Vossieck. We prove Theorem A and B in Section 3, where an explicit example is presented. In Section 4, we prove that one-point extensions and coextensions of algebras preserve their singularity categories. We introduce the notion of cyclicization of an algebra, which is used in the proof of Theorem C in Section 5.

For artin algebras, we refer to [1]. For triangulated categories, we refer to [11] and [9].

2. Preliminaries

In this section, we collect some facts on singularity categories of artin algebras. We recall a basic result due to Keller and Vossieck ([12]), which is applied to Ω^∞-finite algebras.

Let A be an artin algebra over a commutative artinian ring R. Recall that A-mod denotes the category of finitely generated left A-modules. We denote by A-proj the full subcategory formed by projective modules, and by A-$\underline{\text{mod}}$ the stable category of A-mod modulo projective modules ([1, p.104]). The morphism space $\underline{\text{Hom}}_A(M, N)$ of two modules M and N in A-$\underline{\text{mod}}$ is defined to be $\text{Hom}_A(M, N)/\mathbf{p}(M, N)$, where $\mathbf{p}(M, N)$ denotes the R-submodule formed by morphisms that factor through projective modules.

Recall that for an A-module M, its syzygy $\Omega(M)$ is the kernel of its projective cover $P \to M$. This gives rise to the *syzygy functor* $\Omega\colon A\text{-}\underline{\text{mod}} \to A\text{-}\underline{\text{mod}}$ ([1, p.124]). Set $\Omega^0(M) = M$ and $\Omega^{i+1}(M) = \Omega^i(\Omega(M))$ for $i \geq 0$. Denote by $\Omega^i(A\text{-mod})$ the full subcategory of A-mod formed by modules of the form $P \oplus \Omega^i(M)$ for some module M and projective module P. Then an A-module X belongs to $\Omega^i(A\text{-mod})$ if and only if there is an exact sequence $0 \to X \to P^{1-i} \to \cdots \to P^{-1} \to P^0$ with each P^j projective.

Recall that $\mathbf{D}^b(A\text{-mod})$ denotes the bounded derived category of A-mod, whose shift functor is denoted by [1]. For $n \in \mathbb{Z}$, $[n]$ denotes the n-th power of [1]. The module category A-mod is viewed as a full subcategory of $\mathbf{D}^b(A\text{-mod})$ by identifying an A-module with the corresponding stalk complex concentrated at degree zero ([11, Proposition I.4.3]). Recall that a complex in $\mathbf{D}^b(A\text{-mod})$ is *perfect* provided that it is isomorphic to a bounded complex consisting of projective modules; these complexes form a full triangulated subcategory $\text{perf}(A)$. Recall that, via an obvious functor, $\text{perf}(A)$ is triangle equivalent to the bounded homotopy category $\mathbf{K}^b(A\text{-proj})$; compare [3, 1.1-1.2].

Following [14], we call the quotient triangulated category

$$\mathbf{D}_{\text{sg}}(A) = \mathbf{D}^b(A\text{-mod})/\text{perf}(A)$$

the *singularity category* of A. Denote by $q\colon \mathbf{D}^b(A\text{-mod}) \to \mathbf{D}_{\text{sg}}(A)$ the quotient functor.

The following two results are known; compare [14, Lemma 1.11] and [3, Lemma 2.2.2].

Lemma 2.1. *Let $X^\bullet$ be a complex in $\mathbf{D}_{\text{sg}}(A)$ and $n_0 > 0$. Then for any n large enough, there exists a module M in $\Omega^{n_0}(A\text{-mod})$ such that $X^\bullet \simeq q(M)[n]$.*

Proof. Take a quasi-isomorphism $P^\bullet \to X^\bullet$ with $P^\bullet$ a bounded above complex of projective modules ([11, Lemma I.4.6]). Take $n \geq n_0$ such that $H^i(X^\bullet) = 0$ for all $i < n_0 - n$, where $H^i(X^\bullet)$ denotes the i-th cohomology of $X^\bullet$. Consider the good truncation $\sigma^{\geq -n}P^\bullet = \cdots \to 0 \to M \to P^{1-n} \to P^{2-n} \to \cdots$ of $P^\bullet$, which is quasi-isomorphic to $P^\bullet$. Then the cone of the obvious chain map $\sigma^{\geq -n}P^\bullet \to M[n]$ is perfect, which becomes an isomorphism in $\mathbf{D}_{\text{sg}}(A)$. This shows that $X^\bullet \simeq q(M)[n]$. We observe that M lies in $\Omega^{n_0}(A\text{-mod})$. □

Lemma 2.2. *Let $0 \to M \to P^{1-n} \to \cdots \to P^0 \to N \to 0$ be an exact sequence with each P^i projective. Then we have an isomorphism $q(N) \simeq q(M)[n]$ in $\mathbf{D}_{\text{sg}}(A)$. In particular, for an A-module M, we have a natural isomorphism $q(\Omega^n(M)) \simeq q(M)[-n]$.*

Proof. The stalk complex N is quasi-isomorphic to $\cdots \to 0 \to M \to P^{1-n} \to \cdots \to P^0 \to 0 \to \cdots$. This gives rise to a morphism $N \to M[n]$ in $\mathbf{D}^b(A\text{-mod})$, whose cone is $\cdots \to 0 \to P^{1-n} \to \cdots \to P^0 \to 0 \to \cdots$ with P^0 at degree -1; it is perfect. Then the morphism $N \to M[n]$ becomes an isomorphism in $\mathbf{D}_{\rm sg}(A)$. □

Consider the composite $q' \colon A\text{-mod} \hookrightarrow \mathbf{D}^b(A\text{-mod}) \xrightarrow{q} \mathbf{D}_{\rm sg}(A)$; it vanishes on projective modules. Then it induces uniquely a functor $A\text{-}\underline{\rm mod} \to \mathbf{D}_{\rm sg}(A)$, which is still denoted by q'. Then Lemma 2.2 yields, for each $n \geq 0$, the following commutative diagram

$$\begin{array}{ccc} A\text{-}\underline{\rm mod} & \xrightarrow{\Omega^n} & A\text{-}\underline{\rm mod} \\ \downarrow{\scriptstyle q'} & & \downarrow{\scriptstyle q'} \\ \mathbf{D}_{\rm sg}(A) & \xrightarrow{[-n]} & \mathbf{D}_{\rm sg}(A). \end{array}$$

We refer to [3, Lemma 2.2.2] for a similar statement.
The functor q' induces a natural map

$$\Phi^0 \colon \underline{\rm Hom}_A(M,N) \to {\rm Hom}_{\mathbf{D}_{\rm sg}(A)}(q(M), q(N))$$

for any modules M, N. Let $n \geq 1$. Lemma 2.2 yields a natural isomorphism $\theta_M \colon q(M) \xrightarrow{\sim} q(\Omega^n(M))[n]$. Then we have a map

$$\Phi^n \colon \underline{\rm Hom}_A(\Omega^n(M), \Omega^n(N)) \to {\rm Hom}_{\mathbf{D}_{\rm sg}(A)}(q(M), q(N))$$

given by $\Phi^n(f) = (\theta^n_N)^{-1} \circ (\Phi^0(f)[n]) \circ \theta^n_M$.
Consider the chain of maps

$$\underline{\rm Hom}_A(\Omega^n(M), \Omega^n(N)) \to \underline{\rm Hom}_A(\Omega^{n+1}(M), \Omega^{n+1}(N))$$

induced by the syzygy functor. It is routine to verify that Φ^n are compatible with this chain of maps. Then we have an induced map

$$\Phi \colon \varinjlim \underline{\rm Hom}_A(\Omega^n(M), \Omega^n(N)) \longrightarrow {\rm Hom}_{\mathbf{D}_{\rm sg}(A)}(q(M), q(N)).$$

We recall the following basic result.

Proposition 2.3. (Keller-Vossieck) *Let M, N be A-modules as above. Then the map Φ is an isomorphism.*

Proof. The statement follows from [12, Exemple 2.3]. We refer to [2, Corollary 3.9(1)] for a detailed proof. □

Recall that an additive category $\mathcal{A}$ is *idempotent split* provided that each idempotent $e \colon X \to X$ splits, that is, it admits a factorization $X \xrightarrow{u} Y \xrightarrow{v} X$ with $u \circ v = {\rm Id}_Y$. For example, a Krull-Schmidt category is idempotent split ([6, Appendix A]). In particular, for an artin algebra A, the stable category $A\text{-}\underline{\rm mod}$ is idempotent split.

Corollary 2.4. *The singularity category $\mathbf{D}_{\rm sg}(A)$ of an artin algebra A is idempotent split.*

Proof. By Lemma 2.1 it suffices to show that for each module M, an idempotent $e\colon q(M) \to q(M)$ splits in $\mathbf{D}_{\rm sg}(A)$. The above proposition implies that for a large n, there is an idempotent $e^n\colon \Omega^n(M) \to \Omega^n(M)$ in A-$\underline{\rm mod}$ which is mapped by Φ to e. The idempotent e^n splits as $\Omega^n(M) \stackrel{u}{\to} Y \stackrel{v}{\to} \Omega^n(M)$ with $u \circ v = {\rm Id}_Y$ in A-$\underline{\rm mod}$. Then the idempotent e splits as $q(M) \stackrel{(q(u)[n])\circ\theta^n_M}{\longrightarrow} q(Y)[n] \stackrel{(\theta^n_M)^{-1}\circ(q(v)[n])}{\longrightarrow} q(M)$. □

Let $\mathcal{A}$ be an additive category. For a subcategory $\mathcal{C}$, denote by add $\mathcal{C}$ the full subcategory of $\mathcal{A}$ formed by direct summands of finite direct sums of objects in $\mathcal{C}$. For any algebra Γ, denote by proj Γ the category of finitely generated right projective Γ-modules. We observe that proj $\Gamma = $ add Γ_Γ.
An artin algebra A is called Ω^∞*-finite* provided that there exists a module E and $n \geq 0$ such that $\Omega^n(A\text{-mod}) \subseteq$ add $(A \oplus E)$. In this case, we call E an Ω^∞*-generator* of A.

Proposition 2.5. *Let A be an Ω^∞-finite algebra with an Ω^∞-generator E. Then we have $\mathbf{D}_{\rm sg}(A) =$ add $q(E)$. Consequently, we have an equivalence of categories*

$$\mathbf{D}_{\rm sg}(A) \simeq \text{proj End}_{\mathbf{D}_{\rm sg}(A)}(q(E)),$$

which sends $q(E)$ to ${\rm End}_{\mathbf{D}_{\rm sg}(A)}(q(E))$.

Proof. Observe that $\Omega^{n+1}(A\text{-mod}) \subseteq \Omega^n(A\text{-mod})$. Then we may assume that add $(A \oplus E) \supseteq$ add $\Omega^{n_0}(A\text{-mod}) =$ add $\Omega^{n_0+1}(A\text{-mod}) = \cdots$ for n_0 large enough.
For the first statement, it suffices to show that each object $X^\bullet$ in $\mathbf{D}_{\rm sg}(A)$ belongs to add $q(E)$. By Lemma 2.1, $X^\bullet \simeq q(M)[n_1]$ for a module $M \in \Omega^{n_0}(A\text{-mod})$ and $n_1 > 0$. Since add $\Omega^{n_0}(A\text{-mod}) =$ add $\Omega^{n_0+n_1}(A\text{-mod})$, we may assume that $M \oplus N \in \Omega^{n_0+n_1}(A\text{-mod})$ for some module N. Take an exact sequence $0 \to M \oplus N \to P^{1-n_1} \to \cdots \to P^0 \to L \to 0$ with each P^i projective and $L \in \Omega^{n_0}(A\text{-mod})$. By Lemma 2.2, $q(L) \simeq q(M \oplus N)[n_1]$ and then $X^\bullet$ is a direct summand of $q(L)$. Observing that $L \in$ add $(A \oplus E)$, we are done with the first statement.
The second statement follows from the projectivization; see [1, Proposition II.2.1]. The functor is given by ${\rm Hom}_{\mathbf{D}_{\rm sg}(A)}(q(E), -)$. We point out that Corollary 2.4 is needed here. □

3. Algebras with radical square zero

In this section, we study the singularity category of an algebra with radical square zero, and prove Theorem A and B. An explicit example is given at the end.
Let A be an artin algebra. Denote by $\mathbf{r}$ the Jacobson radical of A. The algebra A is said to be with *radical square zero* provided that $\mathbf{r}^2 = 0$. In this case, $\mathbf{r}$ has an $A/\mathbf{r}$-$A/\mathbf{r}$-bimodule structure, which is induced from the multiplication of A.

Denote by A-$\underline{\mathrm{ssmod}}$ the full subcategory of A-$\underline{\mathrm{mod}}$ formed by semisimple modules. We observe that $\mathbf{r}\otimes_{A/\mathbf{r}} S=0$ for a simple projective module S. Then the functor $\mathbf{r}\otimes_{A/\mathbf{r}} -\colon A\text{-}\underline{\mathrm{ssmod}}\to A\text{-}\underline{\mathrm{ssmod}}$ is well defined. We observe that the syzygy functor Ω sends semisimple modules to semisimple modules, and then we have the restricted functor $\Omega\colon A\text{-}\underline{\mathrm{ssmod}}\to A\text{-}\underline{\mathrm{ssmod}}$.

The following result is implicitly contained in the proof of [1, Lemma X.2.1].

Lemma 3.1. *There is a natural isomorphism $\Omega\simeq \mathbf{r}\otimes_{A/\mathbf{r}} -$ of functors on A-$\underline{\mathrm{ssmod}}$.*

Proof. Let X be a semisimple A-module. Take a projective cover $P\to X$. Tensoring P with the natural exact sequence of A-A-bimodules $0\to\mathbf{r}\to A\to A/\mathbf{r}\to 0$ yields $\Omega(X)\simeq \mathbf{r}\otimes_A P$. Using isomorphisms $\mathbf{r}\otimes_A P\simeq \mathbf{r}\otimes_{A/\mathbf{r}} P/\mathbf{r}P$ and $P/\mathbf{r}P\simeq X$, we get an isomorphism $\Omega(X)\simeq \mathbf{r}\otimes_{A/\mathbf{r}} X$. It is routine to verify that this isomorphism is natural in X. □

Recall that an algebra Γ is *regular* in the sense of von Neumann provided that for each element a there exists a' such that $aa'a=a$. For example, a semisimple algebra is regular. Then a direct limit of semisimple algebras is regular. For details, we refer to [7, Theorem and Definition 11.24].

Recall that for an artin algebra A with radical square zero, there is a chain of algebra homomorphisms $\mathrm{End}_{A/\mathbf{r}}(\mathbf{r}^{\otimes i})\to \mathrm{End}_{A/\mathbf{r}}(\mathbf{r}^{\otimes i+1})$ induced by $\mathbf{r}\otimes_{A/\mathbf{r}} -$. Here, $\mathbf{r}^{\otimes 0}=A/\mathbf{r}$ and $\mathbf{r}^{\otimes i+1}=\mathbf{r}\otimes_{A/\mathbf{r}}(\mathbf{r}^{\otimes i})$. We set $\Gamma(A)$ to be the direct limit of this chain. Since each algebra $\mathrm{End}_{A/\mathbf{r}}(\mathbf{r}^{\otimes i})$ is semisimple, the algebra $\Gamma(A)$ is regular. It is called the *associated regular algebra* of A. We refer to [8, 19.26B, Example] for a related construction.

We recall the *associated* $\Gamma(A)$-$\Gamma(A)$-*bimodules* $K^n(A)$ of A, $n\in\mathbb{Z}$. For $i\geq\max\{0,n\}$, $\mathrm{Hom}_{A/\mathbf{r}}(\mathbf{r}^{\otimes i},\mathbf{r}^{\otimes i-n})$ has a natural $\mathrm{End}_{A/\mathbf{r}}(\mathbf{r}^{\otimes i-n})$-$\mathrm{End}_{A/\mathbf{r}}(\mathbf{r}^{\otimes i})$-bimodule structure. Consider a chain of maps $\mathrm{Hom}_{A/\mathbf{r}}(\mathbf{r}^{\otimes i},\mathbf{r}^{\otimes i-n})\to \mathrm{Hom}_{A/\mathbf{r}}(\mathbf{r}^{\otimes i+1},\mathbf{r}^{\otimes i+1-n})$, which are induced by $\mathbf{r}\otimes_{A/\mathbf{r}} -$, and define $K^n(A)$ to be its direct limit. Then $K^n(A)$ is naturally a $\Gamma(A)$-$\Gamma(A)$-bimodule for each $n\in\mathbb{Z}$. Observe that $K^0(A)={}_{\Gamma(A)}\Gamma(A)_{\Gamma(A)}$ as $\Gamma(A)$-$\Gamma(A)$-bimodules.

Proposition 3.2. *Let A be an artin algebra with radical square zero. Then there is a natural isomorphism*

$$K^n(A)\simeq \mathrm{Hom}_{\mathbf{D}_{\mathrm{sg}}(A)}(q(A/\mathbf{r}),q(A/\mathbf{r})[n])$$

for each $n\in\mathbb{Z}$.

Proof. Consider the case $n\leq 0$ first. In this case, by Lemmas 2.2 and 3.1 we have $q(A/\mathbf{r})[n]\simeq q(\Omega^{-n}(A/\mathbf{r}))\simeq q(\mathbf{r}^{\otimes -n})$. Then Proposition 2.3 yields an isomorphism $\mathrm{Hom}_{\mathbf{D}_{\mathrm{sg}}(A)}(q(A/\mathbf{r}),q(A/\mathbf{r})[n])\simeq \varinjlim \underline{\mathrm{Hom}}_A(\Omega^i(A/\mathbf{r}),\Omega^i(\mathbf{r}^{\otimes -n}))$. By Lemma 3.1 again we have $\Omega^i(A/\mathbf{r})\simeq\mathbf{r}^{\otimes i}$ and $\Omega^i(\mathbf{r}^{\otimes -n})=\mathbf{r}^{\otimes i-n}$. Then we have a surjective map $\psi\colon K^n(A)\to \mathrm{Hom}_{\mathbf{D}_{\mathrm{sg}}(A)}(q(A/\mathbf{r}),q(A/\mathbf{r})[n])$. On the other hand, every morphism $f\colon\mathbf{r}^{\otimes i}\to\mathbf{r}^{\otimes i-n}$ that is zero in A-$\underline{\mathrm{mod}}$ necessarily factors through a semisimple projective module. However, the functor $\mathbf{r}\otimes_{A/\mathbf{r}} -$ vanishes on semisimple projective modules. Then $\mathbf{r}\otimes_{A/\mathbf{r}} f$ is zero. This forces that ψ is injective. We are done in this case.

For the case $n > 0$, we observe that $\mathrm{Hom}_{\mathbf{D}_{\mathrm{sg}}(A)}(q(A/\mathbf{r}), q(A/\mathbf{r})[n])$ is isomorphic to $\mathrm{Hom}_{\mathbf{D}_{\mathrm{sg}}(A)}(q(A/\mathbf{r})[-n], q(A/\mathbf{r}))$, and by the same argument as above, it is isomorphic to $\varinjlim \underline{\mathrm{Hom}}_{A/\mathbf{r}}(\mathbf{r}^{\otimes i+n}, \mathbf{r}^{\otimes i})$. Then we get a surjective map $K^n(A) \to \mathrm{Hom}_{\mathbf{D}_{\mathrm{sg}}(A)}(q(A/\mathbf{r}), q(A/\mathbf{r})[n])$. Similarly as above, we have that this map is injective. □

Remark 3.3. In the case $n = 0$, the above isomorphism is an isomorphism $K^0(A) = \Gamma(A) \simeq \mathrm{End}_{\mathbf{D}_{\mathrm{sg}}(A)}(q(A/\mathbf{r}))$ of algebras. Then for an arbitrary n, the above isomorphism becomes an isomorphism of $\Gamma(A)$-$\Gamma(A)$-bimodules.

Recall that an abelian category $\mathcal{A}$ is *semisimple* provided that each short exact sequence splits. For example, for a regular algebra Γ, the category proj Γ of finitely generated right projective Γ-modules is a semisimple abelian category. Here, we use the fact that all finitely presented Γ-modules are projective; see [7, Theorem and Definition 11.24(a)].
The following observation is well known.

Lemma 3.4. *Let $\mathcal{A}$ be a semisimple abelian category, and let Σ be an auto-equivalence on $\mathcal{A}$. Then there is a unique triangulated structure on $\mathcal{A}$ with Σ the shift functor.*

The obtained triangulated category in this lemma will be denoted by $(\mathcal{A}, \Sigma)$.

Proof. We use the fact that each morphism in $\mathcal{A}$ is isomorphic to a direct sum of morphisms of the forms $K \to 0$, $I \stackrel{\mathrm{Id}_I}{\to} I$ and $0 \to C$. Then all possible triangles are a direct sum of the following trivial triangles $K \to 0 \to \Sigma(K) \stackrel{\mathrm{Id}_{\Sigma(K)}}{\to} \Sigma(K)$, $I \stackrel{\mathrm{Id}_I}{\to} I \to 0 \to \Sigma(I)$ and $0 \to C \stackrel{\mathrm{Id}_C}{\to} C \to \Sigma(0)$. □

Proposition 3.5. *Let A be an artin algebra with radical square zero, and let $\Gamma(A)$ be its associated regular algebra. Then there is a triangle equivalence*

$$\Psi\colon \mathbf{D}_{\mathrm{sg}}(A) \simeq (\mathrm{proj}\ \Gamma(A), \Sigma)$$

for some auto-equivalence Σ on proj $\Gamma(A)$*, which sends $q(A/\mathbf{r})$ to $\Gamma(A)$.*

Proof. We observe that for any A-module M, its syzygy $\Omega(M)$ is semisimple. Hence we have $\Omega^1(A\text{-mod}) \subseteq \mathrm{add}\ (A \oplus A/\mathbf{r})$. We apply Proposition 2.5 to obtain an equivalence of categories $\mathbf{D}_{\mathrm{sg}}(A) \simeq \mathrm{proj}\ \mathrm{End}_{\mathbf{D}_{\mathrm{sg}}(A)}(q(A/\mathbf{r}))$. By Proposition 3.2 this yields an equivalence of categories $\mathbf{D}_{\mathrm{sg}}(A) \simeq \mathrm{proj}\ \Gamma(A)$.
By transport of structures, the shift functor [1] on $\mathbf{D}_{\mathrm{sg}}(A)$ corresponds to an auto-equivalence Σ on proj $\Gamma(A)$, and then proj $\Gamma(A)$ becomes a triangulated category. However, by Lemma 3.4 the semisimple abelian category proj $\Gamma(A)$ has a unique triangulated structure with Σ the shift functor. Then this structure necessarily coincides with the transported one. Then we are done. □

We are interested in the auto-equivalence Σ above. The following result characterizes it using the bimodules $K^n(A)$.

Lemma 3.6. *Use the notation as above. Then for each $n \in \mathbb{Z}$, the auto-equivalence Σ^n is isomorphic to $- \otimes_{\Gamma(A)} K^n(A)\colon \mathrm{proj}\ \Gamma(A) \to \mathrm{proj}\ \Gamma(A)$.*

Proof. Recall that the above equivalence Ψ is given by $\mathrm{Hom}_{\mathbf{D}_{\mathrm{sg}}(A)}(q(A/\mathbf{r}), -)$, which sends $q(A/\mathbf{r})$ to $\Gamma(A)$. The auto-equivalence Σ^n corresponds, via Ψ, to $[n]$ on $\mathbf{D}_{\mathrm{sg}}(A)$. Then by Proposition 3.2 we have an isomorphism $\phi\colon K^n(A) \xrightarrow{\sim} \Sigma^n(\Gamma(A))$ of right $\Gamma(A)$-modules. Recall that $\Sigma^n(\Gamma(A))$ has a natural $\Gamma(A)$-$\Gamma(A)$-bimodule structure such that Σ^n is isomorphic to $- \otimes_{\Gamma(A)} \Sigma^n(\Gamma(A))$. Thanks to Remark 3.3, the isomorphism ϕ is an isomorphism of bimodules. This proves the lemma. □

Recall that for an algebra Γ, a Γ-Γ-bimodule K is *invertible* provided that the functor $-\otimes_\Gamma K$ induces an auto-equivalence on the category of right Γ-modules. For details, we refer to [7, Definition and Proposition 12.13].

We recall that for an artin algebra A with radical square zero, the associated $\Gamma(A)$-$\Gamma(A)$-bimodules $K^n(A)$ are defined to be $\varinjlim \mathrm{Hom}_{A/\mathbf{r}}(\mathbf{r}^{\otimes i}, \mathbf{r}^{\otimes i-n})$, where $i \geq \max\{0, n\}$. Then composition of maps between the $A/\mathbf{r}$-modules $\mathbf{r}^{\otimes j}$ yields morphisms

$$\phi^{n,m}\colon K^n(A) \otimes_{\Gamma(A)} K^m(A) \longrightarrow K^{n+m}(A)$$

of $\Gamma(A)$-$\Gamma(A)$-bimodules, for all $n, m \in \mathbb{Z}$. More precisely, let $f \in K^n(A)$ and $g \in K^m(A)$ be represented by $f'\colon \mathbf{r}^{\otimes j-m} \to \mathbf{r}^{\otimes j-m-n}$ and $g'\colon \mathbf{r}^{\otimes j} \to \mathbf{r}^{\otimes j-m}$ for some large j, respectively. Then $\phi^{n,m}(f \otimes g)$ is represented by the composite $f' \circ g'$.

The following result is Theorem A.

THEOREM 3.7. *Let A be an artin algebra with radical square zero. Use the above notation. Then for all $n, m \in \mathbb{Z}$, the $\Gamma(A)$-$\Gamma(A)$-bimodules $K^n(A)$ are invertible and the morphisms $\phi^{n,m}$ are isomorphisms.*

Proof. By Lemma 3.6 the functor $- \otimes_{\Gamma(A)} K^n(A)\colon \mathrm{proj}\ \Gamma(A) \to \mathrm{proj}\ \Gamma(A)$ is an auto-equivalence for each $n \in \mathbb{Z}$. This functor extends naturally to an auto-equivalence on the category of all right $\Gamma(A)$-modules. Then $K^n(A)$ is an invertible bimodule. The second statement follows from Lemma 3.6 and the fact that $\Sigma^m\Sigma^n$ is isomorphic to Σ^{n+m}. Here, we use [7, Proposition 12.9] implicitly. □

We now have Theorem B. Denote the functor $- \otimes_{\Gamma(A)} K^1(A)\colon \mathrm{proj}\ \Gamma(A) \to \mathrm{proj}\ \Gamma(A)$ by Σ_A.

THEOREM 3.8. *Let A be an artin algebra with radical square zero. Use the above notation. Then we have a triangle equivalence*

$$\mathbf{D}_{\mathrm{sg}}(A) \simeq (\mathrm{proj}\ \Gamma(A), \Sigma_A),$$

which sends $q(A/\mathbf{r})$ to $\Gamma(A)$.

Proof. It follows from Proposition 3.5 and Lemma 3.6. □

Let A be an artin algebra with radical square zero. For each $n \geq 1$, we consider the artin algebra $G^n = A/\mathbf{r} \oplus \mathbf{r}^{\otimes n}$, which is the *trivial extension* of the $A/\mathbf{r}$-$A/\mathbf{r}$-bimodule $\mathbf{r}^{\otimes n}$ ([1, p.78]). All these algebras G^n have radical square zero. The following observation seems to be of independent interest.

Proposition 3.9. *Use the above notation. Then for each $n \geq 1$, we have a triangle equivalence*

$$\mathbf{D}_{\rm sg}(G^n) \simeq (\text{proj } \Gamma(A), \Sigma_A^n).$$

In particular, we have a triangle equivalence $\mathbf{D}_{\rm sg}(A) \simeq \mathbf{D}_{\rm sg}(G^1)$.

Proof. Write $G^n = A'$. Then from the very definition, we have a natural identification $\Gamma(A') = \Gamma(A)$. Moreover, the $\Gamma(A')$-$\Gamma(A')$-bimodule $K^1(A')$ corresponds to the $\Gamma(A)$-$\Gamma(A)$-bimodule $K^n(A)$. Then by Lemma 3.6 $\Sigma_{A'}$ corresponds to Σ_A^n. Then the result follows from Theorem 3.8 immediately. □

Remark 3.10. We point out that for $n \geq 2$, $\mathbf{D}_{\rm sg}(G^n)$ might not be triangle equivalent to $\mathbf{D}_{\rm sg}(A)$, although the underlying categories are equivalent.

We conclude this section with an example.

Example 3.11. *Let k be a field and let $n \geq 1$. Consider the algebra $A = k[x_1, x_2, \cdots, x_n]/(x_ix_j, 1 \leq i, j \leq n)$, which is with radical square zero. We identify $A/\mathbf{r}$ with k, and $\mathbf{r}$ with the n-dimensional k-space $V = kx_1 \oplus kx_2 \oplus \cdots \oplus kx_n$. Consequently, for each $i \geq 0$, the algebra $\mathrm{End}_{A/\mathbf{r}}(\mathbf{r}^{\otimes i})$ is isomorphic to $\mathrm{End}_k(V^{\otimes i})$, which is identified with the $n^i \times n^i$ total matrix algebra $M_{n^i}(k)$. Then the associated regular algebra $\Gamma(A)$ is isomorphic to the direct limit of the following chain of algebra embeddings*

$$k \longrightarrow M_n(k) \longrightarrow M_{n^2}(k) \longrightarrow M_{n^3}(k) \longrightarrow \cdots$$

Here, for each algebra B, $B \to M_n(B)$ is the algebra embedding sending b to bI_n with I_n the $n \times n$ identity matrix.

We observe that $\Gamma(A)$ is a simple algebra. We point out that this construction is classical; see [8, 19.26 B, Example]. *The algebra A is non-noetherian for $n \geq 2$, while for $n = 1$, it is isomorphic to k.*

Let $1 \leq r, s \leq n$. Define $E_{rs}\colon V \to V$ to be the linear map such that $E_{rs}(x_i) = \delta_{i,s}x_r$, where δ is the Kronecker symbol. Consider, for all $i \geq 0$, the linear maps $- \otimes_k E_{rs}\colon \mathrm{End}_k(V^{\otimes i}) \to \mathrm{End}_k(V^{\otimes i+1})$. Taking the limit, we have the induced linear map $- \otimes_k E_{rs}\colon \Gamma(A) \to \Gamma(A)$ for each pair of r, s. Then we have an isomorphism $\sigma\colon M_n(\Gamma(A)) \to \Gamma(A)$ of algebras, which sends an $n \times n$ matrix (a_{ij}) to $\sum_{1\leq i,j\leq n} a_{ij} \otimes_k E_{ij}$.

The associated $\Gamma(A)$-$\Gamma(A)$-bimodule $K^1(A)$ is described as follows. As a k-space, $K^1(A) = \Gamma(A) \oplus \Gamma(A) \oplus \cdots \oplus \Gamma(A)$ with n copies of $\Gamma(A)$. The left action is given by $a(a_1, a_2, \cdots, a_n) = (aa_1, aa_2, \cdots, aa_n)$, while the right action is given by $(a_1, a_2, \cdots, a_n)a = (a_1, a_2, \cdots, a_n)\sigma^{-1}(a)$.

We remark that the regular algebra $\Gamma(A)$ is related to a quotient abelian category studied in [16], *which might relate to the singularity category $\mathbf{D}_{\rm sg}(A)$ via a version of Koszul duality.*

4. One-point (co)extensions and cyclicizations of algebras

In this section, we prove that one-point extensions and coextensions of algebras preserve their singularity categories. We then introduce the notion of cyclicization of an algebra, which is a repeated operation to remove sources and sinks on the valued quiver. The obtained results will be used in the proof of Theorem C.

Let A be an artin algebra. Let D be a simple artin algebra, and let ${}_AM_D$ be an A-D-bimodule, on which R acts centrally. The *one-point extension* of A by M is the upper triangular matrix algebra $A[M] = \begin{pmatrix} A & M \\ 0 & D \end{pmatrix}$. A left $A[M]$-module is denoted by a column vector $\begin{pmatrix} X \\ V \end{pmatrix}_\phi$, where X and V are a left A-module and D-module, respectively, and that $\phi\colon M \otimes_D V \to X$ is a morphism of A-modules. We sometimes suppress the morphism ϕ, when it is clearly understood. For details, we refer to [1, III.2].

Recall from [1, III.1] the notion of *valued quiver* Q_A for an artin algebra A. We observe that for the unique simple D-module S, the corresponding $A[M]$-module $\begin{pmatrix} 0 \\ S \end{pmatrix}$ is simple injective, which corresponds to a source in the valued quiver $Q_{A[M]}$ of the one-point extension $A[M]$. Indeed, this valued quiver is obtained from Q_A by adding this source together with some valued arrows starting at it.

One-point extensions of algebras preserve singularity categories. Observe the natural exact embedding $i\colon A\text{-mod} \to A[M]\text{-mod}$, which sends ${}_AX$ to $i(X) = \begin{pmatrix} X \\ 0 \end{pmatrix}$.

Proposition 4.1. *Let $A[M]$ be the one-point extension of A as above. Then the exact embedding $i\colon A\text{-mod} \to A[M]\text{-mod}$ induces a triangle equivalence*

$$\mathbf{D}_{\rm sg}(A) \simeq \mathbf{D}_{\rm sg}(A[M]).$$

Proof. The exact functor i extends naturally to a triangle functor $i_*\colon \mathbf{D}^b(A\text{-mod}) \to \mathbf{D}^b(A[M]\text{-mod})$. We observe that $i(A)$ is projective, and then i_* sends perfect complexes to perfect complexes. Then it induces a triangle functor $\bar{i_*}\colon \mathbf{D}_{\rm sg}(A) \to \mathbf{D}_{\rm sg}(A[M])$. We claim that $\bar{i_*}$ is an equivalence.

For the claim, recall that the functor i admits a left adjoint $j\colon A[M]\text{-mod} \to A\text{-mod}$ which sends $\begin{pmatrix} X \\ V \end{pmatrix}_\phi$ to $X/{\rm Im}\phi$. Observe that the corresponding counit $ji \xrightarrow{\sim} {\rm Id}_{A\text{-mod}}$ is an isomorphism. One checks that the cohomological dimension ([11, p.57]) of the functor j is at most one. In particular, the left derived functor $\mathbf{L}^bj\colon \mathbf{D}^b(A[M]\text{-mod}) \to \mathbf{D}^b(A\text{-mod})$ is defined. Moreover, we have the adjoint pair $(\mathbf{L}^bj, i_*)$, and that the counit is an isomorphism. Since the functor j sends projective modules to projective modules, the functor $\mathbf{L}^bj$ preserves perfect complexes. Then it induces a triangle functor $\mathbf{L}^b\bar{j}\colon \mathbf{D}_{\rm sg}(A[M]) \to \mathbf{D}_{\rm sg}(A)$. Moreover, we have the induced adjoint pair $(\mathbf{L}^b\bar{j}, \bar{i_*})$, whose counit $(\mathbf{L}^b\bar{j})\bar{i_*} \xrightarrow{\sim} {\rm Id}_{\mathbf{D}_{\rm sg}(A)}$ is an isomorphism; see [14, Lemma 1.2]. In particular, the functor $\bar{i_*}$ is fully faithful.

It remains to show the denseness of $\bar{i_*}$. We now view the essential image ${\rm Im}\, \bar{i_*}$ of $\bar{i_*}$ as a full triangulated subcategory of $\mathbf{D}_{\rm sg}(A[M])$. It suffices to show that for each $A[M]$-module $\begin{pmatrix} X \\ V \end{pmatrix}$, its image in $\mathbf{D}_{\rm sg}(A[M])$ lies in $\operatorname{Im} \bar{i_*}$; see Lemma 2.1. Observe that $\Omega(\begin{pmatrix} 0 \\ V \end{pmatrix})$ lies in $\operatorname{Im} i$, and then by Lemma 2.2, $q(\begin{pmatrix} 0 \\ V \end{pmatrix})$ lies in $\operatorname{Im} \bar{i_*}$. The following natural exact sequence induces a triangle in $\mathbf{D}_{\rm sg}(A[M])$

$$0 \longrightarrow i(X) \longrightarrow \begin{pmatrix} X \\ V \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ V \end{pmatrix} \longrightarrow 0.$$

This triangle implies that $q(\begin{pmatrix} X \\ V \end{pmatrix})$ lies in $\operatorname{Im} \bar{i_*}$. Then we are done. □

Let D be a simple artin algebra, and let ${}_DN_A$ be a D-A-bimodule, on which R acts centrally. The *one-point coextension* of A by N is the upper triangular matrix algebra $[N]A = \begin{pmatrix} D & N \\ 0 & A \end{pmatrix}$. A left $[N]A$-module is written as $\begin{pmatrix} V \\ X \end{pmatrix}_\phi$, where V and X are a left D-module and A-module, respectively, and that $\phi\colon M \otimes_A X \to V$ is a morphism of D-modules. The valued quiver $Q_{[N]A}$ is obtained from Q_A by adding a sink together with some valued arrows ending at it, where the sink corresponds to the simple projective $[N]A$-module $\begin{pmatrix} S \\ 0 \end{pmatrix}$ for a simple D-module S.

For the one-point coextension $[N]A$, we have an exact embedding $i\colon A\text{-mod} \to [N]A\text{-mod}$, which sends ${}_AX$ to $i(X) = \begin{pmatrix} 0 \\ X \end{pmatrix}$.

The following result is similar to Proposition 4.1, while the proof is simpler. This result is closely related to [4, Theorem 4.1(1)].

Proposition 4.2. *Let $[N]A$ be the one-point coextension as above. Then the embedding $i\colon A$-mod $\to [N]A$-mod induces a triangle equivalence*

$$\mathbf{D}_{\rm sg}(A) \simeq \mathbf{D}_{\rm sg}([N]A).$$

Proof. We observe that $i(A)$ has projective dimension at most one. Then the obviously induced functor $i_*\colon \mathbf{D}^b(A\text{-mod}) \to \mathbf{D}^b([N]A\text{-mod})$ preserves perfect complexes, and it induces the required functor $\bar{i_*}\colon \mathbf{D}_{\rm sg}(A) \to \mathbf{D}_{\rm sg}([N]A)$.

The functor i admits an exact left adjoint $j\colon [N]A\text{-mod} \to A\text{-mod}$, which sends $\begin{pmatrix} V \\ X \end{pmatrix}$ to X; moreover, j preserves projective modules. Then it induces a triangle functor $\bar{j_*}\colon \mathbf{D}_{\rm sg}([N]A) \to \mathbf{D}_{\rm sg}(A)$, which is left adjoint to $\bar{i_*}$. Then as in the proof of Proposition 4.1, we have that $\bar{i_*}$ is fully faithful. The denseness of $\bar{i_*}$ follows from the natural exact sequence

$$0 \longrightarrow \begin{pmatrix} V \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} V \\ X \end{pmatrix} \longrightarrow i(X) \longrightarrow 0,$$

for each $[N]A$-module $\begin{pmatrix} V \\ X \end{pmatrix}$, and that the $[N]A$-module $\begin{pmatrix} V \\ 0 \end{pmatrix}$ is projective. We omit the details. □

We use the above two propositions to reduce the study of the singularity category of arbitrary artin algebras to cyclic-like ones.
Let A be an artin algebra. Consider the valued quiver Q_A. A vertex e is called *cyclic* provided that there is an oriented cycle containing it, and the corresponding simple A-module is called *cyclic*. More generally, a vertex e is called *cyclic-like* provided that there is a path through e, which starts with a cyclic vertex and ends at a cyclic vertex, while the corresponding simple A-module is called *cyclic-like*. An artin algebra A is called *cyclic-like* provided that its valued quiver Q_A is cyclic-like. This is equivalent to that A has neither simple projective nor simple injective modules.
For an artin algebra A, its *cyclicization* is an artin algebra A_c which is either simple or cyclic-like, such that there is a sequence $A_c = A_0, A_1, \cdots, A_r = A$ satisfying that each A_{i+1} is a one-point (co)extension of A_i.
The following is an immediate consequence of the definition.

Lemma 4.3. *Let A be an artin algebra with its cyclicization A_c. Then we have a triangle equivalence*

$$\mathbf{D}_{\mathrm{sg}}(A_c) \simeq \mathbf{D}_{\mathrm{sg}}(A).$$

Proof. Apply Propositions 4.1 and 4.2, repeatedly. □

The following result seems to be well known.

Proposition 4.4. *The following statements hold.*

(1) *Each artin algebra has a cyclicization.*
(2) *Let A_c and $A_{c'}$ be two cyclicizations of A. Then if A_c is simple, so is $A_{c'}$. Otherwise, we have an isomorphism $A_c \simeq A_{c'}$ of algebras.*

Proof. (1) It follows from the well-known fact that the existence of a simple injective (*resp.* projective) module of A implies that A is a one-point extension (*resp.* coextension) of A'. Moreover, the valued quiver $Q_{A'}$ of A' is obtained from the one of A by deleting the relevant source (*resp.* sink).
(2) The first statement follows from the observation that passing from A to A' in (1), the set of cyclic-like vertices stays the same.
For the isomorphism of algebras, it suffices to observe that A_c-mod is equivalent to the smallest Serre subcategory ([7, Chapter 15]) of A-mod containing the cyclic-like simple A-modules S; moreover, the multiplicity of $P_{A_c}(S)$ in the indecomposable decomposition of A_c equals the multiplicity of $P(S)$ in the one of A. Here, $P(S)$ and $P_{A_c}(S)$ denote the projective cover of S as an A-module and A_c-module, respectively. □

5. Hom-finiteness of singularity categories

In this section, we study the Hom-finiteness of the singularity category of an artin algebra with radical square zero, and prove Theorem C.
Throughout, A is an artin R-algebra such that its Jacobson radical $\mathbf{r}$ satisfies $\mathbf{r}^2 = 0$. Recall that in this case, the syzygy $\Omega(X)$ of any A-module X is semisimple.

Lemma 5.1. *Suppose that A is cyclic-like. Then we have*

(1) *each simple A-module has infinite projective dimension;*
(2) *for each $i \geq 0$, the algebra homomorphism* $\mathrm{End}_{A/\mathbf{r}}(\mathbf{r}^{\otimes i}) \to \mathrm{End}_{A/\mathbf{r}}(\mathbf{r}^{\otimes i+1})$ *induced by* $\mathbf{r} \otimes_{A/\mathbf{r}} -$ *is injective.*

Proof. (1) Recall that a cyclic-like algebra does not have simple projective modules. Then the statement follows from the observation that for a simple module S with finite projective dimension, we have that $\mathrm{proj.dim}\ \Omega(S) = \mathrm{proj.dim}\ S - 1$.
(2) We recall that A-$\underline{\mathrm{ssmod}}$ is the full subcategory of A-$\underline{\mathrm{mod}}$ consisting of semisimple modules. Then by (1), A-$\underline{\mathrm{ssmod}}$ is naturally equivalent to A-ssmod, and the syzygy functor $\Omega\colon A\text{-}\underline{\mathrm{ssmod}} \to A\text{-}\underline{\mathrm{ssmod}}$ is faithful. Now the result follows from Lemma 3.1. □

Recall that the singularity category $\mathbf{D}_{\mathrm{sg}}(A)$ is naturally R-linear. We are interested in the problem when it is Hom-finite, that is, all the Hom sets are finitely generated R-modules.

Theorem 5.2. *Let A be an artin algebra with radical square zero. Then the following statements are equivalent:*

(1) *the singularity category $\mathbf{D}_{\mathrm{sg}}(A)$ is Hom-finite;*
(2) *the associated regular algebra $\Gamma(A)$ is semisimple;*
(3) *the cyclicization A_c of A is either simple or isomorphic to a finite product of self-injective algebras.*

We point out that the cyclicization A_c of A is necessarily with radical square zero. Recall that an indecomposable non-simple artin algebra with radical square zero is self-injective if and only if its valued quiver is an oriented cycle with the trivial valuation; see [1, Proposition IV.2.16] or the proof of [5, Corollary 1.3]. Then the statement (3) above is equivalent to the corresponding one in Theorem C.

Proof. Recall from Proposition 3.2 the isomorphism $\Gamma(A) \simeq \mathrm{End}_{\mathbf{D}_{\mathrm{sg}}(A)}(q(A/\mathbf{r}))$. Then we have the implication "(1)⇒(2)", since an artin regular algebra is necessarily semisimple.
For "(2)⇒(1)", consider the cyclicization A_c of A, whose Jacobson radical is denoted by $\mathbf{r}_c$. Then by Lemma 4.3 we have an equivalence $\mathbf{D}_{\mathrm{sg}}(A_c) \simeq \mathbf{D}_{\mathrm{sg}}(A)$. Applying Proposition 3.5 we have an equivalence $\mathrm{proj}\ \Gamma(A_c) \simeq \mathrm{proj}\ \Gamma(A)$, that is, $\Gamma(A_c)$ and $\Gamma(A)$ are Morita equivalent. Then $\Gamma(A_c)$ is also semisimple. Recall that $\Gamma(A_c) = \varinjlim \mathrm{End}_{A_c/\mathbf{r}_c}(\mathbf{r}_c^{\otimes i})$. By Lemma 5.1 all the canonical maps $\mathrm{End}_{A_c/\mathbf{r}_c}(\mathbf{r}_c^{\otimes i}) \to \Gamma(A_c)$ are injective. Recall that for a semisimple algebra, the number of pairwise orthogonal idempotents is bounded. Then the R-lengths of the algebras $\mathrm{End}_{A_c/\mathbf{r}_c}(\mathbf{r}_c^{\otimes i})$ are uniformly bounded. Consequently, the algebra $\Gamma(A)$ is an artin R-algebra. By Proposition 3.5 the singularity category $\mathbf{D}_{\mathrm{sg}}(A_c)$ is Hom-finite. Then we are done by Lemma 4.3.

Recall from [15, Theorem 2.1] that the singularity category of a self-injective algebra is equivalent to its stable category. In particular, it is Hom-finite. Then the implication "(3)⇒(1)" follows from Lemma 4.3.

It remains to show "(1)⇒(3)". Without loss of generality, we assume that the algebra A is cyclic-like such that $\mathbf{D}_{\rm sg}(A)$ is Hom-finite. We will show that A is self-injective.

We claim that the sysygy $\Omega(S)$ of any cyclic simple A-module S is simple. Then there is only one arrow starting at S in Q_A, which is valued by $(1, b)$ for some natural number b. Since Q_A is cyclic-like, this forces that Q_A is a disjoint union of oriented cycles. In each oriented cycle, every arrow has valuation $(1, b_i)$ for some b_i. Then the symmetrization condition implies that all these b_i's are necessarily one; compare the proof of [1, Proposition VIII. 6.4]. As we point out above, this implies that A is self-injective.

We prove the claim. Since by Corollary 2.4 $\mathbf{D}_{\rm sg}(A)$ is idempotent split, we have that $\mathbf{D}_{\rm sg}(A)$ is a Krull-Schmidt category ([6, Appendix A]). In particular, each object is uniquely decomposed as a direct sum of finitely many indecomposable objects. We observe that for each semisimple module X, $lX \leq l\Omega(X)$. Here, l denotes the composition length. Consider a cyclic simple A-module S, and take a path $S = S_1 \to S_2 \to \cdots \to S_r \to S_{r+1} = S$ in Q_A. Assume on the contrary that $l\Omega(S) \geq 2$. Then we have $\Omega(S) = S_2 \oplus X$ for some nonzero semisimple module X. Observe that S is a direct summand of $\Omega^{r-1}(S_2)$, and then we have $\Omega^r(S) = S \oplus X'$ for a nonzero semisimple module X'. Consequently, we have $\Omega^{nr}(S) = S \oplus X' \oplus \Omega^r(X') \oplus \cdots \oplus \Omega^{(n-1)r}(X')$. Then the lengths of the semisimple modules $\Omega^{nr}(S)$ tend to the infinity, when n goes to the infinity. By Lemma 5.1(1), $q(T)$ is not zero for any simple A-module T. Recall from Lemma 2.2 that $q(S) \simeq q(\Omega^{nr}(S))[nr]$, and then $q(S) \simeq q(S)[nr] \oplus q(X')[nr] \oplus q(\Omega^r(X'))[nr] \oplus \cdots \oplus q(\Omega^{(n-1)r}(X'))[nr]$ for each $n \geq 1$. This contradicts to the Krull-Schmidt property of $\mathbf{D}_{\rm sg}(A)$, and we are done with the claim. □

ACKNOWLEDGEMENTS. The author thanks Professor Zhaoyong Huang, Professor Yu Ye and Dr. Longgang Sun for their helpful comments.

REFERENCES

[1] M. AUSLANDER, I. REITEN AND S.O. SMALØ, Representation Theory of Artin Algebras, Cambridge Studies in Adv. Math. 36, Cambridge Univ. Press, Cambridge, 1995.

[2] A. BELIGIANNIS, *The homological theory of contravariantly finite subcategories: Auslander-Buchweitz contexts, Gorenstein categories and (co-)stabilization,* Comm. Algebra 28(10) (2000), 4547–4596.

[3] R.O. BUCHWEITZ, Maximal Cohen-Macaulay Modules and Tate Cohomology over Gorenstein Rings, Unpublished Manuscript, 1987. Availble at: http://hdl.handle.net/1807/16682.

[4] X.W. CHEN, *Singularity categories, Schur functors and triangular matrix rings*, Algebr. Represent. Theor. 12 (2009), 181–191.

[5] X.W. Chen, *Algebras with radical square zero are either self-injective or CM-free*, Proc. Amer. Math. Soc. 140 (1) (2012), 93–98.
[6] X.W. Chen, Y. Ye and P. Zhang, *Algebras of derived dimension zero*, Comm. Algebra 36 (2008), 1–10.
[7] C. Faith, Algebra I, Rings, Modules and Categories, Springer-Verlag, Berlin Heidelberg New York, 1973.
[8] C. Faith, Algebra II, Ring Theory, Springer-Verlag, Berlin Heidelberg New York, 1976.
[9] D. Happel, Triangulated Categories in the Representation Theory of Finite Dimensional Algebras, London Math. Soc., Lecture Notes Ser. 119, Cambridge Univ. Press, Cambridge, 1988.
[10] D. Happel, *On Gorenstein algebras*, In: Progress in Math. 95, Birkhäuser Verlag, Basel, 1991, 389–404.
[11] R. Hartshorne, Duality and Residue, Lecture Notes in Math. 20, Springer, Berlin, 1966.
[12] B. Keller and D. Vossieck, *Sous les catégories dérivées,* C.R. Acad. Sci. Paris, t. 305 Série I (1987) 225–228.
[13] H. Krause, *The stable derived category of a noetherian scheme,* Compositio Math. 141 (2005), 1128–1162.
[14] D. Orlov, *Triangulated categories of singularities and D-branes in Landau-Ginzburg models*, Trudy Steklov Math. Institute 204 (2004), 240–262.
[15] J. Rickard, *Derived categories and stable equivalence,* J. Pure Appl. Algebra 61 (1989), 303–317.
[16] S.P. Smith, *The non-commutative scheme having a free algebra as a homogeneous coordinate ring*, arXiv:1104.3822v1.

Xiao-Wu Chen
Department of Mathematics
University of Science
and Technology of China
Wu Wen-Tsun Key Laboratory
of Mathematics, USTC
Chinese Academy of Sciences
PR China
http://mail.ustc.edu.cn/~xwchen