

THE UNIRATIONALITY OF HURWITZ SPACES
OF 6-GONAL CURVES OF SMALL GENUS

FLORIAN GEISS

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ABSTRACT. In this short note we prove the unirationality of Hurwitz spaces of 6-gonal curves of genus g with $5 \leq g \leq 28$ or $g = 30, 31, 35, 36, 40, 45$. Key ingredient is a liaison construction in $\mathbb{P}^1 \times \mathbb{P}^2$. By semicontinuity, the proof of the dominance of this construction is reduced to a computation of a single curve over a finite field.

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1 INTRODUCTION

The study of the birational geometry of moduli spaces of curves with additional structures such as marked points or line bundles is a central topic in algebraic geometry, see for example the books [HM98] and [ACG11]. The Hurwitz space $\mathcal{H}(d, w)$ parametrizes d -sheeted branched simple covers of the projective line by smooth curves of genus g with branch divisor of degree $w = 2g + 2d - 2$ up to isomorphism,

$$\mathcal{H}(d, 2g + 2d - 2) = \{C \xrightarrow{d:1} \mathbb{P}^1 \text{ simply branched} \mid C \text{ smooth of genus } g\} / \sim .$$

It is a classical result by Arbarello and Cornalba [AC81] based on a work of Segre [Seg28] that these spaces are unirational for all $d \leq 5$ and all $g \geq d - 1$ and in few cases for higher gonality, namely for $d = 6$ and $5 \leq g \leq 10$ or $g = 12$ and for $d = 7$ and $g = 7$.

In this paper we present the following extension of this result to significantly higher genus for 6-gonal curves.

THEOREM 1.1. *Over an algebraically closed field of characteristic zero, the Hurwitz spaces $\mathcal{H}(6, 2g + 10)$ of 6-gonal curves of genus g are unirational for*

$$5 \leq g \leq 28 \text{ or } g = 30, 31, 33, 35, 36, 40, 45. \quad (1)$$

Our proof is based on the observation that a general 6-gonal curve in $\mathbb{P}^1 \times \mathbb{P}^2$ can be linked in two steps to the union of a rational curve and a collection of lines. It turns out that for small genera this process can be reversed by starting with a general rational curve and general lines.

To show that the described construction yields a parametrization of the Hurwitz space, we only need to run the construction for a single curve over a finite field. Semicontinuity then ensures that all assumptions we made actually hold for an open dense subset of $\mathcal{H}(6, 2g+10)$ in characteristic zero. Since the construction works a priori only for finitely many genera we settle for a computer-aided verification using the computer algebra system *Macaulay2* [GS].

An immediate consequence of our approach is that in the considered cases the general 6-gonal curve has a plane model as expected from Brill-Noether theory.

COROLLARY 1.2. *For g among (1) and $d = \lceil \frac{2}{3}g + 2 \rceil$ the Brill-Noether locus $W_d^2(C)$ of a general curve $C \in \mathcal{H}(6, 10 + 2g)$ has a smooth generically reduced component of expected dimension $\rho(g, 2, d)$.*

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2 PRELIMINARIES

Throughout this paper, we fix the following notation: Let $\mathbb{P} = \mathbb{P}^1 \times \mathbb{P}^2$ be the product of the projective line and the projective plane over a field K with projections $\pi_1 : \mathbb{P} \rightarrow \mathbb{P}^1$ and $\pi_2 : \mathbb{P} \rightarrow \mathbb{P}^2$. For $a, b \in \mathbb{Z}$ we write

$$\mathcal{O}_{\mathbb{P}}(a, b) = \mathcal{O}_{\mathbb{P}^1}(a) \boxtimes \mathcal{O}_{\mathbb{P}^2}(b) = \pi_1^* \mathcal{O}_{\mathbb{P}^1}(a) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^2}(b)$$

and denote with $R = \bigoplus_{i,j} H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(i, j)) \cong K[x_0, x_1, y_0, y_1, y_2]$ the bihomogeneous coordinate ring of \mathbb{P} . By a curve C in \mathbb{P} , we mean an equidimensional subscheme of codimension 2 which is locally a complete intersection. We say that C is (*geometrically*) *linked* to a curve $C' \subset \mathbb{P}$ by a complete intersection $X \subset \mathbb{P}$ if C and C' have no common components and $C \cup C' = X$. The Chow ring of \mathbb{P} is generated by classes α and β which are the pullback of a point in \mathbb{P}^1 and the pullback of a hyperplane in \mathbb{P}^2 , respectively. The bidegree (d_1, d_2) of a curve C is given by $d_1 = [C].\alpha$ and $d_2 = [C].\beta$.

As in the classical setting of liaison of subschemes in \mathbb{P}^n , we have the following

PROPOSITION 2.1 (Exact sequence of liaison). *Let C be a curve of bidegree (d_1, d_2) that is linked to C' via a complete intersection X defined by forms of bidegree (a_1, b_1) and (a_2, b_2) . We set $a = a_1 + a_2$ and $b = b_1 + b_2$.*

(a) *There is an exact sequence*

$$0 \rightarrow \omega_C \rightarrow \omega_X \rightarrow \mathcal{O}_{C'}(a - 2, b - 3) \rightarrow 0.$$

(b) *The curve C' has bidegree $(d'_1, d'_2) = (b_1b_2 - d_1, a_1b_2 + a_2b_1 - d_2)$ and arithmetic genus $p_a(C') = p_a(X) - (d_1(a - 2) + d_2(b - 3) + (1 - p_a(C)))$.*

Proof. To prove the first part, consider the standard exact sequence

$$0 \rightarrow \mathcal{I}_{C/X} \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_C \rightarrow 0$$

and apply $\text{Hom}_{\mathcal{O}_{\mathbb{P}}}(-, \omega_{\mathbb{P}})$. From the long exact sequence, we get

$$0 \rightarrow \omega_C \rightarrow \omega_X \rightarrow \mathcal{E}xt^2(\mathcal{I}_{C/X}, \omega_{\mathbb{P}}) \rightarrow 0$$

but $\mathcal{E}xt^2(\mathcal{I}_{C/X}, \omega_{\mathbb{P}}) = \mathcal{O}_{C'}(a - 2, b - 3)$ since C and C' are linked by X . The formula for the genus follows immediately. To compute the bidegree, note that $[C] + [C'] = [X] = (b_1b_2)\beta^2 + (a_1b_2 + a_2b_1)\alpha\beta$ in the Chow ring of \mathbb{P} . \square

Recall the following well-known fact about minimal resolutions of points in the plane.

PROPOSITION 2.2. *Let Δ be a collection of δ general points in \mathbb{P}^2 and let k be maximal under the condition $\varepsilon = \delta - \binom{k+1}{2} \geq 0$. Then the minimal free resolution of \mathcal{O}_{Δ} is of the form*

$$0 \rightarrow \mathcal{G} \rightarrow \mathcal{F} \rightarrow \mathcal{O}_{\mathbb{P}^2} \rightarrow \mathcal{O}_{\Delta} \rightarrow 0$$

with $\mathcal{F} = \mathcal{O}(-k)^{k+1-\varepsilon}$ and $\mathcal{G} = \mathcal{O}(-k-1)^{k-2\varepsilon} \oplus \mathcal{O}(-k-2)^{\varepsilon}$ if $2\varepsilon \leq k$ and $\mathcal{F} = \mathcal{O}(-k)^{k+1-\varepsilon} \oplus \mathcal{O}(-k-1)^{2\varepsilon-k}$ and $\mathcal{G} = \mathcal{O}(-k-2)^{\varepsilon}$ else.

Proof. [Gae51] \square

We also note the following simple but useful criterion for the irreducibility of plane curves. Recall that a variety over a field K is called absolutely irreducible if it is irreducible as a variety over the algebraic closure \overline{K} .

PROPOSITION 2.3. *Let C be a plane curve of degree d with $\delta \leq \frac{d(d-3)}{2}$ ordinary double points and no other singularities. If the singular locus Δ of C has a resolution as in 2.2 then C is absolutely irreducible.*

Proof. Assume that C decomposes into two curves C_1 and C_2 of degree d_1 and d_2 defined by homogeneous polynomials f_1 and f_2 . By assumption, C_1 and C_2 intersect transversely in $d_1 \cdot d_2$ distinct points. First, we reduce to the case $d_1, d_2 \leq k$ where $k = \lceil (\sqrt{9 + 8\delta} - 3)/2 \rceil$ is the minimal degree of generators of

I_Δ . Clearly, the case that one of the generators has degree strictly larger than $k+1$ is not possible since $I_\Delta \subset (f_1, f_2)$ is generated in degree k and (possibly) $k+1$. The cases $d_1 = k+1$, say, and $d_2 \leq k+1$ can be excluded by considering the number of minimal generators of I_Δ in degrees k and $k+1$.

We are left with the case $d_1, d_2 \leq k$. Trivially, we can assume that $\delta - d_1 d_2 \geq 0$. A polynomial of the form $sf_1 + tf_2$ of degree k lies in I_Δ if it vanishes at the remaining $\delta - d_1 d_2$ points. Hence,

$$\begin{aligned} h^0(\mathcal{I}_\Delta(k)) &\geq \binom{k-d_1+2}{2} + \binom{k-d_2+2}{2} - \delta + d_1 d_2 \\ &= 2 \binom{k+2}{2} + \binom{d-1}{2} - (dk+1) - \delta \end{aligned}$$

But this is strictly larger than $\binom{k+2}{2} - \delta$ since $d \geq k+3$. \square

Recall from [ACGH85] the following facts from Brill-Noether theory: For a fixed smooth curve C of genus g , the Brill-Noether locus

$$W_d^r(C) = \{L \in \text{Pic}^d(C) \mid h^0(L) \geq r+1\} \quad (2)$$

is of dimension at least equal to the Brill-Noether number

$$\rho(g, r, d) = g - (r+1)(g-d+r). \quad (3)$$

The tangent space at a linear series $L \in W_d^r(C) \setminus W_d^{r+1}(C)$ is the dual of the cokernel of the Petri-map

$$\mu_L : H^0(C, L) \otimes H^0(C, \omega_C \otimes L^{-1}) \rightarrow H^0(C, \omega_C) \quad (4)$$

Hence, $W_d^r(C)$ is smooth of dimension ρ at L if and only if μ_L is injective.

PROPOSITION 2.4. *Let C be a smooth curve of genus $g \geq 3$ with $|D|$ a base point free \mathfrak{g}_d^2 , $d = \lceil \frac{2g}{3} + 2 \rceil$, such that the image of C under the associated map is a plane curve with $\delta = \binom{d-1}{2} - g$ ordinary double points and no other singularities. If the singular locus Δ has a resolution as in 2.2 then $|D|$ is a smooth point in $W_d^2(C)$.*

Proof. By adjunction, the Petri map for $\mathcal{O}(D)$ can be identified with

$$H^0(\mathbb{P}^2, \mathcal{O}(1)) \otimes H^0(\mathbb{P}^2, \mathcal{I}_\Delta(d-4)) \rightarrow H^0(\mathbb{P}^2, \mathcal{I}_\Delta(d-3)).$$

Under the given assumptions the minimal degree of generators of I_Δ is precisely $k = d - 4$. As $2\varepsilon \leq k$ we see from the minimal free resolution of I_Δ that the Petri map is injective since there are no linear relations among the generators of degree k and $k+1$. \square

3 LIAISON CONSTRUCTION

For $g \geq 5$, let $f : C \rightarrow \mathbb{P}^1$ be an element of $\mathcal{H}(6, 10 + 2g)$ and let $\mathcal{O}(D_1) = f^*\mathcal{O}_{\mathbb{P}^1}(1)$ be the 6-gonal bundle. We assume that C has a line bundle $\mathcal{O}(D_2)$ such that $|D_2|$ is a complete base point free \mathfrak{g}_d^2 with $d = d(g) = \lceil \frac{2g}{3} + 2 \rceil$ minimal under the condition that the Brill-Noether number $\rho(g, 2, d) \geq 0$. Suppose further that the map

$$\varphi : C \xrightarrow{|D_1|, |D_2|} \mathbb{P}H^0(\mathcal{O}(D_1)) \times \mathbb{P}H^0(\mathcal{O}(D_2)) = \mathbb{P} \tag{5}$$

is an embedding. In particular, this is the case when we assume that the plane model has only ordinary double points and no other singularities and for any node p the points in the preimage of p under $C \rightarrow \mathbb{P}^2$ are not identified under the map to \mathbb{P}^1 .

Hence, we can identify C with its image under φ . Furthermore, we assume that the map $H^0(\mathcal{O}_{\mathbb{P}}(a, 3)) \rightarrow H^0(\mathcal{O}_C(a, 3))$ is of maximal rank for all $a \geq 1$. To simplify matters, assume $g \equiv 0 \pmod{12}$ for the moment. By the maximal rank assumption, we have

$$a_{\text{Cubic}} := \min\{a \mid H^0(\mathcal{I}_C(a, 3)) \neq 0\} = \frac{g}{4} \tag{6}$$

and $h^0(\mathcal{I}_C(a_{\text{Cubic}}, 3)) = 3$. Let $X = V(f_1, f_2)$ be the complete intersection defined by two general sections $f_i \in H^0(\mathcal{I}_C(a_i, b_i))$ of bidegrees $(a_1, b_1) = (a_2, b_2) = (a_{\text{Cubic}}, 3)$. The curve C' , obtained by liaison of C by X , is smooth of bidegree $(3, \frac{5}{6}g - 2)$ and genus $g' = \frac{g}{2} - 3$ with $h^0(\mathcal{I}_{C'}(a_{\text{Cubic}}, 3)) \geq 2$.

The geometric situation is understood best when thinking of C as a family of collections of plane points over \mathbb{P}^1 . We expect the general fiber of C to be a collection of 6 points in \mathbb{P}^2 which are cut out by 4 cubics. We expect a finite number ℓ of distinguished fibers where the points lie on a conic as this is a codimension 1 condition on the points. Since the residual three points under liaison are collinear exactly in the distinguished fibers we can compute ℓ by examining the geometry of C' . The projection of C' to \mathbb{P}^2 yields a divisor D'_2 of degree $d' > g' + 2$. Our claim is that $\ell = d' - (g' + 2)$. Indeed, the image of C' under the associated map

$$\psi : C' \rightarrow \mathbb{P}^1 \times \mathbb{P}H^0(C', \mathcal{O}(D'_2)) = \mathbb{P}^1 \times \mathbb{P}^{d'-g'} \tag{7}$$

lies on the graph of the projection $S \rightarrow \mathbb{P}^1$ where S is a 3-dimensional scroll of degree $d' - g' - 2$ swept out by the 3-gonal series $|D'_1|$, i.e.

$$\psi(C') \subset \mathbb{P}^1 \times S = \bigcup_{D \in |D'_1|} \{D\} \times \overline{D}. \tag{8}$$

See [Sch86] for a proof of this fact. C' is obtained from $\psi(C')$ by projection from a linear subspace $\mathbb{P}^1 \times V \subset \mathbb{P}^1 \times \mathbb{P}^{d'-g'}$ of codimension 3. A general space V intersects S in precisely $d' - g' - 2$ points lying in distinct fibers over \mathbb{P}^1 .

Clearly, under the projection the points of $D \in |D_1|$ are mapped to 3 collinear points if and only if V meets the corresponding fiber of S .

To keep things neat, we consider again the case $g \equiv 0$ (12) which implies $\ell = \frac{1}{3}g - 1$. Suppose further that $\ell \equiv 1$ (3). If we assume that the map $H^0(\mathcal{O}_{\mathbb{P}}(a, 2)) \rightarrow H^0(\mathcal{O}_{C'}(a, 2))$ is of maximal rank for all $a \geq 1$ then

$$a_{\text{Conic}} = \min\{a \mid H^0(\mathcal{I}_{C'}(a, 2)) \neq 0\} = \frac{g' + 2\ell + 1}{3} \quad (9)$$

and $h^0(\mathcal{I}_{C'}(a_{\text{Conic}}, 2)) = 2$. Let $X' = V(f'_1, f'_2)$ be defined by two general forms $f'_i \in H^0(\mathcal{I}(a'_i, b'_i))$ of bidegrees $(a'_1, b'_1) = (a'_2, b'_2) = (a_{\text{Conic}}, 2)$ and let C'' denote the curve that is linked to C' via X' . The general fiber of C'' consists of a single point. In a distinguished fiber the conics of the complete intersection are reducible and have the line spanned by the points of the fiber of C' as a common factor. Hence, C'' is a rational curve together with ℓ lines. The rational curve has degree

$$d'' = \frac{g' + 2\ell - 2}{3} = \frac{7}{18}g - \frac{7}{3}. \quad (10)$$

Turning things around we see that the difficulty lies in reversing the first linkage step. Indeed, a simple counting argument shows that for any g , the union C'' of ℓ general lines in \mathbb{P} and the graph of a general rational normal curve of degree d'' we have

$$\min\{a \in \mathbb{Z} \mid H^0(\mathcal{I}_{C''}(a, 2)) \neq 0\} = \left\lceil \frac{2d'' + 3\ell}{5} \right\rceil - 1 \leq a_{\text{Conics}}.$$

Hence, we always obtain a trigonal curve C' as desired. However, for general choices of C'' and X' we expect that the map $H^0(\mathcal{O}_{\mathbb{P}}(a, 3)) \rightarrow H^0(\mathcal{O}_{C'}(a_{\text{Cubic}}, 3))$ is of maximal rank. In the case $g \equiv 0$ (12), this restriction yields $h^0(\mathcal{I}_{C'}(a_{\text{Cubic}}, 3)) = -\frac{g}{4} + 12$, hence $g < 48$. Checking all congruency classes of g , we expect that C' can be linked to a general curve C exactly in the cases

$$5 \leq g \leq 28 \text{ or } g = 30, 31, 33, 35, 36, 40, 45. \quad (11)$$

Table 1 lists the appearing numbers for all values of g in (11).

Summarizing, we obtain for g among (11) the following unirational construction for curves in $\mathcal{H}(6, 10 + 2g)$:

1. We start with a general rational curve of degree d'' in \mathbb{P} together with a collection of ℓ general lines. Call the union C'' .
2. We choose two general forms $f'_i \in H^0(\mathcal{I}_{C''}(a'_i, b'_i))$, $i = 1, 2$, that define a complete intersection X' and obtain a trigonal curve $C' = \overline{X' \setminus C''}$ of degree d' and genus g' .
3. We choose two general forms $f_i \in H^0(\mathcal{I}_{C'}(a_i, b_i))$, $i = 1, 2$, that define a complete intersection X and obtain a 6-gonal curve $C = \overline{X \setminus C'}$.

It remains to show that the construction actually yields a parametrization of the Hurwitz spaces.

g	d	$(a_1, b_1), (a_2, b_2)$	g'	d'	$(a'_1, b'_1), (a'_2, b'_2)$	ℓ	d''
5	6	(2, 3), (2, 3)	2	6	(3, 2), (2, 2)	2	2
6	6	(2, 3), (1, 3)	0	3	(1, 2), (1, 2)	1	0
7	7	(2, 3), (2, 3)	1	5	(2, 2), (2, 2)	2	1
8	8	(3, 3), (2, 3)	2	7	(3, 2), (3, 2)	3	2
9	8	(2, 3), (2, 3)	0	4	(2, 2), (2, 2)	2	2
10	9	(3, 3), (3, 3)	4	9	(4, 2), (4, 2)	3	4
11	10	(3, 3), (3, 3)	2	8	(4, 2), (4, 2)	4	4
12	10	(3, 3), (3, 3)	3	8	(4, 2), (3, 2)	3	3
13	11	(4, 3), (3, 3)	4	10	(5, 2), (4, 2)	4	4
14	12	(4, 3), (4, 3)	5	12	(6, 2), (5, 2)	5	5
15	12	(4, 3), (4, 3)	6	12	(5, 2), (5, 2)	4	4
16	13	(4, 3), (4, 3)	4	11	(5, 2), (5, 2)	5	4
17	14	(5, 3), (5, 3)	8	16	(7, 2), (7, 2)	6	6
18	14	(5, 3), (4, 3)	6	13	(6, 2), (6, 2)	5	6
19	15	(5, 3), (5, 3)	7	15	(7, 2), (7, 2)	6	7
20	16	(6, 3), (5, 3)	8	17	(8, 2), (8, 2)	7	8
21	16	(5, 3), (5, 3)	6	14	(7, 2), (6, 2)	6	6
22	17	(6, 3), (6, 3)	10	19	(9, 2), (8, 2)	7	8
23	18	(6, 3), (6, 3)	8	18	(9, 2), (8, 2)	8	8
24	18	(6, 3), (6, 3)	9	18	(8, 2), (8, 2)	7	7
25	19	(7, 3), (6, 3)	10	20	(9, 2), (9, 2)	8	8
26	20	(7, 3), (7, 3)	11	22	(10, 2), (10, 2)	9	9
27	20	(7, 3), (7, 3)	12	22	(10, 2), (10, 2)	8	10
28	21	(7, 3), (7, 3)	10	21	(10, 2), (10, 2)	9	10
30	22	(8, 3), (7, 3)	12	23	(11, 2), (10, 2)	9	10
31	23	(8, 3), (8, 3)	13	25	(12, 2), (11, 2)	10	11
33	24	(8, 3), (8, 3)	12	24	(11, 2), (11, 2)	10	10
35	26	(9, 3), (9, 3)	14	28	(13, 2), (13, 2)	12	12
36	26	(9, 3), (9, 3)	15	28	(13, 2), (13, 2)	11	13
40	29	(10, 3), (10, 3)	16	31	(15, 2), (14, 2)	13	14
45	32	(11, 3), (11, 3)	18	34	(16, 2), (16, 2)	14	16

Table 1: Numerical data for all cases of the construction

4 PROOF OF THE DOMINANCE

THEOREM 4.1. *For all (g, d) as in Table 1, there is a unirational component H_g of the Hilbert scheme $\text{Hilb}_{(6,d),g}(\mathbb{P})$ of curves in \mathbb{P} of bidegree $(6, d)$ and genus g . The generic point of H_g corresponds to a smooth absolutely irreducible curve C such that the map $H^0(\mathcal{O}_{\mathbb{P}}(a, 3)) \rightarrow H^0(\mathcal{O}_C(a, 3))$ is of maximal for all $a > 1$.*

Proof. The crucial part is to prove the existence of a curve with the desired properties. Code 5.1 implements the construction above for any given value of g in (11) and establishes the existence of a smooth and absolutely irreducible curve C_p of given genus and bidegree defined over a prime field \mathbb{F}_p . This computation can be regarded as the reduction of a computation over \mathbb{Q} which yields some curve C_0 . This curve is already defined over the rationals, since all construction steps invoke only Groebner basis computations. By semicontinuity, C_0 is also smooth, absolutely irreducible and of maximal rank.

Again, by semicontinuity, there is a Zariski open neighborhood $U \subset \text{Hilb}_{(6,d),g}(\mathbb{P})$ of points corresponding to smooth absolutely irreducible curves that fulfill the maximal rank condition. Let \mathbb{A}^N be the parameter-space for all the choices made in the construction, i.e. the space of coefficients of the polynomials defining C'' and the complete intersections X and X' . The construction then translates to a rational map $\mathbb{A}^N \dashrightarrow U$ defined over \mathbb{Q} and we set H_g to be the closure of the image of this map. \square

REMARK 4.2. *We want to point out two issues concerning the computational verification:*

1. *The restriction to finite fields in the Macaulay2 computation in the appendix is only due to limitations in computational power. For very small values of g , i.e. $g \leq 15$, it is still possible to compute examples over the rationals if all coefficients are chosen among integers of small absolute value.*
2. *The reduction of C_0 modulo p gives a curve C_p with desired properties for p in an open part of $\text{Spec}(\mathbb{Z})$. Hence, the main theorem is also true in almost all characteristics p . One way to extend it to all prime numbers would be to keep track of all denominators in a computation over the rationals and check case by case the primes where a bad reduction happens. This is computationally also out of reach at the moment.*

It remains to show that there exists a dominant rational map from H_g to the Hurwitz-scheme.

THEOREM 4.3. *For g among (11) and H_g as in Theorem 4.1 there is a dominant rational map*

$$H_d \dashrightarrow \mathcal{H}(6, 10 + 2g).$$

Proof. Using Code 5.1 again, we check for any given value of g in (11) the existence of a point in H_g corresponding to a smooth absolutely irreducible

curve $C \subset \mathbb{P}^1$ such that the projection onto \mathbb{P}^1 is simply branched and the bundle $L_2 = \varphi^* \mathcal{O}_{\mathbb{P}^1}(0, 1)$ is a smooth point in the corresponding $W_d^2(C)$. By semicontinuity, the loci of curves with this property is open and dense in H_g . Hence, we have a rational map $H_g \dashrightarrow \mathcal{H}(6, 10 + 2g)$. The locus of curves in $\mathcal{H}(6, 10 + 2g)$ having a smooth component of the Brill-Noether loci of expected dimension is also open and contains the image of $[C]$ under this map. Since $\mathcal{H}(6, 10 + 2g)$ is irreducible this locus is dense. This proves the theorem. \square

We want to emphasize the last statement in the proof:

COROLLARY 4.4. *For g among (11) and $d = \lceil \frac{2}{3}g + 2 \rceil$ the Brill-Noether locus $W_d^2(C)$ of a general curve $C \in \mathcal{H}(6, 10 + 2g)$ has a smooth generically reduced component of expected dimension ρ .*

5 COMPUTATIONAL VERIFICATION

The following Code for *Macaulay2* [GS] realizes the unirational construction of a 6-gonal curve of genus g as in (11) over a finite field $K = \mathbb{F}_p$ with random choices for all parameters.

In order to explain the single steps in the computation, we also print the most relevant parts of the output for the example case $g = 24$.

CODE 5.1. *We start with the following initialization:*

```
i1 : Fp=ZZ/32009; -- a finite field
    S=Fp[x_0,x_1,y_0..y_2,Degrees=>{2:{1,0},3:{0,1}}];
    -- Cox-ring of P^1 x P^2
m=ideal basis({1,1},S);
    -- irrelevant ideal
setRandomSeed("HurwitzSpaces");
    -- initialization of the random number generator
```

The following functions handle the numerics of the construction:

```
i2 : expHilbFuncIdealSheaf=(g,d,a)->
    max(0,(a_0+1)*(a_1+2)*(a_1+1)/2-(a_0*d_0+a_1*d_1+1-g))
    -- expected number of sections of the ideal sheaf

linkedGenus=(g,d,F,G)->(
    pX:=binomial(F_0+G_0-1,1)*binomial(F_1+G_1-1,2)-
        (F_0-1)*binomial(F_1-1,2)-(G_0-1)*binomial(G_1-1,2);
    -- genus of the complete intersection
    pX-d_0*(F_0+G_0-2)-d_1*(F_1+F_1-3)-1+g)
    -- genus of the linked curve

linkedDegree=(g,d,F,G)->{F_1*G_1-d_0,F_0*G_1+G_0*F_1-d_1}
    -- bidegree of the linked curve
```

The first step is to determine the degree d'' of the rational curve and the number of lines ℓ . We start by computing the bidegrees of the forms that define the complete intersection for the linkage to the trigonal curve:

```

i3 : g=24;
     d={6,ceiling(-g/3+g+2)};
     -- choose the second degree Brill-Noether general
a=for i from 0 do
    if expHilbFuncIdealSheaf(g,d,{i,3})!=0 then break i;
    -- find the minimal value a s.t. H^0(IC(a,3)) nonzero
if expHilbFuncIdealSheaf(g,d,{a,3})==1 then
    fX={{a+1,3},{a,3}} else fX={{a,3},{a,3}};
    -- choose bidegrees of forms for the complete intersection
    (d,fX)

```

```
o3 = ({6, 18}, {{6, 3}, {6, 3}})
```

The genus and degree of the trigonal curve and the number of lines:

```

i4 : g'=linkedGenus(g,d,fX_0,fX_1);
     d'=linkedDegree(g,d,fX_0,fX_1);
     l=d'_1-g'-2;
     (g',d',l)

```

```
o4 = (9,{3,18},7)
```

We compute the bidegrees for the complete intersection for the linkage to the rational curve

```

i5 : b=for i from 0 do
    if expHilbFuncIdealSheaf(g',d',{i,2})!=0 then break i;
if expHilbFuncIdealSheaf(g',d',{b,2})==1 then
    fX'={{b+1,2},{b,2}} else fX'={{b,2},{b,2}};
d''=linkedDegree(g'+2*1,d'+{0,1},fX'_0,fX'_1);
dRat={{ceiling(d''_1/2),1},{floor(d''_1/2),1}};
(fX',d'')

```

```
o5 = ({8, 2}, {8, 2}), {1, 7})
```

The second step is the actual construction: First, we choose a rational curve and random lines and compute the saturated vanishing ideal $I_{C''}$ of their union:

```

i6 : ICrat=saturate(ideal random(S^1,S^(-dRat)),m);
     ILines=apply(1,i->ideal random(S^1,S^{{-1,0},{0,-1}}));
     time IC''=saturate(intersect(ILines|{ICrat}),ideal(x_0*y_0));
     -- used 1.29537 seconds

```

Next, we choose random forms in $I_{C''}$ of degree b (resp. of $b+1$) that define the complete intersection X' and compute the saturated vanishing ideal $I_{C'}$ of the trigonal curve C' .

```

i7 : IX'=ideal(gens IC'' * random(source gens IC'',S^(-fX')));
     IC'=IX':ICrat;
     time scan(1,i->IC'=IC':ILines_i);
     time IC'sat=saturate(IC',ideal(x_0*y_0));
     -- used 2.06236 seconds
     -- used 23.7319 seconds

```

In the final step, we compute the vanishing ideal of the 6-gonal curve C by linking C' with a complete intersection X given by random forms in IC' of degree a (resp. $a + 1$).

```
i8 : IX=ideal(gens IC'sat * random(source gens IC'sat,S^(-fX)));
      time IC=IX:IC';
      time ICsat=saturate(IC,ideal(x_0*y_0));
      -- used 15.7815 seconds
      -- used 3.84807 seconds
```

We check that C is of maximal rank in the degrees $(a, 3)$ by looking at the minimal generators of the saturated vanishing ideal:

```
i9 : tally degrees ideal mingens gb ICsat

o9 = Tally{{0, 18} => 1}
      {1, 14} => 5
      {1, 15} => 4
      {2, 8} => 2
      {2, 9} => 8
      {3, 6} => 9
      {4, 4} => 2
      {4, 5} => 8
      {5, 4} => 7
      {6, 3} => 3
      {7, 3} => 1
```

In order to check irreducibility, we compute the plane model Γ of C :

```
i10 : Sel=Fp[x_0,x_1,y_0..y_2,MonomialOrder=>Eliminate 2];
      -- elimination order
      R=Fp[y_0..y_2]; -- coordinate ring of P^2
      IGammaC=sub(ideal selectInSubring(1,gens gb sub(ICsat,Sel)),R);
      -- ideal of the plane model
```

We check that Γ is a curve of desired degree and genus and its singular locus Δ consists only of ordinary double points:

```
i11 : distinctPoints=(J)->(
      singJ:=minors(2,jacobian J)+J;
      codim singJ==3)

i12 : IDelta=ideal jacobian IGammaC + IGammaC; -- singular locus
      distinctPoints(IDelta)

o12 = true

i13 : delta=degree IDelta;
      dGamma=degree IGammaC;
```

```
gGamma=binomial(dGamma-1,2)-delta;
(dGamma,gGamma)==(d_1,g)
```

```
o13 = true
```

We compute the free resolution of I_Δ :

```
i14 : time IDelta=saturate IDelta;
      betti res IDelta
      -- used 55.063 seconds
```

```
          0 1 2
o14 = total: 1 8 7
          0: 1 . .
          1: . . .
          2: . . .
          3: . . .
          4: . . .
          5: . . .
          6: . . .
          7: . . .
          8: . . .
          9: . . .
         10: . . .
         11: . . .
         12: . . .
         13: . 8 .
         14: . . 7
```

This is the resolution as expected. Hence, C is absolutely irreducible by Proposition 2.2 and $\mathcal{O}(D_2)$ is a smooth point of the Brill-Noether loci by Proposition 2.4.

It remains to verify that C is actually smooth and simply branched. We compute the vanishing ideal $I_B \subset K[x_0, x_1]$ of the locus B in \mathbb{P}^1 of points with non-reduced fiber.

```
i15 : gensICsat=flatten entries mingens ICsat;
      Icubics=ideal select(gensICsat,f->(degree f)_1==3);
      -- select the cubic forms
      Jacobian=diff(matrix{{y_0}..{y_2}},gens Icubics);
      -- compute the jacobian w.r.t. to vars of P^2
      IGraphB=minors(2,Jacobian)+Icubics;
      time IGraphBsat=saturate(IGraphB,ideal(x_0*y_0));
      -- used 60.2963 seconds
```

We check that the fibers over B are disjoint from the preimages of the double points of the plane model. This shows that C is smooth:

```
i16 : time ISing=saturate(sub(IDelta,S)+IGraphBsat,ideal(S_0*S_2));
      degree ISing==0
```

```
o16 = true
```

Finally, we verify that B is reduced of expected degree $2g + 10$ and hence that C is simply branched.

```
i17 : time IGraphBsat=saturate(IGraphB,ideal(x_0*y_0));
      gensIGraphBsat=flatten entries mingens IGraphBsat;
      IB=ideal select(gensIGraphBsat,f->(degree f)_1==0);
      degree radical IB==2*g+10
```

```
o17 = true
```

This code is available in form of a *Macaulay2*-file from [G11] for download. It takes approximately 5 hours CPU-time on a 2.4 GHz processor to check all cases.

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Florian Geiß
Mathematik und Informatik
Universität des Saarlandes,
Campus E2 4,
66123 Saarbrücken,
Germany
fg@math.uni-sb.de