Duality for $\mathbb{Z}$-Constructible Sheaves on Curves over Finite Fields

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Abstract. We prove a duality theorem for Weil-etale cohomology of $\mathbb{Z}$-constructible sheaves on curves over finite fields.

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1 Introduction

Let X be a (possibly singular) curve over a finite field and $\mathcal{F}$ be a $\mathbb{Z}$-constructible sheaf on X. In [3], Deninger defined a dualizing complex $\mathbb{G}$, and in [2] he proves a duality between the groups $H^i_c(X_{\text{et}}, \mathcal{F})$ and $\text{Ext}^{3-i}_{\text{et}}(\mathcal{F}, \mathbb{G})$. The cohomology groups are finitely generated for $i = 0, 1$, of cofinite type for $i = 2, 3$, and conversely for the extension groups). If X is smooth, Lichtenbaum [9] proved a duality of finitely generated groups between Weil-etale cohomology groups with compact support $H^i_c(X_W, \mathbb{Z})$ and Weil-etale cohomology groups $H^i(X_W, \mathbb{G}_m)$.

In this paper, we generalize and unify these results. We define, for any curve X over a finite field and any $\mathbb{Z}$-constructible sheaf $\mathcal{F}$, Weil-etale Borel-Moore homology groups $H^c_i(X_{\text{ar}}, \mathcal{F})$, and construct a pairing of finitely generated abelian groups between Weil-etale cohomology with compact supports $H^i_c(X_W, \mathcal{F})$ and $H^c_i(X_{\text{ar}}, \mathcal{F})$. More precisely, for $0 \le i \le 2$, there are pairings of finitely generated free groups

$$H^c_i(X_{\text{ar}}, \mathcal{F})/\text{tor} \times H^i_c(X_W, \mathcal{F})/\text{tor} \to \mathbb{Z};$$

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and for $0 \leq i \leq 3$ there are pairings of finite groups

$$H^c_{i-1}(X_{\mathrm{ar}}, \mathcal{F})_{\mathrm{tor}} \times H^i_c(X_W, \mathcal{F})_{\mathrm{tor}} \to \mathbb{Q}/\mathbb{Z}.$$

All other cohomology and homology groups vanish.
If X is smooth, then the groups $H^c_i(X_{\mathrm{ar}}, \mathbb{Z})$ are isomorphic to Weil-etale cohomology with $\mathbb{G}_m$-coefficients, and we recover Lichtenbaum's result. In fact, the author's original motivation was to understand the asymmetry between the coefficients $\mathbb{Z} = \mathbb{Z}(0)$ and $\mathbb{G}_m = \mathbb{Z}(1)[1]$ in Lichtenbaum's result: the groups $H^i(X_W, \mathbb{G}_m)$ are finitely generated, but in general $H^i(X_W, \mathbb{Z})$ is not. For this reason, one also cannot expect duality results for positive weights. The relationship to Deninger's result is given by long exact sequences relating etale and Weil-etale cohomology and extension groups [4], see below.
The strategy of proof of our duality result is to first show finite generation of the groups involved, and then reduce to the case of torsion coefficients, which was treated in [6] for arbitrary schemes over finite fields.
We note that for higher dimensional schemes, Weil-etale cohomology groups are not well-behaved. Instead one has to use the eh-topology, see the discussion and example in [5]. But in this case, the standard methods to construct the pairing fail. Moreover, a duality result as the above would for $\mathcal{F} = \mathbb{Z}$ imply that $CH_0(X, i)_{\mathbb{Q}} = 0$ for $i > 0$ and X smooth and projective. This is a special case of Parshin's conjecture that $K_i(X)_{\mathbb{Q}} = 0$ for $i > 0$ and X smooth and projective over a finite field (Parshin's conjecture is known if X is a curve).
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2 Arithmetic homology and cohomology

2.1 The dualizing complex

We recall some properties of Bloch's higher Chow complex [1]. For a scheme X essentially of finite type over a fixed field k, $z_0(X, i)$ is defined as the free abelian group generated by closed integral subschemes of dimension i on $X \times_k \Delta^i$ which meet all faces properly. If $z_0(X, *)$ is the (homological) complex of abelian groups obtained by taking the alternating sum of intersection with face maps as differentials, then varying X we obtain a complex of sheaves $z_0(-, *)$ for the etale topology. By definition, the higher Chow group $CH_0(X, i)$ is the homology in degree i of $z_0(X, *)$. We define $\mathbb{Z}^c_X = z_0(-, *)$ to be the (cohomological) complex which is the etale sheaf $z_0(-, i)$ in degree $-i$, and omit X if there is no ambiguity. For a quasi-finite, flat map $f : X \to Y$, there is a pull-back $f^*\mathbb{Z}^c_Y \to \mathbb{Z}^c_X$, and for a proper map $f : X \to Y$ there is a push-forward $f_*\mathbb{Z}^c_X \to \mathbb{Z}^c_Y$. For a closed embedding $i : Z \to X$ over k, the isomorphism $\mathbb{Z}^c_Z \cong Ri^!\mathbb{Z}^c_X$ on the Zariski-site is called purity or localization property. It implies an isomorphism [1] between cohomology and hypercohomology

$$CH_0(X, i) \cong H^c_i(X_{\mathrm{Zar}}, \mathbb{Z}^c) := H^{-i}(X_{\mathrm{Zar}}, \mathbb{Z}^c). \tag{1}$$

The following result is proved in [6]:

Theorem 2.1 *a) Over an algebraically closed base field, the complex $\mathbb{Z}^c_X$ has etale hypercohomological descent, i.e. if $\mathbb{Z}^c \to I^\cdot$ is an injective resolution of etale sheaves, then $\mathbb{Z}^c(U) \to I^\cdot(U)$ is a quasi-isomorphism for every $U \to X$ etale.*
b) If $f : X \to Y$ is a proper map over a perfect field k, then there is a functorial push-forward $f_* : Rf_*\mathbb{Z}^c_X \to \mathbb{Z}^c_Y$ in the derived category of etale sheaves.
c) If $i : Z \to X$ is a closed embedding over a perfect field k, then we have a quasi-isomorphism $\mathbb{Z}^c_Z \xrightarrow{\sim} Ri^!\mathbb{Z}^c_X$ of etale sheaves on Z.

If $f : X \to k$ is proper over a perfect field, then the trace map agrees on the stalk $\operatorname{Spec}\bar{k}$ with the map sending a complex to its highest cohomology group, composed with the degree map,

$$tr : Rf_*\mathbb{Z}^c_X(\bar{k}) \xleftarrow{\sim} \mathbb{Z}^c_X(X_{\bar{k}}) \to CH_0(X_{\bar{k}}) \xrightarrow{\deg_{\bar{k}}} \mathbb{Z}.$$

By a result of Nart [11], $\mathbb{Z}^c_X$ it is quasi-isomorphic to a shift of Deninger's complex $\mathbb{G}$, but we prefer to work with Bloch's complex, as it yields the correct dualizing complex in higher dimensions.

2.2 Arithmetic Borel-Moore homology and cohomology

We fix a finite field $\mathbb{F}_q$ with Galois group $\hat{G} = \operatorname{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ and let $G \subset \hat{G}$ be its Weil group, the free abelian subgroup of rank 1 generated by the Frobenius endomorphism φ. Given a separated scheme of finite type X over $\mathbb{F}_q$, let $\bar{X} = X \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q$.
For a sheaf $\mathcal{F}$ on the small etale site $\mathrm{Et}^\sim_X$ of X, we define arithmetic Borel-Moore homology groups as the cohomology groups of the complex

$$R\operatorname{Hom}_{\mathrm{ar}}(\mathcal{F}, \mathbb{Z}^c_X) := R\Gamma_G R\operatorname{Hom}_{\mathrm{et}}(\mathcal{F}|_{\bar{X}}, \mathbb{Z}^c_{\bar{X}})[1]$$

so that $H^c_i(X_{\mathrm{ar}}, \mathcal{F}) := Ext^{-i}_{\mathrm{ar}}(\mathcal{F}, \mathbb{Z}^c)$. This generalizes the groups $H^c_i(X_{\mathrm{ar}}, \mathbb{Z})$ considered in [7]. The Leray spectral sequence degenerates to

$$0 \to \operatorname{Ext}^{-i}_{\mathrm{et}}(\mathcal{F}|_{\bar{X}}, \mathbb{Z}^c_{\bar{X}})_G \to H^c_i(X_{\mathrm{ar}}, \mathcal{F}) \to \operatorname{Ext}^{1-i}_{\mathrm{et}}(\mathcal{F}|_{\bar{X}}, \mathbb{Z}^c_{\bar{X}})^G \to 0. \qquad (2)$$

Let $f : X \to Y$ be a map, $\mathcal{F} \in \mathrm{Et}^\sim_Y$ and $\mathcal{G} \in \mathrm{Et}^\sim_X$. If f is proper, then the map $Rf_*\mathbb{Z}^c_X \to \mathbb{Z}^c_Y$ induces by adjunction covariant functoriality $H^c_i(X_{\mathrm{ar}}, f^*\mathcal{F}) \to H^c_i(Y_{\mathrm{ar}}, \mathcal{F})$. If f is a closed embedding, then the quasi-isomorphism $Rf^!\mathbb{Z}^c_Y \cong \mathbb{Z}^c_X$ induces by adjunction an isomorphism $H^c_i(X_{\mathrm{ar}}, \mathcal{G}) \xrightarrow{\sim} H^c_i(Y_{\mathrm{ar}}, f_*\mathcal{G})$. If f is flat and quasi-finite, then the map $f^*\mathbb{Z}^c_Y \to \mathbb{Z}^c_X$ induces contravariant functoriality $H^c_i(Y_{\mathrm{ar}}, \mathcal{F}) \to H^c_i(X_{\mathrm{ar}}, f^*\mathcal{F})$. If f is etale, then since $f^*\mathbb{Z}^c_Y \cong \mathbb{Z}^c_X$ we obtain an isomorphism

$$H^c_i(X_{\mathrm{ar}}, \mathcal{G}) = \operatorname{Ext}^{-i}_{\mathrm{ar}}(\mathcal{G}, \mathbb{Z}^c_X) \cong \operatorname{Ext}^{-i}_{\mathrm{ar}}(f_!\mathcal{G}, \mathbb{Z}^c_Y) = H^c_i(Y_{\mathrm{ar}}, f_!\mathcal{G}). \qquad (3)$$

For a closed subscheme $i : Z \to X$ with open complement $j : U \to X$ and $\mathcal{F} \in \mathrm{Et}^\sim_X$, the short exact sequence $0 \to j_!j^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F} \to 0$ induces a localization sequence

$$\to H^c_{i+1}(U_{\mathrm{ar}}, \mathcal{F}|_U) \to H^c_i(Z_{\mathrm{ar}}, \mathcal{F}|_Z) \to H^c_i(X_{\mathrm{ar}}, \mathcal{F}) \to H^c_i(U_{\mathrm{ar}}, \mathcal{F}|_U) \to . \qquad (4)$$

THEOREM 2.2 *For every etale sheaf $\mathcal{F} \in \mathrm{Et}_X^{\sim}$, there is a long exact sequence*

$$\to \mathrm{Ext}^{1-i}_{\mathrm{et}}(\mathcal{F}, \mathbb{Z}^c_X) \to H^c_i(X_{\mathrm{ar}}, \mathcal{F}) \to \mathrm{Ext}^{-i}_{\mathrm{et}}(\mathcal{F}, \mathbb{Z}^c_X)_{\mathbb{Q}} \xrightarrow{\delta} \mathrm{Ext}^{2-i}_{\mathrm{et}}(\mathcal{F}, \mathbb{Z}^c_X) \to,$$

and the map δ has torsion image. In particular, for any torsion sheaf $\mathcal{F}$,

$$\mathrm{Ext}^{1-i}_{\mathrm{et}}(\mathcal{F}, \mathbb{Z}^c_X) \cong H^c_i(X_{\mathrm{ar}}, \mathcal{F}).$$

If $\mathcal{F}$ is a sheaf of $\mathbb{Q}$-vector spaces, then the long exact sequence is split by (2), *and we obtain*

$$H^c_i(X_{\mathrm{ar}}, \mathcal{F})_{\mathbb{Q}} \cong \mathrm{Ext}^{1-i}_{\mathrm{et}}(\mathcal{F}, \mathbb{Z}^c_X)_{\mathbb{Q}} \oplus \mathrm{Ext}^{-i}_{\mathrm{et}}(\mathcal{F}, \mathbb{Z}^c_X)_{\mathbb{Q}}.$$

Proof. By [4], any complex of $\hat{G}$-modules gives rise to a distinguished triangle $R\Gamma_{\hat{G}} C^\cdot \to R\Gamma_G C^\cdot \to R\Gamma_{\hat{G}} C^\cdot \otimes \mathbb{Q}[-1]$. We apply this to the complex of $\hat{G}$-modules $R\operatorname{Hom}_{\mathrm{et}}(\mathcal{F}|_{\bar{X}}, \mathbb{Z}^c_{\bar{X}})$, and it suffices to show that $R\Gamma_{\hat{G}} R\operatorname{Hom}_{\mathrm{et}}(\mathcal{F}|_{\bar{X}}, \mathbb{Z}^c_{\bar{X}}) \cong R\operatorname{Hom}_{\mathrm{et}}(\mathcal{F}, \mathbb{Z}^c)$. But if $\mathbb{Z}^c \to I^\cdot$ is an injective resolution, then $\operatorname{Hom}_{\bar{X}}(\mathcal{F}, I^j)$ is acyclic for $(-)^{\hat{G}}$ [10, III Cor. 2.13c)], hence the claim follows from $\operatorname{Hom}_{\bar{X}}(\mathcal{F}, \mathcal{G})^{\hat{G}} \cong \operatorname{Hom}_X(\mathcal{F}, \mathcal{G})$ for etale sheaves. □

If $j : U \to X$ is a compactification of the curve U and $\mathcal{F}$ in $\mathrm{Et}_U^{\sim}$, then Weil-etale cohomology is defined as $H^i(U_W, \mathcal{F}) := H^i R\Gamma_G R\Gamma(\bar{U}_{\mathrm{et}}, \mathcal{F})$, and Weil-etale cohomology with compact support as

$$H^i_c(U_W, \mathcal{F}) := H^i R\Gamma_G R\Gamma_c(\bar{X}_{\mathrm{et}}, j_! \mathcal{F}). \tag{5}$$

LEMMA 2.3 *Consider a cartesian diagram*

$$\begin{array}{ccc} Z' & \longrightarrow & X' \\ \downarrow & & \downarrow f \\ Z & \xrightarrow{i} & X. \end{array}$$

with i a closed embedding, f finite, and such that f induces an isomorphism of dense open subsets $X' - Z' \to X - Z$. Then for any etale sheaf $\mathcal{F}$ on X, there is an exact triangle

$$R\Gamma(X_{\mathrm{et}}, \mathcal{F}) \to R\Gamma(Z_{\mathrm{et}}, \mathcal{F}|_Z) \oplus R\Gamma(X'_{\mathrm{et}}, \mathcal{F}|_{X'}) \to R\Gamma(Z'_{\mathrm{et}}, \mathcal{F}|_{Z'}).$$

Proof. This follows because for finite maps f, the functor f_* is exact. Concretely, if d is the map $Z' \to X$, then $R^s f_* \mathcal{F}|_{X'} = R^s d_* \mathcal{F}|_{Z'}$ for $s > 0$. Hence the triangle is induced by the short exact sequence of sheaves on X, $0 \to \mathcal{F} \to i_* \mathcal{F}|_Z \oplus f_* \mathcal{F}|_{X'} \to d_* \mathcal{F}|_{Z'} \to 0$, which is easily checked on stalks. □

PROPOSITION 2.4 *The complex $R\Gamma_c(\bar{X}_{\mathrm{et}}, j_! \mathcal{F})$, hence definition* (5), *is independent of the choice of the compactification $j : U \to X$.*

Proof. Given two compactifications : $U \to X$ and $j' : U' \to X'$, we consider the closure C of U in $X \times X'$ and by comparing the compactifications $p_1 : C \to X$ and $p_2 : C \to X'$, we can assume that there is a finite map $f : X' \to X$ with $fj' = j$. Then by the Lemma

$$\begin{aligned} R\Gamma(\bar{X}_{\mathrm{et}} j_! \mathcal{F}) &= \mathrm{cone} R\Gamma(\bar{X}_{\mathrm{et}}, j_* \mathcal{F}) \to R\Gamma(\bar{Z}_{\mathrm{et}}, j_* \mathcal{F}|_Z) \\ &= \mathrm{cone} R\Gamma(\bar{X}_{\mathrm{et}}, f_* j'_* \mathcal{F}) \to R\Gamma(\bar{Z}_{\mathrm{et}}, f'_*(j'_* \mathcal{F}|_{Z'})) \\ &= \mathrm{cone} R\Gamma(\bar{X}'_{\mathrm{et}}, j'_* \mathcal{F}) \to R\Gamma(\bar{Z}'_{\mathrm{et}}, (j'_* \mathcal{F})|_{Z'}) = R\Gamma(\bar{X}'_{\mathrm{et}}, j'_! \mathcal{F}) \end{aligned}$$

□

For a closed subscheme Z of the curve X with open complement U, we obtain a localization sequence

$$\to H^{i-1}_c(Z_W, \mathcal{F}|_Z) \to H^i_c(U_W, \mathcal{F}|_U) \to H^i_c(X_W, \mathcal{F}) \to H^i_c(Z_W, \mathcal{F}|_Z) \to, \quad (6)$$

and the Leray spectral sequence degenerates to

$$0 \to H^{i-1}_c(\bar{X}_{\mathrm{et}}, \mathcal{F})_G \to H^i_c(X_W, \mathcal{F}) \to H^i_c(\bar{X}_{\mathrm{et}}, \mathcal{F})^G \to 0. \quad (7)$$

The analog of Theorem 2.2 holds, in particular, there is a long exact sequence

$$\to H^i_c(X_{\mathrm{et}}, \mathcal{F}) \to H^i_c(X_W, \mathcal{F}) \to H^{i-1}_c(X_{\mathrm{et}}, \mathcal{F})_{\mathbb{Q}} \xrightarrow{\delta} H^{i+1}_c(X_{\mathrm{et}}, \mathcal{F}) \to, \quad (8)$$

and we have

$$H^i_c(X_W, \mathcal{F})_{\mathbb{Q}} \cong H^i_c(X_{\mathrm{et}}, \mathcal{F})_{\mathbb{Q}} \oplus H^{i-1}_c(X_{\mathrm{et}}, \mathcal{F})_{\mathbb{Q}}. \quad (9)$$

Remark. The definition (5) given here agrees with the definition in [5] only for curves. For schemes of higher dimension Proposition 2.4 does not hold, see [5]. Thus the etale topology has to be replaced by the eh-topology in order to obtain good properties.

2.3 Finite generation

Lemma 2.5 *Let $\mathcal{F}$ be a $\mathbb{Z}$-constructible etale sheaf on a zero-dimensional scheme P.*
a) The groups $H^c_i(P_{\mathrm{ar}}, \mathcal{F})$ are finite for $i = -1$, finitely generated for $i = 0$, finitely generated free for $i = 1$, and trivial otherwise.
b) The groups $H^i_c(P_W, \mathcal{F})$ are finitely generated for $i = 0, 1$, and trivial otherwise.

Proof. We may assume that $P = \mathrm{Spec}\, \mathbb{F}_{q^r}$.
a) Since $\mathbb{Z}^c_P \cong \mathbb{Z}$, the group $\mathrm{Ext}^0_{\mathrm{et}}(\mathcal{F}, \mathbb{Z}^c_{\bar{P}}) = \mathrm{Hom}_{Ab}(\mathcal{F}_{\bar{P}}/\mathrm{tor}, \mathbb{Z})$ is free, the group $\mathrm{Ext}^1_{\mathrm{et}}(\mathcal{F}, \mathbb{Z}^c_{\bar{P}}) = ({}_{\mathrm{tor}}\mathcal{F}_{\bar{P}})^*$ is finite, and the other extension groups vanish. The Lemma follows with (2).
b) This follows from (7) because $H^i_c(\bar{P}_{\mathrm{et}}, \mathcal{F})$ is finitely generated for $i = 0$, and trivial for $i \neq 0$. □

Proposition 2.6 *Let $\mathcal{F}$ be a $\mathbb{Z}$-constructible etale sheaf on a curve X. Then the groups $H^i_c(X_W,\mathcal{F})$ and $H^c_i(X_{\mathrm{ar}},\mathcal{F})$ are finitely generated.*

Proof. Using (6) and (4) we can (by shrinking X) assume that $\mathcal{F}$ is locally constant. Shrinking X further, we can assume that there is a finite etale Galois covering $f:Y\to X$ with Galois group A such that $\mathcal{F}|_Y$ is constant. Since $R\Gamma_X\cong R\Gamma_A R\Gamma_Y$ and since group cohomology of finite groups with finitely generated coefficients is finitely generated, we can assume that $\mathcal{F}$ is constant. Increasing X again we can assume that X is smooth and proper. The groups $H^i(X_W,\mathbb{Z})$ are finitely generated by [4, Prop.7.4], and the groups $H^c_i(X_{\mathrm{ar}},\mathbb{Z})$ are finitely generated by [7]. This implies finite generation for finitely generated constant coefficients. □

Lemma 2.7 *Let $\bar{X}$ be a curve over an algebraically closed field and $\mathcal{F}$ a sheaf on $\bar{X}$. Then the groups $H^i(\bar{X}_{\mathrm{et}},\mathcal{F})$ and $H^i_c(\bar{X}_{\mathrm{et}},\mathcal{F})$ are torsion for $i>1$ and vanish for $i>3$..*
In particular, for X a curve over a finite field, $H^i_c(X_W,\mathcal{F})$ is torsion for $i>2$ and vanishes for $i>4$.

Proof. Clearly the result for cohomology implies the result for cohomology with compact support. Writing $\mathcal{F}$ as a colimit of $\mathbb{Z}$-constructible sheaves we can assume that $\mathcal{F}$ is $\mathbb{Z}$-constructible. For a closed embedding $i:Z\to X$ with open complement $j:U\to X$, we have a short exact sequence of sheaves $0\to j_!j^*\mathcal{F}\to\mathcal{F}\to i_*i^*\mathcal{F}\to 0$. Since a zero-dimensional scheme over an algebraically closed field has cohomological dimension 0, we can assume that X is normal and connected, and $\mathcal{F}=M$ is locally constant and finitely generated. Let $g:\eta\to X$ be the embedding of the generic point. Then $j_*M_\eta\cong M$, and $k(\eta)$ has cohomological dimension 1. We consider the case M torsion and M torsion free separately. If M is torsion, then $R^1g_*M_\eta$ is torsion, the higher derived images vanish, and $H^i(k(\eta)_{\mathrm{et}},M_\eta)=0$ for $i>0$. If M is torsion free, then $R^1g_*M_\eta=0$, $R^2g_*M_\eta$ is torsion, the higher derived image are zero, and $H^i(k(\eta)_{\mathrm{et}},M_\eta)=0$ for $i>2$. Now the claim follows by analyzing the spectral sequence $H^s(\bar{X}_{\mathrm{et}},R^tg_*M_\eta)\Rightarrow H^{s+t}(k(\eta)_{\mathrm{et}},M_\eta)$. □

Lemma 2.8 *If $\bar{X}$ is a curve over an algebraically closed field, then the groups $\mathrm{Ext}^i_{\mathrm{et}}(\mathcal{F}|_{\bar{X}},\mathbb{Z}^c_{\bar{X}})$ vanish unless $-1\le i\le 4$. In particular, the groups $H^c_i(\bar{X}_{\mathrm{ar}},\mathcal{F})$ vanish unless $-4\le i\le 2$.*

Proof. The lower bound is obtained by observing that $\mathbb{Z}_{\bar{X}}$ is quasi-isomorphic to a complex concentrated in degrees $-1,0$. By the analog of (4) and Lemma 2.5 for etale extension groups, we can assume that $\mathcal{F}$ is locally constant. Consider the spectral sequence

$$H^s(\bar{X},\mathcal{E}xt^t(\mathcal{F}|_{\bar{X}},\mathbb{Z}^c_{\bar{X}}))\Rightarrow \mathrm{Ext}^{s+t}_{\mathrm{et}}(\mathcal{F}|_{\bar{X}},\mathbb{Z}^c_{\bar{X}}).$$

Since $\mathcal{F}$ locally constant, the $\mathcal{E}\mathfrak{xt}^t$-sheaf can be calculated at stalks [10, III 1.31], and since $\mathbb{Z}^c_{\bar{X}}$ is concentrated in non-positive degrees, the stalks vanish for $t > 1$. On the other hand, we just saw that $H^s(\bar{X}_{\text{et}}, \mathcal{F}) = 0$ for $s > 3$. The final statement follows from (2). □

3 The pairing over an algebraically closed field

Let X be a proper curve over an algebraically closed field k, and $\mathcal{F}^\cdot$ a bounded complex of etale sheaves. Choose injective resolutions $\mathbb{Z}^c_X \to I^\cdot_X$, and $\mathcal{F} \to J^\cdot$. We can assume that $J^\cdot$ is bounded, because X has strict cohomological dimension 3. If $I = \big(\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}\big)$ is the injective resolution of $\mathbb{Z}$ as an abelian group, we obtain by Theorem 2.1 a map

$$I^\cdot_X(X) \xleftarrow{\sim} \mathbb{Z}^c_X(X) \to \mathbb{Z} \cong I$$

in the derived category of bounded above complexes of abelian groups. Hence we obtain a natural transformation of derived functors $D^b(\text{Et}^\sim_X) \to D^-(\text{Ab})$,

$$\begin{aligned}\tau_X(\mathcal{F}^\cdot) : R\operatorname{Hom}_{\text{et}}(\mathcal{F}^\cdot, \mathbb{Z}^c_X) \cong \operatorname{Hom}_{\text{et}}(J^\cdot, I^\cdot_X) &\xrightarrow{\Gamma} \operatorname{Hom}_{\text{Ab}}(J^\cdot(X), I^\cdot_X(X)) \\ &\xrightarrow{tr} \operatorname{Hom}_{\text{Ab}}(\Gamma(X_{\text{et}}, J^\cdot), I) \xrightarrow{\sim} R\operatorname{Hom}_{\text{Ab}}(R\Gamma(X_{\text{et}}, \mathcal{F}^\cdot), \mathbb{Z}).\end{aligned}$$

For an arbitrary separated curve X over k, we choose a compactification $j : X \to X'$ of X, and define

$$\begin{aligned}\tau_X(\mathcal{F}) : R\operatorname{Hom}_{\text{et}}(j_!\mathcal{F}^\cdot, \mathbb{Z}^c_{X'}) &\xrightarrow{\tau_{X'}(j_!\mathcal{F})} \\ R\operatorname{Hom}_{\text{Ab}}(R\Gamma(X'_{\text{et}}, j_!\mathcal{F}^\cdot), \mathbb{Z}) &\cong R\operatorname{Hom}_{\text{Ab}}(R\Gamma_c(X_{\text{et}}, \mathcal{F}^\cdot), \mathbb{Z}). \quad (10)\end{aligned}$$

By compatibility of the trace map with proper push-forward, the usual argument comparing compactifications shows that this is independent of the compactification.

Let $RT_m = R\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/m, -)$ be the right derived functor of the m-torsion functor. Then for every bounded complex $\mathcal{F}^\cdot$, there is a distinguished triangle

$$RT_m\mathcal{F}^\cdot \to \mathcal{F}^\cdot \xrightarrow{\times m} \mathcal{F}^\cdot \to RT_m\mathcal{F}^\cdot[1]. \quad (11)$$

Proposition 3.1 *Let $\mathcal{F}^\cdot$ be a bounded complex of $\mathbb{Z}$-constructible sheaves on a curve X over an algebraically closed field k. Then the map induced by* (10)*,*

$$f_i : \operatorname{Ext}^{-i}_{\text{et}}(\mathcal{F}^\cdot, \mathbb{Z}^c_X) \to R\operatorname{Hom}^{-i}_{\text{Ab}}(R\Gamma_c(X_{\text{et}}, \mathcal{F}^\cdot), \mathbb{Z})$$

has divisible kernel, torsion free cokernel, and $\operatorname{coker} f_i/m \cong {}_m\ker f_{i-1}$.

Proof. If $\mathcal{G}^\cdot$ is a bounded below complex of constructible sheaves, then $\tau_X(\mathcal{G}^\cdot)$ is a quasi-isomorphism by the main theorem of [6]. In particular, $\tau_X(RT_m\mathcal{F}^\cdot)$ is a quasi-isomorphism for every m and $\mathcal{F}^\cdot$. We apply the map τ_X to the triangle (11) and obtain a map of long exact sequences

$$\begin{array}{ccc}
\mathrm{Ext}_X^{-i-1}(RT_m\mathcal{F}^\cdot,\mathbb{Z}^c_X) & \xrightarrow{\sim} & R\operatorname{Hom}_{\mathrm{Ab}}^{-i-1}(R\Gamma_c(X_{\mathrm{et}},RT_m\mathcal{F}^\cdot),\mathbb{Z}) \\
\downarrow & & \downarrow \\
\mathrm{Ext}_X^{-i}(\mathcal{F}^\cdot,\mathbb{Z}^c_X) & \xrightarrow{f_i} & R\operatorname{Hom}_{\mathrm{Ab}}^{-i}(R\Gamma_c(X_{\mathrm{et}},\mathcal{F}^\cdot),\mathbb{Z}) \\
{\scriptstyle\times m}\downarrow & & {\scriptstyle\times m}\downarrow \\
\mathrm{Ext}_X^{-i}(\mathcal{F}^\cdot,\mathbb{Z}^c_X) & \xrightarrow{f_i} & R\operatorname{Hom}_{\mathrm{Ab}}^{-i}(R\Gamma_c(X_{\mathrm{et}},\mathcal{F}^\cdot),\mathbb{Z}) \\
\downarrow & & \downarrow \\
\mathrm{Ext}_X^{-i}(RT_m\mathcal{F}^\cdot,\mathbb{Z}^c_X) & \xrightarrow{\sim} & R\operatorname{Hom}_{\mathrm{Ab}}^{-i}(R\Gamma_c(X_{\mathrm{et}},RT^\cdot_m),\mathbb{Z})
\end{array}$$

By the previous discussion, the upper and lower map are isomorphisms. If K_i and C_i are the kernel and cokernel of f_i, respectively, then a diagram chase shows that we get, for every integer m, an exact sequence

$$0 \to C_i \xrightarrow{\times m} C_i \to K_{i-1} \xrightarrow{\times m} K_{i-1} \to 0.$$

This implies that C_i is torsion free, K_{i-1} is divisible, and ${}_mK_{i-1} = C_i/m$. □

4 The main Theorem

We are going to descend the pairing $\tau_X(\mathcal{F})$ to a pairing of arithmetic cohomology groups.

Lemma 4.1 *If $M^\cdot$ is complex of $\mathbb{Z}[G]$-modules and $N^\cdot$ a complex of abelian groups (viewed as a complex of trivial $\mathbb{Z}[G]$-modules), then there is a quasi-isomorphism*

$$R\Gamma_G R\operatorname{Hom}_{\mathrm{Ab}}(M^\cdot,N^\cdot) = R\operatorname{Hom}_G(M^\cdot,N^\cdot) \cong R\operatorname{Hom}_{\mathrm{Ab}}(R\Gamma_G M^\cdot,N^\cdot)[-1].$$

Proof. For any $\mathbb{Z}[G]$-module M and abelian group N we have $\operatorname{Hom}_{\mathrm{Ab}}(M_G,N) \cong \operatorname{Hom}_G(M,N)$. Since the total left derived functor $L(-)_G$ of the coinvariant functor agrees with the shift of the total right derived functor $R(-)^G[-1]$ of the invariant functor, we get the adjunction $R\operatorname{Hom}_{\mathrm{Ab}}(R\Gamma_G M,N) \cong R\operatorname{Hom}_G(M,N[1])$. The lemma follows. □

Restricting the action of the Galois group $\hat{G}$ on the source and target of $\tau_X(\mathcal{F})$ to the Weil group G, and applying $R\Gamma_G$, we get by Lemma 4.1 the duality homomorphism

$$\begin{aligned}
\delta_X(\mathcal{F}) : R\operatorname{Hom}_{\mathrm{ar}}(\mathcal{F},\mathbb{Z}^c) = R\Gamma_G R\operatorname{Hom}_{\mathrm{et}}(\mathcal{F},\mathbb{Z}^c|_{\bar{X}})[1] &\xrightarrow{R\Gamma_G\tau_X(\mathcal{F})} \\
R\Gamma_G R\operatorname{Hom}_{\mathrm{Ab}}(R\Gamma_c(\bar{X}_{\mathrm{et}},\mathcal{F}),\mathbb{Z})[1] &\cong \\
R\operatorname{Hom}_{\mathrm{Ab}}(R\Gamma_G R\Gamma_c(\bar{X}_{\mathrm{et}},\mathcal{F}),\mathbb{Z}) = R\operatorname{Hom}_{\mathrm{Ab}}(R\Gamma_c(X_W,\mathcal{F}),\mathbb{Z}). &\quad (12)
\end{aligned}$$

THEOREM 4.2 *Let $\mathcal{F}^{\cdot}$ be a bounded complex of $\mathbb{Z}$-constructible sheaves on the curve X over $\mathbb{F}_q$. Then the pairing* (12) *induces a quasi-ismorphism*

$$R\operatorname{Hom}_{\mathrm{ar}}(\mathcal{F}, \mathbb{Z}^c) \xrightarrow{\sim} R\operatorname{Hom}_{\mathrm{Ab}}(R\Gamma_c(X_W, \mathcal{F}), \mathbb{Z}).$$

Proof. We apply δ to the triangle (11) note that the map $\delta_X(RT_m\mathcal{F}) \cong R\Gamma_G\tau_X(RT_m\mathcal{F})$ is a quasi-isomorphism for every m by the main theorem of [6]. If K_i and C_i are the kernel and cokernel of $H_i^c(X_{\mathrm{ar}}, \mathcal{F}) \to R\operatorname{Hom}_{\mathrm{Ab}}^{-i}(R\Gamma_c(X_{\mathrm{ar}}, \mathcal{F}), \mathbb{Z})$, respectively, then the argument of Proposition 3.1 gives, for every integer m, an exact sequence

$$0 \to C_{i+1} \xrightarrow{\times m} C_{i+1} \to K_i \xrightarrow{\times m} K_i \to 0.$$

Since K_i is divisible, finite generation implies that K_i is trivial, and then C_{i+1} is divisible hence trivial. □

COROLLARY 4.3 *We have perfect pairings of finitely generated groups*

$$H_i^c(X_{\mathrm{ar}}, \mathcal{F})/\mathrm{tor} \times H_c^i(X_W, \mathcal{F})/\mathrm{tor} \to \mathbb{Z}; \tag{13}$$

$${}_{\mathrm{tor}}H_{i-1}^c(X_{\mathrm{ar}}, \mathcal{F}) \times {}_{\mathrm{tor}}H_c^i(X_W, \mathcal{F}) \to \mathbb{Q}/\mathbb{Z}. \tag{14}$$

The torsion free groups vanish unless $0 \leq i \leq 2$, and the torsion groups vanish unless $0 \leq i \leq 3$.

Proof. Taking the map induced on $-i$th cohomology groups by $\delta_X(\mathcal{F})$, we get an isomorphism

$$H_i^c(X_{\mathrm{ar}}, \mathcal{F}) \cong H^{-i}R\operatorname{Hom}_{\mathrm{Ab}}(R\Gamma_c(X_W, \mathcal{F}), \mathbb{Z}).$$

The Leray spectral sequence degenerates into

$$0 \to \operatorname{Ext}^1_{\mathrm{Ab}}(H_c^{i+1}(X_W, \mathcal{F}), \mathbb{Z}) \to H^{-i}R\operatorname{Hom}_{\mathrm{Ab}}(R\Gamma_c(X_W, \mathcal{F}), \mathbb{Z}) \\ \to \operatorname{Hom}_{\mathrm{Ab}}(H_c^i(X_W, \mathcal{F}), \mathbb{Z}) \to 0.$$

For a finitely generated abelian group M, the surjection $\operatorname{Hom}_{\mathrm{Ab}}(M, \mathbb{Q}/\mathbb{Z}) \to \operatorname{Ext}^1_{\mathrm{Ab}}(M, \mathbb{Z})$ induces an isomorphism $\operatorname{Hom}_{\mathrm{Ab}}({}_{\mathrm{tor}}M, \mathbb{Q}/\mathbb{Z}) \cong \operatorname{Ext}^1_{\mathrm{Ab}}(M, \mathbb{Z})$. Since this is torsion and $\operatorname{Hom}_{\mathrm{Ab}}(-, \mathbb{Z})$ is torsion free, we have

$$H^{-i}R\operatorname{Hom}_{\mathrm{Ab}}(R\Gamma_c(X_W, \mathcal{F}))/\mathrm{tor} = \operatorname{Hom}_{\mathrm{Ab}}(H_c^i(X_W, \mathcal{F}), \mathbb{Z})$$

and

$${}_{\mathrm{tor}}H^{-i}R\operatorname{Hom}_{\mathrm{Ab}}(R\Gamma_c(X_W, \mathcal{F})) = \operatorname{Ext}^1_{\mathrm{Ab}}(H_c^{i+1}(X_W, \mathcal{F}), \mathbb{Z}) \\ = \operatorname{Hom}_{\mathrm{Ab}}({}_{\mathrm{tor}}H_c^{i+1}(X_W, \mathcal{F}), \mathbb{Q}/\mathbb{Z}).$$

The vanishing follows by Lemma's 2.7 and 2.8. □

Corollary 4.4 *The pairing*

$$\delta_X(\mathcal{F}) : R\operatorname{Hom}_{\mathrm{ar}}(\mathcal{F}, \mathbb{Z}^c) \to R\operatorname{Hom}_{\mathrm{Ab}}(R\Gamma_c(X_W, \mathcal{F}), \mathbb{Z})$$

is a quasi-isomorphism for every sheaf $\mathcal{F} \in \mathrm{Et}_X^\sim$.

Proof. Given $\mathcal{F}$ on $\mathrm{Et}_X^\sim$, we can write $\mathcal{F}$ as the filtered colimit of $\mathbb{Z}$-constructible sheaves, $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$. Since the etale site is noetherian, $R\Gamma(X_{\mathrm{et}}, -)$ commutes with colimits, the duality pairing can be identified with

$$\begin{array}{ccc}
R\operatorname{Hom}_{\mathrm{ar}}(\mathcal{F}, \mathbb{Z}_X^c) & = & R\operatorname{Hom}_{\mathrm{ar}}(\operatorname{colim} \mathcal{F}_i, \mathbb{Z}_X^c) \\
\downarrow & & \downarrow \\
R\operatorname{Hom}_{\mathrm{Ab}}(R\Gamma_c(X_W, \mathcal{F}), \mathbb{Z}) & = & R\operatorname{Hom}_{\mathrm{Ab}}(\operatorname{colim} R\Gamma_c(X_W, \mathcal{F}_i), \mathbb{Z})
\end{array}$$

The right hand map induces a map between the spectral sequences of [12, Theorem 1]

$$\begin{array}{ccc}
E_2^{s,t} = \lim^s \operatorname{Ext}^t_{\mathrm{ar}}(\mathcal{F}_i, \mathbb{Z}_X^c) & \Rightarrow & \operatorname{Ext}^{s+t}_{\mathrm{ar}}(\operatorname{colim} \mathcal{F}_i, \mathbb{Z}^c) \\
\downarrow & & \downarrow \\
E_2^{s,t} = \lim^s \operatorname{Ext}^t_{\mathrm{Ab}}(R\Gamma_c(X_W, \mathcal{F}_i), \mathbb{Z}) & \Rightarrow & \operatorname{Ext}^{s+t}_{\mathrm{Ab}}(\operatorname{colim} R\Gamma_c(X_W, \mathcal{F}_i), \mathbb{Z}).
\end{array}$$

The map on E_2-terms is an isomorphism, and the spectral sequences converge, by Lemma's 2.7 and 2.8. Hence we get an isomorphism on the abutment. □

5 Constant coefficients

In this section we connect our results to the result of Lichtenbaum [9]. Recall that if X is smooth, then $\mathbb{Z}_X^c \cong \mathbb{G}_m[1]$, hence $H_i^c(X_{\mathrm{ar}}, \mathbb{Z}) \cong H^{2-i}(X_W, \mathbb{G}_m)$. For an abelian group A, let $A^* = \operatorname{Hom}(A, \mathbb{Q}/\mathbb{Z})$.

Proposition 5.1 *(Lichtenbaum [9, Thm.6.1c,d]) Let U be the complement of $s > 0$ points of a connected smooth and proper curve X. Then*

$$H_c^i(U_W, \mathbb{Z}) = \begin{cases} 0 & i = 0; \\ \mathbb{Z}^s/\mathbb{Z} & i = 1; \\ \operatorname{Hom}(\Gamma(U, \mathcal{O}_U)^\times, \mathbb{Z}) \oplus \operatorname{Pic}(U)^* & i = 2; \\ \Gamma(X, \mathcal{O}_X^\times)^* & i = 3. \end{cases}$$

$$H_i^c(U_{\mathrm{ar}}, \mathbb{Z}) = \begin{cases} 0 & i = 0; \\ \operatorname{Pic}(U) \oplus \ker(\mathbb{Z}^s \to \mathbb{Z}) & i = 1; \\ \Gamma(U, \mathcal{O}_U^\times) & i = 2. \end{cases}$$

Proof. For cohomology, this follows from the long exact sequence (6) comparing the cohomology and homology of U to its compactification X. For homology we have $H_2(X_{\mathrm{ar}},\mathbb{Z}) = H^0(X_W,\mathbb{G}_m) = H^0(X_{\mathrm{et}},\mathbb{G}_m)$ by (7), and the long exact sequence (4) gives the other two groups. Note that $\Gamma(U,\mathcal{O}_U^\times)\cong\Gamma(X,\mathcal{O}_X^\times)\oplus\ker(\mathbb{Z}^{\oplus s}\to\mathbb{Z})$. □

The dual graph of a proper curve X is defined as follows. Let X' be the normalization of X, S be the set of singular points of X and $S' = S\times_X X'$. Then the dual graph is a bipartite graph with vertices the points of S' and the connected components of X', and an edge for each point in S' connecting its image in S with the component it is contained it. If X is connected, then Γ is connected, hence $H^0(\Gamma,\mathbb{Z})=\mathbb{Z}$ and $H^1(\Gamma,\mathbb{Z})$ is free of rank $|\pi_0(S)|-|\pi_0(S')|+|\pi_0(X')|$. The following proposition generalizes Lichtenbaum [9, Thm.6.1a,b], since for smooth and proper curves, $CH_0(X)\cong\mathrm{Pic}(X)$, $CH_0(X,1)=\Gamma(X,\mathcal{O}_X^\times)$ and $H^1(\Gamma,\mathbb{Z})=0$.

Proposition 5.2 *Let X be a connected proper curve with normalization $X'=\coprod_i X_i$ and dual graph Γ. Then the non-vanishing cohomology and homology groups are*

$$H^i_c(X_W,\mathbb{Z}) = \begin{cases} H^0(\Gamma,\mathbb{Z}) & i=0;\\ H^0(\Gamma,\mathbb{Z})\oplus H^1(\Gamma,\mathbb{Z}) & i=1;\\ H^1(\Gamma,\mathbb{Z})\oplus\coprod\mathrm{Pic}^0(X_i)^* & i=2;\\ \coprod\Gamma(X_i,\mathcal{O}_{X_i}^\times)^* & i=3.\end{cases}$$

$$H_i^c(X_{\mathrm{ar}},\mathbb{Z}) = \begin{cases} H_0(\Gamma,\mathbb{Z}) & i=0;\\ CH_0(X)\oplus H_1(\Gamma,\mathbb{Z}) & i=1;\\ CH_0(X,1) & i=2.\end{cases}$$

Proof. In the smooth, proper case, the result for cohomology follows from (7) and $H^i(X_W,\mathbb{Z})\cong H^{i-1}(X_{\mathrm{et}},\mathbb{Q}/\mathbb{Z})$ for $i\geq 2$. The calculation for homology can also be found in [7]. In the general case, one uses the long exact sequences (4) and (6) arising from the cartesian square

$$\begin{array}{ccc} S' & \longrightarrow & \coprod X_i\\ \downarrow & & \downarrow\\ S & \longrightarrow & X. \end{array}$$

□

6 Comparison to Deninger's results

By Nart [11], Deninger's [3] dualizing complex $\mathbb{G}$ is quasi-isomorphic to the shift $\mathbb{Z}^c_X[-1]$. According to Deninger [2], if X is smooth and proper, then the groups $H^i(X_{\text{et}},\mathcal{F})$ and $\text{Ext}^i_{\text{et}}(\mathcal{F},\mathbb{G}) = \text{Ext}^{i-1}_{\text{et}}(\mathcal{F},\mathbb{Z}^c_X)$ are finitely generated for $i=0,1$ and of cofinite type for $i=2,3$. We are going to recover and improve Deninger's result in this section. Let

$$\begin{aligned} r &= \text{rank}\, H^0_c(X_W,\mathcal{F}) \\ s &= \text{rank}\, H^2_c(X_W,\mathcal{F}). \end{aligned}$$

Lemma 6.1 *Let X be a separated curve over a finite field.*
a) The groups $H^i_c(X_{\text{et}},\mathcal{F})$ and $\text{Ext}^{i-1}_{\text{et}}(\mathcal{F},\mathbb{Z}^c)$ are torsion for $i\neq 0,1$.
b) We have

$$\begin{aligned} \text{rank}\, H^1_c(X_W,\mathcal{F}) &= r+s \\ \text{rank}\, H^0_c(X_{\text{et}},\mathcal{F}) &= r \\ \text{rank}\, H^1_c(X_{\text{et}},\mathcal{F}) &= s. \end{aligned}$$

Proof. a) The first statement is Lemma 2.7. By Theorem 2.2, we have $\text{Ext}^{i-1}_{\text{et}}(\mathcal{F},\mathbb{Z}^c)_{\mathbb{Q}} \subseteq H^c_{2-i}(X_{\text{ar}},\mathcal{F})_{\mathbb{Q}}$ as well as $\text{Ext}^{i-1}_{\text{et}}(\mathcal{F},\mathbb{Z}^c)_{\mathbb{Q}} \subseteq H^c_{1-i}(X_{\text{ar}},\mathcal{F})_{\mathbb{Q}}$ and we can conclude with Corollary 4.3
b) This follows from a) and (9). □

Now consider the long exact sequence (8)

$$\begin{aligned} 0\to H^1_c(X_{\text{et}},\mathcal{F})\to H^1_c(X_W,\mathcal{F}) \xrightarrow{\alpha} H^0_c(X_{\text{et}},\mathcal{F})_{\mathbb{Q}} \xrightarrow{\delta_0} H^2_c(X_{\text{et}},\mathcal{F})\to \\ H^2_c(X_W,\mathcal{F})\to H^1_c(X_{\text{et}},\mathcal{F})_{\mathbb{Q}} \xrightarrow{\delta_1} H^3_c(X_{\text{et}},\mathcal{F})\to H^3_c(X_W,\mathcal{F})\to 0. \end{aligned}$$

Some easy considerations together with the fact that Weil-etale cohomology is finitely generated, and that $H^2_c(X_{\text{et}},\mathcal{F})$ and $H^3_c(X_{\text{et}},\mathcal{F})$ are torsion, gives

Theorem 6.2 *Let X be a separated curve over a finite field.*
a) The groups $H^0_c(X_{\text{et}},\mathcal{F})$, $H^1_c(X_{\text{et}},\mathcal{F})$ are finitely generated, the groups $H^2_c(X_{\text{et}},\mathcal{F})$, $H^3_c(X_{\text{et}},\mathcal{F})$ are cofinitely generated of corank r and s, respectively, and all other groups vanish.
b) We have a decomposition into finite and cofree groups

$$\begin{aligned} {}_{\text{tor}}H^1_c(X_{\text{et}},\mathcal{F}) &\cong {}_{\text{tor}}H^1_c(X_W,\mathcal{F}); \\ H^2_c(X_{\text{et}},\mathcal{F}) &\cong {}_{\text{tor}}H^2_c(X_W,\mathcal{F}) \oplus \big(H^1_c(X_W,\mathcal{F})/H^1_c(X_{\text{et}},\mathcal{F})\big)\otimes\mathbb{Q}/\mathbb{Z}; \\ H^3_c(X_{\text{et}},\mathcal{F}) &\cong H^3_c(X_W,\mathcal{F}) \oplus H^2_c(X_W,\mathcal{F})\otimes\mathbb{Q}/\mathbb{Z}. \end{aligned}$$

For example, the image of δ_0 is $H^0_c(X_{\text{et}},\mathcal{F})\otimes\mathbb{Q}/\mathbb{Z}$ because it is torsion and $H^1_c(X_W,\mathcal{F})/\text{tor}$ is a finitely generated abelian group. By the Lemma, Theorem 2.6 and (8), $H^i_c(X_{\text{et}},\mathcal{F})\cong H^i_c(X_W,\mathcal{F})=0$ for $i>3$.
By Lemma 6.1a), the exact sequence from Theorem 2.2 becomes

$$\begin{aligned} 0\to \text{Ext}^0_{\text{et}}(\mathcal{F},\mathbb{Z}^c_X)\to H^c_1(X_{\text{ar}},\mathcal{F}) \xrightarrow{\beta} \text{Ext}^{-1}_{\text{et}}(\mathcal{F},\mathbb{Z}^c_X)_{\mathbb{Q}} \to \text{Ext}^1_{\text{et}}(\mathcal{F},\mathbb{Z}^c_X)\to \\ H^c_0(X_{\text{ar}},\mathcal{F})\to \text{Ext}^0_{\text{et}}(\mathcal{F},\mathbb{Z}^c_X)_{\mathbb{Q}} \to \text{Ext}^2_{\text{et}}(\mathcal{F},\mathbb{Z}^c_X)\to H^c_{-1}(X_{\text{ar}},\mathcal{F})\to 0 \end{aligned}$$

together with an isomorphism $\operatorname{Ext}^{-1}_{\mathrm{et}}(\mathcal{F},\mathbb{Z}^c) = H_2^c(X_{\mathrm{ar}},\mathcal{F})$. A similar argument as above gives

Theorem 6.3 *Let X be a separated curve over a finite field.*
a) Then the groups $\operatorname{Ext}^{-1}_{\mathrm{et}}(\mathcal{F},\mathbb{Z}^c_X)$ and $\operatorname{Ext}^{0}_{\mathrm{et}}(\mathcal{F},\mathbb{Z}^c_X)$ are finitely generated of rank s and r, respectively, the groups $\operatorname{Ext}^{1}_{\mathrm{et}}(\mathcal{F},\mathbb{Z}^c_X)$ and $\operatorname{Ext}^{2}_{\mathrm{et}}(\mathcal{F},\mathbb{Z}^c_X)$ are cofinitely generated of corank s and r, respectively, and all other groups vanish.
b) We have a decomposition into finite and cofree groups

$$\begin{aligned}
{}_{\mathrm{tor}}\operatorname{Ext}^{0}_{\mathrm{et}}(\mathcal{F},\mathbb{Z}^c_X) &\cong {}_{\mathrm{tor}}H_1(X_{\mathrm{ar}},\mathcal{F}),\\
\operatorname{Ext}^{1}_{\mathrm{et}}(\mathcal{F},\mathbb{Z}^c_X) &\cong {}_{\mathrm{tor}}H_0^c(X_{\mathrm{ar}},\mathcal{F}) \oplus \big(H_1^c(X_{\mathrm{ar}},\mathcal{F})/\operatorname{Ext}^{0}_{\mathrm{et}}(\mathcal{F},\mathbb{Z}^c_X)\big)\otimes\mathbb{Q}/\mathbb{Z},\\
\operatorname{Ext}^{2}_{\mathrm{et}}(\mathcal{F},\mathbb{Z}^c_X) &\cong H_{-1}^c(X_{\mathrm{ar}},\mathcal{F}) \oplus H_0(X_{\mathrm{ar}},\mathcal{F})\otimes\mathbb{Q}/\mathbb{Z}.
\end{aligned}$$

Corollary 6.4 *(Deninger) There are isomorphisms of discrete torsion groups of cofinite type:*

$$\begin{aligned}
\operatorname{Ext}^{-1}_{\mathrm{et}}(\mathcal{F},\mathbb{Z}^c_X)^* &\cong H_c^3(X_{\mathrm{et}},\mathcal{F});\\
H_c^0(X_{\mathrm{et}},\mathcal{F})^* &\cong \operatorname{Ext}^{2}_{\mathrm{et}}(\mathcal{F},\mathbb{Z}^c_X)\\
{}_{\mathrm{tor}}\operatorname{Ext}^{0}_{\mathrm{et}}(\mathcal{F},\mathbb{Z}^c)^* &\cong_{\mathrm{cotor}} H_c^2(X_{\mathrm{et}},\mathcal{F})\\
{}_{\mathrm{tor}}H_c^1(X_{\mathrm{et}},\mathcal{F})^* &\cong_{\mathrm{cotor}} \operatorname{Ext}^{1}_{\mathrm{et}}(\mathcal{F},\mathbb{Z}^c)
\end{aligned}$$

Proof. This follows by splicing together the torsion and cotorsion part of Corollary 4.3. For example,

$$\operatorname{Ext}^{-1}_{\mathrm{et}}(\mathcal{F},\mathbb{Z}^c)^* \cong ({}_{\mathrm{tor}}H_2(X_{\mathrm{ar}},\mathcal{F}))^* \oplus (H_2(X_{\mathrm{ar}},\mathcal{F})/\mathrm{tor})^*.$$

The first term is $H_c^3(X_W,\mathcal{F})$, and the second term is

$$\operatorname{Hom}(\operatorname{Hom}(H_c^2(X_W,\mathcal{F})/\mathrm{tor},\mathbb{Z}),\mathbb{Q}/\mathbb{Z}) \cong H_c^2(X_W,\mathcal{F})\otimes\mathbb{Q}/\mathbb{Z}$$

because $H_c^2(X_W,\mathcal{F})$ is a finitely generated abelian group. The last two isomorphisms are

$${}_{\mathrm{tor}}\operatorname{Ext}^{0}_{\mathrm{et}}(\mathcal{F},\mathbb{Z}^c)^* = {}_{\mathrm{tor}}H_1^c(X_{\mathrm{ar}},\mathcal{F})^* = {}_{\mathrm{tor}}H_c^2(X_W,\mathcal{F}) = {}_{\mathrm{cotor}}H_c^2(X_{\mathrm{et}},\mathcal{F}),$$

and

$${}_{\mathrm{tor}}H_c^1(X_{\mathrm{et}},\mathcal{F})^* = {}_{\mathrm{tor}}H_c^1(X_W,\mathcal{F})^* = {}_{\mathrm{tor}}H_0^c(X_{\mathrm{ar}},\mathcal{F}) = {}_{\mathrm{cotor}}\operatorname{Ext}^{1}_{\mathrm{et}}(\mathcal{F},\mathbb{Z}^c).$$

□

We leave it as an open problem to derive the remaining isomorphisms

$$\begin{aligned}
(\operatorname{Ext}^{0}_{\mathrm{et}}(\mathcal{F},\mathbb{Z}^c)/\mathrm{tor})^* &\cong H_c^2(X_{\mathrm{et}},\mathcal{F})/\mathrm{cotor}\\
(H_c^1(X_{\mathrm{et}},\mathcal{F})/\mathrm{tor})^* &\cong \operatorname{Ext}^{1}_{\mathrm{et}}(\mathcal{F},\mathbb{Z}^c)/\mathrm{cotor}
\end{aligned}$$

from our results.
In higher dimension, the etale cohomology and extension groups will be mixed in the sense that they contain both a finitely generated free subgroup as well as a cofinitely generated torsion divisible subgroup.

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