

ON ARTIN REPRESENTATIONS  
AND NEARLY ORDINARY HECKE ALGEBRAS  
OVER TOTALLY REAL FIELDS

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ABSTRACT. We prove many new cases of the strong Artin conjecture for two-dimensional, totally odd, insoluble (icosahedral) representations  $\text{Gal}(\overline{F}/F) \rightarrow GL_2(\mathbf{C})$  of the absolute Galois group of a totally real field  $F$ .

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1 INTRODUCTION

Let  $K$  be a number field. Artin conjectured that the  $L$ -series of any continuous representation  $\rho : \text{Gal}(\overline{K}/K) \rightarrow GL_n(\mathbf{C})$  of the absolute Galois group  $\text{Gal}(\overline{K}/K)$  of  $K$  is holomorphic except a possible pole at  $s = 1$  when the trivial representation is a constituent of  $\rho$ .

A result of Brauer (See [36]) about finite groups immediately implies that  $L(\rho, s)$  has meromorphic continuation and satisfies a certain functional equation relating the values at  $s$  and  $1 - s$ . Any such complex representation is semi-simple, and because Artin showed that  $L(\rho_1 + \rho_2) = L(\rho_1, s)L(\rho_2, s)$ , the conjecture immediately follows from the case where  $\rho$  is irreducible. In the case where  $\rho$  is irreducible, the strong form of this conjecture, known as the strong Artin conjecture, asserts that there is a cuspidal automorphic representation  $\pi$  of  $GL_n(\mathbf{A}_K)$  such that  $L(\pi, s) = L(\rho, s)$ , and Artin conjecture follows from the strong Artin conjecture (See [22], Theorem 8.8 with its proof (p.286) attributed to Ramakrishnan).

When  $n = 2$  and the image of the projective representation  $\text{proj } \rho : \text{Gal}(\overline{K}/K) \rightarrow PGL_2(\mathbf{C}) = GL_2(\mathbf{C})/\mathbf{C}^\times$  is dihedral ( $D_{2n}$  for some  $n \geq 2$ ),  $\rho$  is induced from a character  $\chi$  of the absolute Galois group  $\text{Gal}(\overline{K}/M)$  of a quadratic extension  $M$  of  $\mathbf{Q}$ , and Artin himself proved the conjecture (the holomorphy of  $L(\rho, s) = L(\text{Ind}_{G_M}^{G_K} \chi, s) = L(\chi, s)$  follows from earlier work of Hecke).

When  $n = 2$  and the image of  $\text{proj } \rho$  is tetrahedral ( $A_4$ ) and when  $n = 2$ ,  $K = \mathbf{Q}$ ,  $\rho$  odd, and the projective image of  $\rho$  is octahedral ( $S_4$ ), Langlands [23], using his theory of (cyclic) base change, “deduced” the strong Artin conjecture from the dihedral case. Tunnell, building on work of Langlands, completed the octahedral case  $n = 2$  and general  $K$ . In the octahedral case, in order to “descend” a cuspidal automorphic representation  $\Pi$  of  $GL_2(\mathbf{A}_E)$  such that  $L(\Pi, s) = L(\rho|_{\text{Gal}(\overline{K}/E)}, s)$  to a cuspidal automorphic representation  $\pi$  of  $GL_2(\mathbf{A}_K)$ , where  $E$  is the quadratic extension of  $K$  corresponding to the unique index 2 subgroup ( $\simeq A_4$ ) of  $S_4$ , Langlands uses a theorem of Deligne-Serre (and therefore  $K = \mathbf{Q}$  and  $\rho$  should be necessarily odd) whilst Tunnell uses cubic base change to match up, for all but finitely many places  $v$  of  $K$ , the restriction of  $\rho$  to the decomposition group at  $v$  and the local representation  $\pi_v$ .

The icosahedral ( $A_5$ ) case had remained largely intractable until Buzzard-Dickinson-Shepherd-Barron-Taylor [4] proved many new cases of the strong Artin conjecture for odd  $\rho : \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \rightarrow GL_2(\mathbf{C})$ .

[4] follows the program of Taylor ([37]), which may be succinctly described as an approach to deduce results about weight one forms from results about weight two forms (more specifically the idea of Wiles in [42]), and it is a culmination of a series of work: “ $R = T$  theorem for 2-adic ordinary finite flat representations” by Dickinson [10], “modularity of mod 2 icosahedral representations” by Shepherd-Barron and Taylor [33], and “modular lifting theorem for two-dimensional  $p$ -adic Artin representations unramified at  $p$  (for any prime  $p$ )” by Buzzard and Taylor [5]. Buzzard [3] later extended [5] to treat almost all two-dimensional  $p$ -adic Artin representations potentially unramified at  $p$  (the image of the inertia group at  $p$  is finite) and subsequently it led to modularity of two-dimensional “5-adic” icosahedral Artin representations by Taylor [39]. The strong Artin conjecture for odd two-dimensional representations of  $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$  is now completely proved by work of Khare-Wintenberger and Kisin on Serre’s conjecture for odd two-dimensional mod  $p$  representations of  $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ .

In this paper, we push through Taylor’s program and generalise them to treat new cases of the strong Artin conjecture for two-dimensional, totally odd, icosahedral Artin representations of the absolute Galois group of a totally real field. More precisely, we prove the following theorems.

**THEOREM 1** *Let  $F$  be a totally real field. Suppose that 5 splits completely in  $F$ . Suppose that  $\rho : \text{Gal}(\overline{F}/F) \rightarrow GL_2(\mathbf{C})$  is a totally odd, irreducible, continuous representation satisfying the following conditions.*

- *The image of the projective representation  $\text{proj } \rho$  of  $\rho$  is  $A_5$ .*

- The projective image of the decomposition group at every place of  $F$  above 5 has order 2.

Then  $\rho$  arises from a holomorphic cuspidal Hilbert modular eigenform of weight 1 and the Artin  $L$ -function  $L(\rho, s)$  is entire.

**THEOREM 2** Let  $F$  be a totally real field. Suppose that 2 splits completely in  $F$  and that  $[F(\zeta_5) : F] = 4$ . Suppose that  $\rho : \text{Gal}(\overline{F}/F) \rightarrow GL_2(\mathbf{C})$  is a totally odd, irreducible, continuous representation satisfying the following conditions.

- The image of the projective representation  $\text{proj } \rho$  of  $\rho$  is  $A_5$ .
- At every place  $\mathfrak{p}$  of  $F$  above 2, the projective representation of  $\rho$  is unramified, and the image of  $\text{Frob}_{\mathfrak{p}}$  has order 3.

Then  $\rho$  arises from a holomorphic cuspidal Hilbert modular eigenform of weight 1 and the Artin  $L$ -function  $L(\rho, s)$  is entire.

These are corollaries of the following theorems, first of which is about “if  $\overline{\rho} : \text{Gal}(\overline{F}/F) \rightarrow GL_2(\overline{\mathbf{F}}_p)$  is modular, then  $\rho : \text{Gal}(\overline{F}/F) \rightarrow GL_2(\overline{\mathbf{Q}}_p) \simeq GL_2(\mathbf{C})$  is modular”:

**THEOREM 3** Let  $p$  be a rational prime. Let  $K$  be a finite extension of  $\mathbf{Q}_p$  with ring  $\mathcal{O}$  of integers and maximal ideal  $\mathfrak{m}$ . Let  $F$  be a totally real field. Suppose that  $p$  splits completely in  $F$ . Let  $\rho : \text{Gal}(\overline{F}/F) \rightarrow GL_2(\mathcal{O})$  be a continuous representation satisfying the following conditions.

- $\rho$  ramifies at only finitely many primes.
- $\overline{\rho} = (\rho \bmod \mathfrak{m})$  is absolutely irreducible when restricted to  $\text{Gal}(\overline{F}/F(\zeta_p))$ , and has a modular lifting which is potentially ordinary and potentially Barsotti-Tate at every prime of  $F$  above  $p$ .
- For every prime  $\mathfrak{p}$  of  $F$  above  $p$ , the restriction  $\rho|_{G_{\mathfrak{p}}}$  to the decomposition group  $G_{\mathfrak{p}}$  at  $\mathfrak{p}$  is the direct sum of 1-dimensional characters  $\chi_{\mathfrak{p},1}$  and  $\chi_{\mathfrak{p},2}$  of  $G_{\mathfrak{p}}$  such that the images of the inertia subgroup at  $\mathfrak{p}$  are finite and  $(\chi_{\mathfrak{p},1} \bmod \mathfrak{m}) \neq (\chi_{\mathfrak{p},2} \bmod \mathfrak{m})$ .

If  $p = 2$ , assume moreover the following conditions.

- The image of the complex conjugation, with respect to every embedding of  $F$  into  $\mathbf{R}$ , is not the identity matrix.
- $\overline{\rho}$  has insoluble image.
- For every prime  $\mathfrak{p}$  of  $F$  above 2,  $\rho$  is unramified at  $\mathfrak{p}$ .

Then there exists an embedding  $\iota : K \hookrightarrow \overline{\mathbf{Q}}_p \simeq \mathbf{C}$  and a classical holomorphic cuspidal Hilbert modular eigenform  $f$  of weight 1 such that  $\iota \circ \rho$  is isomorphic to the representation associated to  $f$  by Rogawski-Tunnell [28].

In proving the theorem, we shall firstly establish  $R = T$  theorems for Hida  $p$ -ordinary families over a finite soluble totally real extension  $F_\Sigma$  of  $F$  in which  $p \geq 2$  remains split completely—for lack of reference we shall prove them. Since  $\bar{\rho}$  has a potentially  $p$ -Barsotti-Tate and potentially  $p$ -ordinary modular lifting, one can deduce  $R = T$  in  $p$ -adic families from Kisin's  $R = T$  theorems in the  $p$ -Barsotti-Tate case. Note that, unfortunately, it is not possible to make appeal to Geraghty's  $R = T$  theorems in  $p$ -ordinary families as they assume that  $p > 2$  and that  $\bar{\rho}$  is trivial at every prime of  $F$  above  $p$ . This is because one can not eliminate the possibility that, upon 'soluble' base-changing to  $F_\Sigma$  to set  $\bar{\rho}|_{\text{Gal}(\bar{F}/F_\Sigma)}$  trivial at every prime of  $F_\Sigma$  above  $p$ ,  $F_\Sigma$  may no longer be split at  $p$ , which is crucial in constructing weight one forms in our approach. In the light of [1], the condition about the existence of a potentially ordinary Barsotti-Tate lifting of  $\bar{\rho}$  can be weaker, more precisely, it suffices to assume ' $\bar{\rho}$  is modular'. It is not necessary to make appeal to their results however.

The next two theorems are about modularity of  $\bar{\rho}$ .

**THEOREM 4** *Let  $F$  be a totally real field. Suppose that 5 is unramified in  $F$ . Let  $\bar{\rho} : \text{Gal}(\bar{F}/F) \rightarrow \text{GL}_2(\bar{\mathbf{F}}_5)$  be a continuous representation of satisfying the following conditions.*

- $\bar{\rho}$  is totally odd.
- $\bar{\rho}$  has projective image  $A_5$ .

*The there exists a cuspidal Hilbert modular eigenform of weight 2 such that its associated 5-adic Galois representation is potentially Barsotti-Tate and potentially ordinary at every prime of  $F$  above 5, and its associated mod 5 Galois representation is isomorphic to  $\bar{\rho}$ .*

The idea is exactly the same as that of Taylor—to prove modularity of  $\bar{\rho}$ , one firstly finds an elliptic curve  $E$  over a finite *soluble* totally real field extension  $F_\Sigma$  of  $F$  such that the action of  $\text{Gal}(\bar{F}/F_\Sigma)$  on the 5-torsion points of  $E$  is isomorphic to  $\bar{\rho}|_{\text{Gal}(\bar{F}/F_\Sigma)}$ ; secondly one proves  $E$  modular, therefore  $\bar{\rho}|_{\text{Gal}(\bar{F}/F_\Sigma)}$  modular; and finally it follows from Khare-Wintenberger [18] and Kisin [20] that  $\bar{\rho}|_{\text{Gal}(\bar{F}/F_\Sigma)}$  has a characteristic zero lifting which is modular. The 'automorphic descent' works as in [39].

In proving  $E$  is modular, we make some technical improvements on a 'naive' generalisation over totally real fields of the main theorem of Taylor in [39] by making appeal to the main result of Kisin [20] rather than the main result of Skinner-Wiles [35]. While Taylor/Skinner-Wiles requires the mod 3 representation  $E[3](\bar{F}_\Sigma)$  of  $\text{Gal}(\bar{F}/F_\Sigma)$  to be reducible with distinct characters on the diagonal at every prime of  $F_\Sigma$  above 3, we no longer requires this and consequently remove the '3-distinguishedness condition' in the main theorems of [39]. The key observation is that the weight 2 specialisation  $F_{H,2}$  of the Hida family  $F_H$ , whose weight 1 specialisation  $F_{H,1}$  renders  $E[3](\bar{F}_\Sigma)$  modular by

Langlands-Tunnell, does indeed render the 3-adic Barsotti-Tate representation  $T_3E$  ‘strongly residually modular’ in the sense of Kisin [20] if  $E[3](\overline{F}_\Sigma)$  is unramified at every prime above 3.

As is clear from its proof, what we are proving indeed is modularity of general mod 5 representations  $\text{Gal}(\overline{F}/F) \rightarrow GL_2(\mathbf{F}_5)$ , and this allows us to work with the prime 2—proving modularity of  $\overline{\rho}_2 : \text{Gal}(\overline{F}/F) \rightarrow SL_2(\mathbf{F}_4)$  with  $\text{proj } \overline{\rho}_2 \simeq A_5$ —instead of the prime 5, going back to the original approach of Buzzard-Dickinson-Shepherd-Barron-Taylor; in [4] one firstly finds an abelian surface  $A$  over  $F$  with real multiplication  $\mathbf{Z}[(1 + \sqrt{5})/2]$  such that  $A(\overline{F})[2] \simeq \overline{\rho}_2$ ; secondly proves the mod 5 representation  $\text{Gal}(\overline{F}/F) \rightarrow GL_2(A(\overline{F})[\sqrt{5}]) \simeq GL_2(\mathbf{F}_5)$  is modular; and deduce  $A$  is modular by a modular lifting theorem.

**THEOREM 5** *Let  $F$  be a totally real field. Suppose that  $[F(\zeta_5) : F] = 4$ . Let  $\overline{\rho} : \text{Gal}(\overline{F}/F) \rightarrow SL_2(\mathbf{F}_4)$  be a continuous representation. Then there exists a cuspidal Hilbert modular eigenform of weight 2 such that its associated 2-adic Galois representation is potentially Barsotti-Tate and potentially unramified at every prime of  $F$  above 2 and its associated mod 2 Galois representation is isomorphic to  $\overline{\rho}$ .*

Lastly it might come in useful comparing our work and others. After the first draft of this paper was written in 2010, Kassaei announced a result proving an analogue of the main theorem 3 in the case when  $p$  is odd,  $p$  is unramified in  $F$ , and  $\chi_{\mathfrak{p},1}/\chi_{\mathfrak{p},2}$  and  $\chi_{\mathfrak{p},2}/\chi_{\mathfrak{p},1}$  are both unramified at every prime  $\mathfrak{p}$  of  $F$  above  $p$ . Pilloni, on the other hand, seems to have proved a slightly stronger analogue in which  $p$  is allowed to ramify a little in  $F$ . The fundamental ideas in both works and ours are essentially the same and are due to Buzzard, more specifically to Buzzard’s Theorem 9.1 in [3]. In forthcoming joint work with Kassaei and Tian, we extend Kassaei’s work to the case where  $\chi_{\mathfrak{p},1}/\chi_{\mathfrak{p},2}$  and  $\chi_{\mathfrak{p},2}/\chi_{\mathfrak{p},1}$  are of conductor  $\mathfrak{p}$  for every prime  $\mathfrak{p}$  of  $F$  above  $p$  (unramified in  $F$ ) and prove many new cases of the strong Artin conjecture for  $\rho : \text{Gal}(\overline{F}/F) \rightarrow GL_2(\mathbf{C})$  in the insoluble case as above.

To prove an analogue of the main theorem 3 in the case where  $\chi_{\mathfrak{p},1}/\chi_{\mathfrak{p},2}$  and  $\chi_{\mathfrak{p},2}/\chi_{\mathfrak{p},1}$  are of conductor  $\mathfrak{p}^r$  with  $r > 1$  for every prime  $\mathfrak{p}$  of  $F$  above  $p$ , one needs to know precise geometry of Hilbert modular varieties of level  $p^r$  and, unless  $p$  splits completely in  $F$  which we solve, this may not even be possible. Calculating  $q$ -expansions at cusps to glue weight one forms does not seem to depend on the ramification of  $p$  in  $F$  and, for that, this work is very useful in general. On the other hand, in order to prove the general case ( $p$  ramifies arbitrarily in  $F$ ), the author [30] considers new moduli spaces of Hilbert-Blumenthal abelian varieties; and he expects to make progress in the general case in his forthcoming work.

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## 2 MODULARITY OF MOD 5 ICOSAHEDRAL REPRESENTATIONS OF $\text{Gal}(\overline{F}/F)$

LEMMA 6 *Let  $F$  be a totally real field. Suppose that 5 is unramified in  $F$ . Suppose that  $\overline{\rho} : \text{Gal}(\overline{F}/F) \rightarrow \text{GL}_2(\overline{\mathbf{F}}_5)$  is a continuous representation satisfying the following conditions.*

- $\overline{\rho}$  is totally odd.
- $\overline{\rho}$  has projective image  $A_5$ .

*Then there is a finite soluble totally real field extension  $F_\Sigma$  of  $F$  and an elliptic curve  $E$  over  $F_\Sigma$  such that*

- $F(\sqrt{5}) \subset F_\Sigma \subset \overline{F}$ , and  $\sqrt{5}$  splits completely in  $F_\Sigma$ ;
- $E$  has good ordinary reduction at every prime of  $F$  above 3 and has potentially ordinary reduction at every prime of  $F$  above 5;
- $\overline{\rho}_{E,5} : \text{Gal}(\overline{F}/F_\Sigma) \rightarrow \text{Aut}(E(\overline{F}_\Sigma)[5])$  is equivalent to a twist of  $\overline{\rho}|_{\text{Gal}(\overline{F}/F_\Sigma)}$ ;
- $\overline{\rho}_{E,3}|_{\text{Gal}(\overline{F}/F_\Sigma(\zeta_3))} : \text{Gal}(\overline{F}/F_\Sigma(\zeta_3)) \rightarrow \text{Aut}(E(\overline{F}_\Sigma)[3])$  is absolutely irreducible.

*Proof.* Firstly, as in [39], find a biquadratic totally real extension  $K_1 \subset \overline{F}$  of  $F$ , which is a quadratic totally real extension of  $F(\sqrt{5})$  in which  $\sqrt{5}$  splits completely, such that  $\text{proj } \overline{\rho} : \text{Gal}(\overline{F}/K_1) \rightarrow \text{PSL}_2(\mathbf{F}_5) \simeq A_5$  lifts to a representation  $\overline{\rho}_1 : \text{Gal}(\overline{F}/K_1) \rightarrow \text{GL}_2(\mathbf{F}_5)$  with determinant the mod 5 cyclotomic character  $\epsilon$ . Choose, by class field theory, a finite soluble totally real extension  $K_2 \subset \overline{F}$  of  $K_1$  such that  $\overline{\rho}_1|_{\text{Gal}(\overline{F}/K_2)}$  is trivial when restricted to the decomposition group at every prime of  $K_2$  above 3. Let  $F_\Sigma$  denote the Galois closure of  $K_2$  over  $F$ . Let  $\overline{\rho}_\Sigma$  denote the restriction of  $\overline{\rho}$  to  $\text{Gal}(\overline{F}/F_\Sigma)$ .

As in section 1 of [33], let  $Y_{\overline{\rho}_\Sigma}/F_\Sigma$  (resp.  $X_{\overline{\rho}_\Sigma}/F_\Sigma$ ) denote the twist of the (resp. compactified) modular curve  $Y_5$  (resp.  $X_5$ ) with full level 5 structure by the cohomology class in  $H^1(\text{Gal}(\overline{F}/F_\Sigma), \text{Aut } X_5)$  defined by an isomorphism  $\overline{\rho}_\Sigma \simeq (\mathbf{Z}/5\mathbf{Z}) \times \mu_5$  of the  $\mathbf{F}_5$ -vector spaces. As proved in Lemma 1.1 in [33], the ‘twist’ cohomology class is indeed trivial, and therefore  $X_{\overline{\rho}_\Sigma} \simeq X_5$  and  $Y_{\overline{\rho}_\Sigma}$  is isomorphic over  $F_\Sigma$  to a Zariski open subset of the projective line  $\mathbf{P}^1$ . In particular,  $Y_{\overline{\rho}_\Sigma}$  has infinitely many rational points.

Let  $Y_{\overline{\rho}_\Sigma,0}(3)$  denote the degree 4 cover over  $Y_{\overline{\rho}_\Sigma}$  which parameterises isomorphism classes of elliptic curves  $E$  equipped with an isomorphism  $E[5] \simeq \overline{\rho}_\Sigma$  taking

the Weil pairing on  $E[5]$  to  $\epsilon : \wedge^2 \bar{\rho}_\Sigma \rightarrow \mu_5$  and a finite flat subgroup scheme  $C \subset E[3]$  of order 3.

Let  $Y_{\bar{\rho}_\Sigma, \text{split}}(3)$  denote the étale cover over  $Y_{\bar{\rho}_\Sigma}$  which parameterises isomorphism classes of elliptic curves  $E$  equipped with an isomorphism  $E[5] \simeq \bar{\rho}_\Sigma$  taking the Weil pairing on  $E[5]$  to  $\epsilon : \wedge^2 \bar{\rho}_\Sigma \rightarrow \mu_5$  and an unordered pair, fixed by  $\text{Gal}(\bar{F}/F_\Sigma)$ , of finite flat subgroup schemes  $C, D \subset E[3]$  of order 3 which intersect trivially. Then it follows from Lemma 12 in [27] that  $Y_{\bar{\rho}_\Sigma, \text{split}}(3)$  and  $Y_{\bar{\rho}_\Sigma, 0}(3)$  has only finitely many rational points.

For every prime  $\mathfrak{p}$  of  $F_\Sigma$  above 3, the elliptic curve  $y^2 = x^3 + x^2 - x$  defines an element of  $Y_{\bar{\rho}_\Sigma}(F_{\Sigma, \mathfrak{p}})$  with good ordinary reduction, and we let  $\mathcal{U}_{\mathfrak{p}} \subset Y_{\bar{\rho}_\Sigma}(F_{\Sigma, \mathfrak{p}})$  denote a (non-empty) open neighbourhood (for the 3-adic topology) of the point, consisting of elliptic curves with good ordinary reduction at  $\mathfrak{p}$ .

For every prime  $\mathfrak{p}$  of  $F_\Sigma$  above 5, we define a non-empty open subset (for the 5-adic topology)  $\mathcal{U}_{\mathfrak{p}} \subset Y_{\bar{\rho}_\Sigma}(F_{\Sigma, \mathfrak{p}})$  as in the proof of Lemma 2.3 in [39].

By Hilbert irreducibility theorem (Theorem 1.3 in [11]; see also Theorem 3.5.7 in [32]), we may then find a rational point in  $Y_{\bar{\rho}_\Sigma}(F_\Sigma)$  which lies in  $\mathcal{U}_{\mathfrak{p}}$  for every  $\mathfrak{p}$  of  $F_\Sigma$  above either 3 or 5 and does *not* lie in the images of  $Y_{\bar{\rho}_\Sigma, 0}(3)(F_\Sigma) \rightarrow Y_{\bar{\rho}_\Sigma}(F_\Sigma)$  and  $Y_{\bar{\rho}_\Sigma, \text{split}}(3)(F_\Sigma) \rightarrow Y_{\bar{\rho}_\Sigma}(F_\Sigma)$ . The elliptic curve over  $F_\Sigma$  corresponding to the rational point is what we are looking for.  $\square$

**THEOREM 7** *Let  $F$  be a totally real field. Suppose that 5 is unramified in  $F$ . Let  $\bar{\rho} : \text{Gal}(\bar{F}/F) \rightarrow \text{GL}_2(\bar{\mathbf{F}}_5)$  be a continuous representation of satisfying the following conditions.*

- $\bar{\rho}$  is totally odd.
- $\bar{\rho}$  has projective image  $A_5$ .

*Then there exists a cuspidal Hilbert modular eigenform of weight 2 such that its associated 5-adic Galois representation is potentially Barsotti-Tate and potentially ordinary at every prime of  $F$  above 5 and its associated mod 5 Galois representation is isomorphic to  $\bar{\rho}$ .*

*Proof.* Choose an elliptic curve over a finite soluble totally real extension  $F_\Sigma$  of  $F$  as in the lemma. Replace  $F_\Sigma$  by its finite soluble totally real extension if necessary to assume that the mod 3 representation  $\bar{\rho}_{E,3}$  of  $\text{Gal}(\bar{F}/F_\Sigma)$  on  $E(\bar{F}_\Sigma)[3]$  is unramified when restricted to the decomposition subgroup at every prime of  $F_\Sigma$  above 3. By the Langlands-Tunnell theorem, there exists a weight 1 holomorphic cuspidal Hilbert modular eigenform  $f_1$  which gives rise to  $\bar{\rho}_{E,3}$ . By 3-adic Hida theory [14], we may find a holomorphic cuspidal Hilbert modular eigenform  $f_2$  of weight 2 and level prime to 3, ordinary at every prime of  $F_\Sigma$  above 3, which gives rise to  $\bar{\rho}_{E,3}$ . As  $E$  is ordinary at 3,  $\rho_{E,3}$  is strongly residually modular in the sense of Kisin [20] (3.5.4), and it follows from Theorem 3.5.5 in [20] that  $T_3 E$  is modular. By Faltings' isogeny theorem,  $E$  is therefore modular. As  $\bar{\rho}_{E,5}$  is modular,  $\bar{\rho}|_{\text{Gal}(\bar{F}/F_\Sigma)}$  is modular. Since  $F_\Sigma$  is a soluble extension of  $F$ ,  $\bar{\rho}_\Sigma$  remains absolutely irreducible when restricted

to  $\text{Gal}(\overline{F}/F_\Sigma(\zeta))$ . Furthermore, since 5 is unramified in  $F$ , the kernel of  $\text{proj } \overline{\rho}_\Sigma$  does not fix  $F_\Sigma(\zeta_5)$ . It then follows from results of Khare-Wintenberger [18] and Kisin [20] that there exists a modular lifting of  $\overline{\rho}_\Sigma$ . The ‘soluble descent’ to  $F$  is exactly as in [39].  $\square$

REMARK. In the forthcoming work with Kassaei and Tian, we remove the assumption that 5 is unramified in  $F$  in Lemma 6, and thereby in Theorem 7. Essentially the same argument works.

### 3 MODULARITY OF MOD 2 ICOSAHEDRAL REPRESENTATIONS OF $\text{Gal}(\overline{F}/F)$

THEOREM 8 *Let  $F$  be a totally real field. Suppose that  $[F(\zeta_5) : F] = 4$ . Let  $\overline{\rho} : \text{Gal}(\overline{F}/F) \rightarrow SL_2(\mathbf{F}_4)$  be a continuous representation. Then there exists a cuspidal Hilbert modular eigenform of weight 2 such that its associated 2-adic Galois representation is potentially Barsotti-Tate and potentially unramified at every prime of  $F$  above 2 and its associated mod 2 Galois representation is isomorphic to  $\overline{\rho}$ .*

*Proof.* By Theorem 3.4 in [33], there exists a principally polarised abelian surface  $A$  over  $F$  with real multiplication by  $\mathbf{Z}[(1 + \sqrt{5})/2]$  compatible with the polarisation such that the action of  $\text{Gal}(\overline{F}/F)$  on  $A(\overline{F})[2] \simeq \mathbf{F}_4^2$  is equivalent to  $\overline{\rho}$ ; and the action of  $\text{Gal}(\overline{F}/F)$  on  $A(\overline{F})[\sqrt{5}] \simeq \mathbf{F}_5^2$  is given via a homomorphism

$$\overline{\rho}_{A, \sqrt{5}} : \text{Gal}(\overline{F}/F) \rightarrow GL_2(\mathbf{F}_5)$$

which is surjective and whose image contains  $SL_2(\mathbf{F}_5)$ . It suffices to prove that  $A$  is modular.

Firstly, the Weil pairing on  $A(\overline{F})[\sqrt{5}]$  shows that  $\det \overline{\rho}_{A, \sqrt{5}}$  is the mod 5 cyclotomic character. Since  $[F(\zeta_5) : F] = 4$ , the determinant is indeed surjective, and therefore  $\overline{\rho}_{A, \sqrt{5}}$  is absolutely irreducible.

If  $\overline{\rho}_{A, \sqrt{5}}$  is irreducible at some place of  $F$  above 5, the absolute irreducibility of  $\overline{\rho}_{A, \sqrt{5}}$  implies the absolute irreducibility of its restriction to  $\text{Gal}(\overline{F}/F(\zeta_5))$ . Otherwise,  $\overline{\rho}_{A, \sqrt{5}}$  is reducible at every place of  $F$  above 5; in which case, it is also equally easy to check that its restriction to  $\text{Gal}(\overline{F}/F(\sqrt{5}))$  of  $\overline{\rho}_{A, \sqrt{5}}$  is absolutely irreducible (See Proposition 7 in [27], for example). It follows from results of Khare-Wintenberger [16] and Kisin [20] that it is possible to construct a modular lifting of  $\overline{\rho}_{A, \sqrt{5}}$ ; more precisely,  $\overline{\rho}_{A, \sqrt{5}}$  is strongly residually modular. The modularity of  $\rho_{A, \sqrt{5}}$  follows from Theorem 3.5.5 in [20] and [12].  $\square$

### 4 HOLOMORPHIC HILBERT MODULAR FORMS AND HIDA THEORY OF MODULAR GALOIS REPRESENTATIONS

Let  $F$  be a totally real field. We let  $\mathcal{O}_F$  denote the ring of integers,  $\mathfrak{d}_F$  the different of  $F$ ,  $\mathbf{A}_F = \mathbf{A}_F^\infty \times F_\infty$ , and  $\mathcal{O}_F^\wedge$  denote  $\mathcal{O}_F \otimes_{\mathbf{Z}} \mathbf{Z}^\wedge \subset \mathbf{A}_F^\infty$ . Let  $S_\infty$



denote the set of infinite places of  $F$ . For an ideal  $\mathfrak{n}$  of  $\mathcal{O}_F$ , let  $F_{\mathfrak{n}}$  denote the strict ray class field of conductor  $\mathfrak{n}S_{\infty}$ .

For an ideal  $\mathfrak{n}$ , let  $U^1(\mathfrak{n})$  (resp.  $U_1(\mathfrak{n})$ ) denote the open compact subgroup of  $GL_2(\mathcal{O}_F^{\wedge})$  consisting of matrices which are congruent modulo  $\mathfrak{n}\mathcal{O}_F^{\wedge}$  to matrices with first column  $(1, 0)$  (resp. the second row  $(0, 1)$ ). Let  $I_{\mathfrak{n}}$  denote  $\mathbf{A}_F^{\times}/(F^{\times}(\mathbf{A}_F^{\infty\times} \cap U^1(\mathfrak{n}))F_{\infty}^{+\times})$ .

For  $k \in \mathbf{Z}$  and an open compact subgroup  $U$  of  $GL_2(\mathcal{O}_F^{\wedge})$ , let  $S_k(U)$  denote the space,  $S_{k,k/2}(U)$  in the sense of Hida [14], of cuspidal holomorphic Hilbert modular forms  $f$  of parallel weight  $k$  and level  $U$  with the Fourier coefficient  $c(\mathfrak{n}, f) \in \mathbf{Z}$  for all ideals  $\mathfrak{n}$  of  $\mathcal{O}_F$ . The spaces  $S_k(U^1(\mathfrak{n}))$  and  $S_k(U_1(\mathfrak{n}))$  for an ideal  $\mathfrak{n}$  of  $\mathcal{O}_F$  come equipped with an action of  $I_{\mathfrak{n}}$  via the diamond operator  $\langle \cdot \rangle$ , and Hecke operators  $T_{\mathfrak{q}}$  for every prime  $\mathfrak{q}$  of  $\mathcal{O}_F$  not dividing  $\mathfrak{n}$  and  $U_{\mathfrak{q}}$  for every prime  $\mathfrak{q}$  dividing  $\mathfrak{n}$ .

Let  $h_k(\mathfrak{n})$  denote the sub  $\mathbf{Z}$ -algebra of  $\text{End}(S_k(U^1(\mathfrak{n})))$  generated over  $\mathbf{Z}$  by all these operators (See Proposition 2.3, Theorem 4.10, and Theorem 4.11 of [14]). For every prime  $\mathfrak{q}$  not dividing  $\mathfrak{n}$ , let  $S_{\mathfrak{q}} = (\mathbf{N}_{F/\mathbf{Q}}\mathfrak{q})^{k-2}\langle \mathfrak{q} \rangle \in h_k(\mathfrak{n})$ ; this corresponds to the action of the scalar matrix with a uniformiser of  $\mathcal{O}_F$  at  $\mathfrak{q}$  on the diagonal. Following [14], for every ideal  $\mathfrak{m}$  of  $\mathcal{O}_F$ , we may define  $T_{\mathfrak{m}} \in h_k(\mathfrak{n})$ .

Let  $p$  be a rational prime and let  $S_p$  denote the set of prime ideals of  $\mathcal{O}_F$  dividing  $p$ . Fix an algebraic closure  $\overline{\mathbf{Q}}_p$ , an isomorphism  $\overline{\mathbf{Q}}_p \simeq \mathbf{C}$ , and an embedding  $\overline{\mathbf{Q}} \rightarrow \overline{\mathbf{Q}}_p$ .

For a ring  $R \subset \overline{\mathbf{Q}}_p$ , we shall let  $S_k(U_1(\mathfrak{n}))_R$  denote  $S_k(U_1(\mathfrak{n})) \otimes_{\mathbf{Z}} R$  and  $h_k(\mathfrak{n})_R$  denote  $h_k(\mathfrak{n}) \otimes_{\mathbf{Z}} R$ ; there is a pairing  $(\cdot, \cdot) : h_k(\mathfrak{n})_R \times S_k(U_1(\mathfrak{n}))_R \rightarrow R$  defined by  $(T, f) = c(\mathcal{O}_F, Tf)$ .

For a ray class character  $\psi : I_{\mathfrak{n}} \rightarrow \overline{\mathbf{Q}}_p^{\times} \text{ mod } \mathfrak{n}S_{\infty}$ , let  $S_{k,\psi}(U_1(\mathfrak{n}))_{\mathbf{Z}_p[\psi]}$  denote the submodule of  $S_k(U_1(\mathfrak{n}))_{\mathbf{Z}_p[\psi]}$  consisting of cuspidal Hilbert modular forms of parallel weight  $k$  and level  $U_1(\mathfrak{n})$  with central character  $\psi$ - $S_{\mathfrak{q}}$  acts via  $\psi$  at  $\mathfrak{q}$ ; the forms in  $S_{k,\psi}(U_1(\mathfrak{n}))_{\mathbf{Z}_p[\psi]}$  may be thought of as  $|I_{\mathfrak{n}}|$ -tuple of classical Hilbert modular forms of ‘level  $\Gamma_1(\mathfrak{n})$ ’ on the  $|I_{\mathfrak{n}}|$ -copies of  $(GL_2(\mathbf{R})/(\mathbf{R}^{\times}SO_2(\mathbf{R})))^{\text{Hom}(F,\mathbf{R})}$  with ‘Dirichlet character mod  $\mathfrak{n}$ ’.

Fix an ideal  $\mathfrak{n}$  of  $\mathcal{O}_F$  coprime to  $p$ . For a finite extension  $K$  of  $\mathbf{Q}_p$  with ring  $\mathcal{O}$  of integers, Hida [14] defines the idempotent  $e$  and we set  $h_{\mathcal{O}}^0(\mathfrak{n})$  to be the inverse limit with respect to  $r \in \mathbf{Z}_{\geq 1}$  of  $h_2(\mathfrak{n}p^r)_{\mathbf{Z}_p} \otimes_{\mathbf{Z}_p} \mathcal{O}$ . Let  $I_{\mathfrak{n}p^{\infty}}$  denote the inverse limit of the  $I_{\mathfrak{n}p^r}$  and the diamond operators  $\langle \cdot \rangle : I_{\mathfrak{n}p^r} \rightarrow eh_2(\mathfrak{n}p^r)_{\mathcal{O}}$  induce

$$\langle \cdot \rangle : I_{\mathfrak{n}p^{\infty}} \rightarrow h_{\mathcal{O}}^0(\mathfrak{n})^{\times}.$$

One can also see  $\langle \cdot \rangle$  as the action of  $(\mathcal{O}_F/\mathfrak{n})^{\times} \times (\mathcal{O}_F \otimes_{\mathbf{Z}} \mathbf{Z}_p)^{\times}$  by the composite:

$$(\mathcal{O}_F/\mathfrak{n})^{\times} \times (\mathcal{O}_F \otimes_{\mathbf{Z}} \mathbf{Z}_p)^{\times} \rightarrow I_{\mathfrak{n}p^{\infty}} \xrightarrow{\langle \cdot \rangle} h_{\mathcal{O}}^0(\mathfrak{n}).$$

We let  $\text{Tor}_{np^\infty}$  (resp.  $\text{Fr}_{np^\infty}$ ) denote the torsion subgroup (resp. the maximal  $\mathbf{Z}_p$  free subgroup of rank  $1 + \delta$  with  $\delta = 0$  if the Leopoldt conjecture holds) of  $I_{np^\infty}$ ; let  $\Lambda$  denote the completed group algebra over  $\mathbf{Z}_p$  of  $\text{Fr}_{np^\infty}$  and  $\Lambda_K = \Lambda \otimes_{\mathbf{Z}_p} \mathcal{O}$ . Then  $h_{\mathcal{O}}^0(\mathfrak{n})$  is a  $\Lambda_K$ -module by  $\langle \rangle$ . We will let

$$\text{Art} : \mathbf{A}_F^\times / \overline{F^\times F_\infty^{+\times}} \simeq \text{Gal}(\overline{F}/F)^{\text{ab}}$$

denote the (global) Artin map, normalised compatibly with the local Artin maps normalised to take uniformisers to arithmetic Frobenius elements. By abuse of notation, we shall let  $\text{Art}$  also denote the induced homomorphism  $I_{np^\infty} \rightarrow \text{Gal}(F_n(\mu_{p^\infty})/F)$  and let  $\epsilon$  denote the cyclotomic character  $\epsilon : \text{Gal}(F_n(\mu_{p^\infty})/F) \rightarrow \mathbf{Z}_p^\times$ .

Hida [14] proves that  $h_{\mathcal{O}}^0(\mathfrak{n})$  is a torsion free  $\Lambda_K$ -module and, for a character  $\psi : I_{np^\infty} \rightarrow K$  which factors through  $I_{np^r}$  for  $r \in \mathbf{Z}_{\geq 1}$ , if  $k \geq 2$ , then  $h_{\mathcal{O}}^0(\mathfrak{n})_{\ker((\epsilon \circ \text{Art})^{k-2}\psi)}$  is isomorphic to the subspace of  $eS_k(U_1(np^r))_{\mathcal{O}}$  where  $\langle \rangle = \psi$  on  $\text{Fr}_{np^\infty}$ .

We will let  $\epsilon^{\text{cyclo}}$  denote the character

$$\text{Gal}(\overline{F}/F) \twoheadrightarrow \text{Gal}(\overline{F}/F)^{\text{ab}} \twoheadrightarrow I_{np^\infty} \hookrightarrow \mathcal{O}[[I_{np^\infty}]]^\times = \Lambda_K[\text{Tor}_{np^\infty}]^\times.$$

Note that  $\mathfrak{q} \mapsto \mathbf{Nq}S_{\mathfrak{q}}$  extends to  $\mathbf{NS} : (\mathcal{O}_F/\mathfrak{n})^\times \times (\mathcal{O}_F \otimes_{\mathbf{Z}} \mathbf{Z}_p)^\times \rightarrow I_{np^\infty} \rightarrow h_{\mathcal{O}}^0(\mathfrak{n})^\times$ . Let  $\mathbf{NS}_\Sigma$  (resp.  $\mathbf{NS}^{\text{P}}$ ) denote the  $\Sigma$  (resp. the prime to  $S_{\text{P}}$ ) part  $\prod_{\mathfrak{p} \in \Sigma} \mathcal{O}_{F_{\mathfrak{p}}}^\times \rightarrow h_{\mathcal{O}}^0(\mathfrak{n})$  (resp.  $(\mathcal{O}_F/\mathfrak{n})^\times \rightarrow h_{\mathcal{O}}^0(\mathfrak{n})$ ) for s subset  $\Sigma$  of  $S_{\text{P}}$ .

If  $\mathfrak{m}$  is a maximal ideal of  $h_{\mathcal{O}}^0(\mathfrak{n})$  with residue field  $k_{\mathfrak{m}}$ , there is a continuous representation

$$\overline{\rho}_{\mathfrak{m}} : G_F = \text{Gal}(\overline{F}/F) \rightarrow \text{GL}_2(k_{\mathfrak{m}})$$

such that, for every prime ideal  $\mathfrak{q}$  of  $\mathcal{O}_F$  not dividing  $np$ ,  $\overline{\rho}_{\mathfrak{m}}$  is unramified at  $\mathfrak{q}$  and  $\text{tr} \overline{\rho}_{\mathfrak{m}}(\text{Frob}_{\mathfrak{q}}) = T_{\mathfrak{q}}$ . Set  $S_{\mathcal{O}}^0(\mathfrak{n}) = \text{Hom}_{\Lambda_K}(h_{\mathcal{O}}^0(\mathfrak{n}), \Lambda_K)$ . For a finite field extension  $L$  of the field  $\text{Frac} \Lambda_K$  of fractions of  $\Lambda_K$  with its integral closure  $\mathcal{O}_L$  of  $\Lambda_K$  in  $L$ , Buzzard-Taylor [5] calls a  $\Lambda_K$ -algebra homomorphism  $F_{\text{H}} \in S_{\mathcal{O}}^0(\mathfrak{n}) \otimes_{\Lambda_K} L = \text{Hom}_{\Lambda_K}(h_{\mathcal{O}}^0, L)$  a  $\Lambda$ -adic *eigenform* (of level  $\mathfrak{n}$ ).

If the unique maximal ideal  $\mathfrak{m}$  above  $\ker F_{\text{H}} \subset h_{\mathcal{O}}^0(\mathfrak{n})$  is non-Eisenstein, i.e.,  $\overline{\rho}_{\mathfrak{m}}$  as above is absolutely irreducible, then there is a continuous representation

$$\rho_{F_{\text{H}}} : G_F \rightarrow \text{GL}_2(h_{\mathcal{O}}^0(\mathfrak{n})_{\mathfrak{m}}) \xrightarrow{F_{\text{H}}} \text{GL}_2(\mathcal{O}_L)$$

which is unramified at every prime ideal  $\mathfrak{q}$  of  $\mathcal{O}_F$  not dividing  $np$  and satisfies  $\text{tr} \rho_{F_{\text{H}}}(\text{Frob}_{\mathfrak{q}}) = T_{\mathfrak{q}}$  and  $\det \rho_{F_{\text{H}}} = (\mathbf{NS}) \circ \epsilon^{\text{cyclo}}$ . Moreover, it is a result of Wiles [43] that, for every place  $\mathfrak{p}$  of  $F$  above  $p$ , the restriction to the decomposition group  $G_{\mathfrak{p}}$  at  $\mathfrak{p}$  of  $\rho_{F_{\text{H}}}$  is of the form

$$\begin{pmatrix} \chi_{F_{\text{H}}, \mathfrak{p}, 2} & * \\ 0 & \chi_{F_{\text{H}}, \mathfrak{p}, 1} \end{pmatrix}$$

where  $\chi_{F_{\mathbf{H},\mathbf{p},1}}$  is an unramified character of  $G_{\mathbf{p}}$  such that  $\chi_{F_{\mathbf{H},\mathbf{p},1}}(\text{Frob}_{\mathbf{p}}) = U_{\mathbf{p}}$  and  $\chi_{F_{\mathbf{H},\mathbf{p},1}}\chi_{F_{\mathbf{H},\mathbf{p},2}} = (F_{\mathbf{H}} \circ \mathbf{NS}) \circ \epsilon^{\text{cycl}}|_{G_{\mathbf{p}}}$ .

DEFINITION. Following [5], we call two  $\Lambda$ -adic eigenforms  $F_{\mathbf{H},1}$  and  $F_{\mathbf{H},2} : h_{\mathcal{O}}^0(\mathbf{n}) \rightarrow \mathcal{O}_L$  of level  $\mathbf{n}$   $\Lambda$ -adic companion form with respect to primes  $\wp_1$  and  $\wp_2$  of  $\mathcal{O}_L$  which do not divide  $p$ , if there are embeddings  $\iota_1 : \mathcal{O}_L/\wp_1 \hookrightarrow \overline{\mathbf{Q}}_p$  and  $\iota_2 : \mathcal{O}_L/\wp_2 \hookrightarrow \overline{\mathbf{Q}}_p$  such that, for every ideal  $\mathbf{m}$  of  $\mathcal{O}_F$  not dividing  $p$ , there exists a subset  $\Sigma$  of  $S_{\mathbf{P}}$  such that  $(F_{\mathbf{H},2}(T_{\mathbf{m}}) \bmod \wp_2) = (F_{\mathbf{H},1}(T_{\mathbf{m}}(\mathbf{NS}_{\Sigma})(\mathbf{m})^{-1}) \bmod \wp_1)$ ; and such that, for every place  $\mathbf{p}$  in  $\Sigma$ ,  $(F_{\mathbf{H},2}(U_{\mathbf{p}}) \bmod \wp_2) = (F_{\mathbf{H},1}(U_{\mathbf{p}}^{-1}(\mathbf{NS}^P)(\mathbf{p})) \bmod \wp_1)$  while, for every  $\mathbf{p}$  in  $S_{\mathbf{P}} - \Sigma$ ,  $(F_{\mathbf{H},2}(U_{\mathbf{p}}) \bmod \wp_2) = (F_{\mathbf{H},1}(U_{\mathbf{p}}) \bmod \wp_1)$ .

5 DEFORMATION RINGS AND HECKE ALGEBRAS

Let  $F$  be a totally real field of even degree in which  $p$  is unramified; if  $p = 2$  assume furthermore that  $2$  splits completely in  $F$ . If  $p$  is odd, suppose  $p \geq 5$ . Let  $D$  be the quaternion algebra over  $F$  which ramifies exactly at a finite set  $\Sigma$  of finite places of  $F$  not dividing  $p$  and the infinite places  $S_{\infty}$  of  $F$ . Let  $\mathcal{O}_D$  denote a maximal order and fix an isomorphism  $\mathcal{O}_{D_{\mathfrak{q}}} \simeq M_2(\mathcal{O}_{F_{\mathfrak{q}}})$  for  $\mathfrak{q}$  not in  $\Sigma$ . Let  $S$  denote the disjoint union of  $\Sigma$ , the set  $S_{\mathbf{P}}$  of places of  $F$  above  $p$ , and the infinite places of  $F$ .

For a topological  $\mathbf{Z}_p$ -algebra  $R$ , let  $\psi : \mathbf{A}_F^{\infty,\times}/F \rightarrow R^{\times}$  be a continuous character such that  $\psi|_{\mathcal{O}_{F_{\mathbf{p}}}^{\times}}$  is trivial for every place  $\mathbf{p}$  of  $F$  above  $p$ , and, for an open compact subgroup  $U = \prod_{\mathfrak{q}} U_{\mathfrak{q}} \subset \prod_{\mathfrak{q}} \mathcal{O}_{D_{\mathfrak{q}}}^{\times}$ , let  $S_{2,\psi}^D(U)_R$  denote the  $R$ -module of  $R$ -valued modular forms on  $D^{\times} \backslash (D \otimes_F \mathbf{A}_F^{\infty})^{\times}$  of weight  $2$  and of level  $U$  in the sense of Taylor [40].

Let  $\mathbf{n}_{\Sigma}$  denote the square-free product of the primes in  $\Sigma$  and define  $U_{\Sigma} \subset (D \otimes_F \mathbf{A}_F^{\infty})^{\times}$  by  $U_{\Sigma,\mathfrak{q}} = GL_2(\mathcal{O}_{F_{\mathfrak{q}}})$  for  $\mathfrak{q}$  not in  $\Sigma$ ; and  $U_{\Sigma,\mathfrak{q}} = \mathcal{O}_{D_{\mathfrak{q}}}^{\times}$  for  $\mathfrak{q} \in \Sigma$ .

We shall write  $S_{2,\psi}^D(\mathbf{n}_{\Sigma})$  for  $S_{2,\psi}^D(U_{\Sigma})$  and  $h_{2,\psi}^D(\mathbf{n}_{\Sigma})_R$  for the  $R$ -subalgebra of  $\text{End}_R(S_{2,\psi}^D(\mathbf{n}_{\Sigma})_R)$  generated by  $T_{\mathfrak{q}}$  and  $S_{\mathfrak{q}}$  for all  $\mathfrak{q}$  not in  $S$ ; and  $T_{\mathbf{p}}$  and  $S_{\mathbf{p}}$  for all  $\mathbf{p}$  in  $S_{\mathbf{P}}$ .

Let  $K$  be a finite extension of  $\mathbf{Q}_p$  and  $\mathcal{O}$  be the ring of integers with maximal ideal  $\mathfrak{m}$  ad residue field  $k$ . Let

$$\rho : \text{Gal}(\overline{F}/F) \rightarrow GL_2(\mathcal{O})$$

be a continuous representation such that

- $\overline{\rho} = (\rho \bmod \mathfrak{m})$  is unramified outside  $S_{\mathbf{P}}$ ,
- $\overline{\rho}$  is not scalar at every place  $\mathbf{p}$  above  $p$ ,
- if  $p$  is odd, the restriction of  $\overline{\rho}$  to  $\text{Gal}(\overline{F}/F(\zeta_p))$  is absolutely irreducible; if  $p = 2$ ,  $\overline{\rho}$  has insoluble image,

- there exists a holomorphic automorphic representation  $\pi$  of  $(D \otimes_F \mathbf{A}_F)^\times$  generated by a cusp form in  $S_{2,\psi}^D(\mathfrak{n}_\Sigma)_\mathcal{O}$  such that  $\pi_{\mathfrak{q}}$  is unramified for every  $\mathfrak{q}$  not in  $\Sigma \cup S_P$ ,  $\pi_{\mathfrak{p}}$  is ordinary at every  $\mathfrak{p}$  in  $S_P$ , for every  $\mathfrak{q} \in \Sigma$ ,  $\pi_{\mathfrak{q}}$  corresponds by the local Jacquet-Langlands correspondence to a special representation of conductor  $\mathfrak{q}$ , and such that  $\bar{\rho}_\pi \simeq \bar{\rho}$ ,
- $\rho$  ramifies at  $\Sigma$  and possibly at  $S_P$ ; for every  $\mathfrak{p}$  in  $S_P$

$$\rho|_{G_{\mathfrak{p}}} \sim \begin{pmatrix} * & * \\ 0 & \chi_{\mathfrak{p}} \end{pmatrix}$$

with  $\chi_{\mathfrak{p}}$  unramified; and for  $\mathfrak{q} \in \Sigma$

$$\rho|_{G_{\mathfrak{q}}} \sim \begin{pmatrix} \epsilon\chi_{\mathfrak{q}} & * \\ 0 & \chi_{\mathfrak{q}} \end{pmatrix}$$

with  $\chi_{\mathfrak{q}}$  unramified such that  $\chi_{\mathfrak{q}}^2 = (\psi \circ \text{Art})|_{G_{\mathfrak{q}}}$ .

Let  $\mathbf{A}_F^\times = \mathbf{A}_F^{\times S} \times \mathbf{A}_{F,S}^\times$  for a finite subset  $S$  of the places of  $F$ . Let  $\psi$  be a character of  $\mathbf{A}_F^{\times S_P}$ . For  $p = 2$  let  $\psi_{p,\pm}$  denote the  $\mathbf{Z}_p$ -linear extension of the norm  $\mathbf{N} : (\mathcal{O}_F \otimes_{\mathbf{Z}} \mathbf{Z}_2)^\times \rightarrow \mathbf{Z}_2^\times$  followed by the character  $\mathbf{Z}_2^\times \rightarrow \mathbf{Z}_p^\times$  whose restriction to  $(\mathbf{Z}/4)^\times = \{\pm 1\}$  sends  $-1$  to  $\mp 1$  and whose restriction to  $(1 + 4\mathbf{Z}_2)^\times$  is trivial. For  $p$  odd, let  $\psi_p$  denote the norm followed by the trivial character on  $\mathbf{Z}_p^\times$ .

### 5.1 (FRAMED) DEFORMATION RINGS $R$

$\underline{\mathfrak{p}}|p$ : if  $p$  is odd, let  $R_{\mathfrak{p}}^{\square,\text{ord}}$  (resp.  $R_{\mathfrak{p}}^{\square,\text{BT},\text{ord}}$ ) denote the  $\mathcal{O}$ -algebra which represents the  $\mathfrak{p}$ -ordinary (resp. Barsotti-Tate  $\mathfrak{p}$ -ordinary) framed deformations of  $\bar{\rho}|_{G_{\mathfrak{p}}}$  of the form

$$\begin{pmatrix} * & * \\ 0 & \chi_{\mathfrak{p}}^{\text{ur}} \end{pmatrix}$$

with an unramified lifting  $\bar{\chi}_{\mathfrak{p}}^{\text{ur}}$  of  $\chi_{\mathfrak{p}}$  (resp. and its determinant is  $\epsilon\psi_P$ ); if  $p = 2$ , we shall write ‘ $\pm$ ’ in shorthand to mean two independent cases—‘ $+$ ’ corresponds to the 2-old case while ‘ $-$ ’ corresponds to the 2-new case in the sense to be made precise below, and let  $R_{\mathfrak{p},\pm}^{\square,\text{ord}}$  (resp.  $R_{\mathfrak{p},\pm}^{\square,\text{BT},\text{ord}}$ ) denote the complete local noetherian  $\mathcal{O}$ -algebra which represents the  $\mathfrak{p}$ -ordinary (resp. Barsotti-Tate  $\mathfrak{p}$ -ordinary) liftings of  $\bar{\rho}|_{G_{\mathfrak{p}}}$  of the form

$$\begin{pmatrix} * & * \\ 0 & \chi_{\mathfrak{p}}^{\text{ur}} \end{pmatrix}$$

with an unramified lifting  $\chi_{\mathfrak{p}}^{\text{ur}}$  of  $\chi_{\mathfrak{p}}$ , and with its determinant corresponding, by the local class field theory, to the norm  $\mathcal{O}_{F_{\mathfrak{p}}}^\times \hookrightarrow (\mathcal{O}_F \otimes_{\mathbf{Z}} \mathbf{Z}_2)^\times \xrightarrow{\mathbf{N}} \mathbf{Z}_2^\times$  followed by the character  $\mathbf{Z}_2^\times \rightarrow \mathbf{Z}_2^\times$  whose restriction to  $(\mathbf{Z}/4)^\times = \{\pm 1\}$  sends  $-1$  to

$\mp 1$  (resp. with its determinant  $\epsilon\psi_{P,\pm}$ ).

Let  $R_{\mathbf{P}}^{\square,\text{ord}} = \bigotimes_{\mathbf{p} \in S_{\mathbf{P}}}^{\wedge} R_{\mathbf{p}}^{\square,\text{ord}}$  (resp.  $R_{\mathbf{P}}^{\square,\text{BT,ord}} = \bigotimes_{\mathbf{p} \in S_{\mathbf{P}}}^{\wedge} R_{\mathbf{p}}^{\square,\text{BT,ord}}$ ) if  $p$  is odd; and  $R_{\mathbf{P},\pm}^{\square,\text{ord}} = \bigotimes_{\mathbf{p} \in S_{\mathbf{P}}}^{\wedge} R_{\mathbf{p},\pm}^{\square,\text{ord}}$  (resp.  $R_{\mathbf{P}}^{\square,\text{BT,ord}} = \bigotimes_{\mathbf{p} \in S_{\mathbf{P}}}^{\wedge} R_{\mathbf{p},\pm}^{\square,\text{BT,ord}}$ ) if  $p = 2$ .

$\mathfrak{q} \in \Sigma$ : let  $R_{\mathfrak{q}}^{\square,\psi}$  denote the domain (see 2.6 in [20], or Proposition 2.12 and 3.3.4 in [18]) parameterising liftings of  $\overline{\rho}|_{G_{\mathfrak{q}}}$  of the form

$$\begin{pmatrix} \epsilon\chi_{\mathfrak{q}}^{\text{ur}} & * \\ 0 & \chi_{\mathfrak{q}}^{\text{ur}} \end{pmatrix}$$

with  $\chi_{\mathfrak{q}}^{\text{ur}}$  an unramified lifting of  $\overline{\chi}_{\mathfrak{q}}$  such that  $(\chi_{\mathfrak{q}}^{\text{ur}})^2 = (\psi \circ \text{Art}^{-1})|_{G_{\mathfrak{q}}}$ .

Let  $R_{\Sigma}^{\square,\psi}$  denote the completed tensor product  $\bigotimes_{\mathfrak{q} \in \Sigma}^{\wedge} R_{\mathfrak{q}}^{\square,\psi}$ .

$\tau|\infty$ : let  $R_{\tau}^{\square,\text{odd}}$  denote the formally smooth ring which represents the liftings of  $\overline{\rho}|_{G_{\tau}}$  which, if  $p$  is odd, are odd; and, if  $p = 2$ , the image of complex conjugation in  $G_{\tau} \simeq \text{Gal}(\mathbf{C}/\mathbf{R})$  is not the identity matrix.

Let  $R_{\infty}^{\square,\text{odd}}$  denote the completed tensor product  $\bigotimes_{\tau|\infty}^{\wedge} R_{\tau}^{\square,\text{odd}}$

Fix a  $k$ -basis of  $\overline{\rho}$  and let

$$\rho^{\square S} : G_F \rightarrow GL_2(R^{\square S})$$

denote the  $S$ -framed universal deformation ring. Let  $R_S^{\square}$  denote the completed tensor product of the local framed deformation rings at places in  $S$ .

Let

$$\begin{aligned} R_S^{\square,\text{ord},\psi} &= R^{\square S} \otimes_{R_S^{\square}}^{\wedge} (R_{\mathbf{P}}^{\square,\text{ord}} \otimes^{\wedge} R_{\Sigma}^{\square,\psi} \otimes^{\wedge} R_{\infty}^{\square,\text{odd}}) \\ R_S^{\square,\text{BT,ord},\psi} &= R^{\square S} \otimes_{R_S^{\square}}^{\wedge} (R_{\mathbf{P}}^{\square,\text{BT,ord}} \otimes^{\wedge} R_{\Sigma}^{\square,\psi} \otimes^{\wedge} R_{\infty}^{\square,\text{odd}}) \end{aligned}$$

if  $p$  is odd; and

$$\begin{aligned} R_{S,\pm}^{\square,\text{ord},\psi} &= R^{\square S} \otimes_{R_S^{\square}}^{\wedge} (R_{\mathbf{P},\pm}^{\square,\text{ord}} \otimes^{\wedge} R_{\Sigma}^{\square,\psi} \otimes^{\wedge} R_{\infty}^{\square,\text{odd}}) \\ R_{S,\pm}^{\square,\text{BT,ord},\psi} &= R^{\square S} \otimes_{R_S^{\square}}^{\wedge} (R_{\mathbf{P},\pm}^{\square,\text{BT,ord}} \otimes^{\wedge} R_{\Sigma}^{\square,\psi} \otimes^{\wedge} R_{\infty}^{\square,\text{odd}}) \end{aligned}$$

if  $p = 2$ .

Let  $R_S^{\text{ord},\psi}$  (resp.  $R_S^{\text{ord,BT},\psi}$ ) denote the subring of  $R_S^{\square,\text{ord},\psi}$  (resp.  $R_S^{\square,\text{BT,ord},\psi}$ ) generated by the traces of  $\rho^{\square S}$ . Similarly define  $R_{S,\pm}^{\text{ord},\psi}$  and  $R_{S,\pm}^{\text{ord,BT},\psi}$ .

5.2 HECKE ALGEBRAS

Since  $\bar{\rho}$  arises from a holomorphic cusp form in  $S_2^D(\mathfrak{n}_\Sigma, \psi)_\mathcal{O}$  on the quaternion algebra  $D$  over  $F_\Sigma$  by assumption, there exists a maximal ideal  $\mathfrak{m}^D \subset h_2^D(\mathfrak{n}_\Sigma, \psi\psi_P)_\mathcal{O}$  if  $p$  odd (resp.  $\mathfrak{m}^D \subset h_2^D(\mathfrak{n}_\Sigma, \psi\psi_{P,+})_\mathcal{O}$  if  $p = 2$ ). It then follows that there exists a maximal ideal  $\mathfrak{m} \subset h_2(\mathfrak{n}_\Sigma p, \psi\psi_P)_\mathcal{O}$  such that

$$h_2(\mathfrak{n}_\Sigma p, \psi)_\mathfrak{m} \simeq h_2^D(\mathfrak{n}_\Sigma, \psi\psi_P)_{\mathfrak{m}^D}$$

if  $p$  odd (resp.  $\mathfrak{m}_+ \subset h_2(\mathfrak{n}_\Sigma 2, \psi\psi_{P,+})_\mathcal{O}$  such that

$$h_2(\mathfrak{n}_\Sigma 2, \psi\psi_{P,+})_{\mathfrak{m}_+} \simeq h_2^D(\mathfrak{n}_\Sigma, \psi\psi_{P,+})_{\mathfrak{m}^D}$$

if  $p = 2$ ). When  $p = 2$ , there also exists  $\mathfrak{m}_- \subset h_2(\mathfrak{n}_\Sigma 4, \psi\psi_{P,-})$  such that

$$h_2(\mathfrak{n}_\Sigma 4, \psi\psi_{P,-})_{\mathfrak{m}_-}/(2) \simeq h_2(\mathfrak{n}_\Sigma 2, \psi\psi_{P,+})_{\mathfrak{m}_+}/(2)$$

This can be proved exactly as in the proof of Lemma 3.2 in [4]; instead use the 0-dimensional Shimura variety corresponding to  $D$  over  $F_\Sigma$ .

For  $p = 2$  define  $e_{\text{BDST},\pm}$  and let  $h^0(\mathfrak{n})_\pm = e_{\text{BDST},\pm} h^0(\mathfrak{n})$ . Let

$$\begin{aligned} h_2^\square(\mathfrak{n}_\Sigma p, \psi\psi_P)_\mathfrak{m} &= h_2(\mathfrak{n}_\Sigma p, \psi\psi_P)_\mathfrak{m} \otimes_{R_S^{\text{BT,ord},\psi}} R_S^{\square,\text{BT,ord},\psi} \\ h^{0\square}(\mathfrak{n}_\Sigma, \psi)_\mathfrak{m} &= h^0(\mathfrak{n}_S, \psi\psi_P)_\mathfrak{m} \otimes_{R_S^{\text{ord},\psi}} R_S^{\square,\text{ord},\psi} \end{aligned}$$

if  $p$  is odd; and let

$$\begin{aligned} h_2^\square(\mathfrak{n}_\Sigma 4, \psi\psi_{P,-})_{\mathfrak{m}_-} &= h_2(\mathfrak{n}_\Sigma 4, \psi\psi_{P,-})_{\mathfrak{m}_-} \otimes_{R_{S,-}^{\text{BT,ord},\psi}} R_{S,-}^{\square,\text{BT,ord},\psi} \\ h^{0\square}(\mathfrak{n}_S, \psi)_{-,\mathfrak{m}_-} &= h^0(\mathfrak{n}_\Sigma, \psi\psi_P)_{-,\mathfrak{m}} \otimes_{R_{S,-}^{\text{ord},\psi}} R_{S,-}^{\square,\text{ord},\psi} \end{aligned}$$

if  $p = 2$ . It then follow from results of Kisin and Khare-Wintenberger that there is a natural surjection

$$R_S^{\square,\text{BT,ord},\psi} \rightarrow h_2^\square(\mathfrak{n}_\Sigma p, \psi\psi_P)_\mathfrak{m}$$

if  $p$  odd and

$$R_{S,-}^{\square,\text{BT,ord},\psi} \rightarrow h_2^\square(\mathfrak{n}_\Sigma 4, \psi\psi_{P,-})_{\mathfrak{m}_-}$$

if  $p = 2$ , which induce isomorphisms

$$R_S^{\square,\text{BT,ord},\psi}[1/p] \simeq h_2^\square(\mathfrak{n}_\Sigma p, \psi\psi_P)_\mathfrak{m}[1/p]$$

if  $p$  odd and

$$R_{S,-}^{\square,\text{BT,ord},\psi}[1/2] \simeq h_2^\square(\mathfrak{n}_\Sigma 4, \psi\psi_{P,-})_{\mathfrak{m}_-}[1/2].$$

The determinant of  $\rho^{\square S}$  defines

$$\text{NS} : I_{\mathfrak{n}_\Sigma p^\infty} \rightarrow R_S^{\text{ord},\psi}$$

and  $R_S^{\text{ord},\psi}/\ker(S - (\psi\psi_{\mathbf{P}} \circ \epsilon^{\text{cyclo}})) \simeq R_S^{\text{BT,ord},\psi}$  if  $p$  odd, and

$$\mathbf{NS} : I_{\mathbf{n}_{\Sigma} p^{\infty}} \rightarrow R_{S,-}^{\text{ord},\psi}$$

induces  $R_{S,-}^{\text{ord},\psi}/\ker(S - (\psi\psi_{\mathbf{P},-} \circ \epsilon^{\text{cyclo}})) \simeq R_{S,-}^{\text{BT,ord},\psi}$  if  $p = 2$ . On the other hand,  $h^0(\mathbf{n}_{\Sigma}, \psi)_{\mathbf{m}}/\ker(S - (\psi\psi_{\mathbf{P}} \circ \epsilon^{\text{cyclo}})) \simeq h_2(\mathbf{n}_{\Sigma} p, \psi\psi_{\mathbf{P}})_{\mathbf{m}}$  and  $h^0(\mathbf{n}_{\Sigma}, \psi)_{-, \mathbf{m}_-}/\ker(S - (\psi\psi_{\mathbf{P},-} \circ \epsilon^{\text{cyclo}})) \simeq h_2(\mathbf{n}_{\Sigma} 4, \psi\psi_{\mathbf{P},-})_{\mathbf{m}_-}$ . Then the surjective  $\Lambda$ -algebra homomorphisms

$$R_S^{\square, \text{ord}, \psi} \rightarrow h^{0\square}(\mathbf{n}_{\Sigma}, \psi)_{\mathbf{m}}$$

if  $p$  odd and

$$R_{S,-}^{\square, \text{ord}, \psi} \rightarrow h^{0\square}(\mathbf{n}_{\Sigma}, \psi)_{-, \mathbf{m}_-}$$

if  $p = 2$  induce the isomorphisms

$$R_S^{\square, \text{ord}, \psi}[1/p] \simeq h^{0\square}(\mathbf{n}_{\Sigma}, \psi)_{\mathbf{m}}[1/p]$$

and

$$R_{S,-}^{\square, \text{ord}, \psi}[1/2] \simeq h^{0\square}(\mathbf{n}_{\Sigma}, \psi)_{-, \mathbf{m}_-}[1/2].$$

## 6 COMPANION FORMS MOD $p$

Let  $F$  be a totally real field and  $p$  be a rational prime. Suppose that  $[F(\zeta_p) : F] > 3$  if  $p > 3$  and that 2 splits completely in  $F$  if  $p = 2$ . Let  $f_2$  be a holomorphic cuspidal Hilbert eigenform of weight  $2 \leq k_2 \leq p$  and of level prime to  $p$ . Assume that the associated  $p$ -adic representation  $\rho_2$  of  $\text{Gal}(\overline{F}/F)$  is crystalline and ordinary at every prime  $\mathfrak{p}$  of  $F$  above  $p$ . It is a well-known theorem of Wiles (Theorem 2.1.4 in [43]) that, for every prime  $\mathfrak{p}$  of  $F$  above  $p$ , the restriction  $\rho|_{G_{\mathfrak{p}}}$  to the decomposition group  $G_{\mathfrak{p}}$  at  $\mathfrak{p}$  is of the form

$$\rho|_{G_{\mathfrak{p}}} \simeq \begin{pmatrix} \epsilon^{k_2-1} \chi_{\mathfrak{p},1} & * \\ 0 & \chi_{\mathfrak{p},2} \end{pmatrix}$$

where  $\chi_{\mathfrak{p},1}$  and  $\chi_{\mathfrak{p},2}$  are unramified characters of  $G_{\mathfrak{p}}$ , and  $\chi_{\mathfrak{p},2}(\text{Frob}_{\mathfrak{p}})$  is a unit  $U_{\mathfrak{p}}$ -eigenvalue of the  $p$ -stabilised newform of  $f_2$ .

**THEOREM 9** *Let  $f_2$  be a holomorphic cuspidal Hilbert eigenform of weight  $2 \leq k_2 \leq p$  and of level prime to  $p$  as above. Let  $k_1 \stackrel{\text{def}}{=} p$  if  $k_2 = p$  and  $k_1 \stackrel{\text{def}}{=} p+1-k_2$  if  $2 \leq k_2 < p$ . Suppose that*

- if  $p > 2$ , the associated mod  $p$  representation  $\overline{\rho}_2 : \text{Gal}(\overline{F}/F) \rightarrow \text{GL}_2(\overline{\mathbf{F}}_p)$  is absolutely irreducible when restricted to  $\text{Gal}(\overline{F}/F(\zeta_p))$ , and if  $p = 2$ ,  $\overline{\rho}_2 : \text{Gal}(\overline{F}/F) \rightarrow \text{GL}_2(\overline{\mathbf{F}}_p)$  has insoluble image;
- if  $p > 2$  and if  $\epsilon^{k_1-2} \overline{\chi}_{\mathfrak{p},2} \neq \overline{\chi}_{\mathfrak{p},1}$ , the ramification index of  $F_{\mathfrak{p}}$  is strictly less than  $p - 1$  for every prime  $\mathfrak{p}$  of  $F$  above  $p$ , and if  $p = 2$ ,  $\overline{\rho}_2$  is unramified at every prime of  $F$  above 2;

- if  $p > 2$ ,  $\bar{\epsilon}^{k_2-1}\bar{\chi}_{\mathfrak{p},1} \neq \bar{\chi}_{\mathfrak{p},2}$  and if  $p = 2$ ,  $\bar{\chi}_{\mathfrak{p},2} \neq \bar{\chi}_{\mathfrak{p},1}$
- $\bar{\rho}_2$  is the direct sum of the characters  $\bar{\epsilon}^{k_2-1}\bar{\chi}_{\mathfrak{p},1}$  and  $\bar{\chi}_{\mathfrak{p},2}$  at every prime  $\mathfrak{p}$  of  $F$  above  $p$ .

Then there exists a holomorphic cuspidal Hilbert eigenform of weight  $2 \leq k_1 \leq p$  and of level prime to  $p$  with its associated mod  $p$  representation  $\rho_1 : \text{Gal}(\bar{F}/F) \rightarrow \text{GL}_2(\bar{\mathbf{F}}_p)$  satisfying  $\bar{\rho}_1 \simeq \bar{\rho}_2 \otimes \bar{\epsilon}^{k_1-1}$  if  $p > 2$  and  $\bar{\rho}_1 \simeq \bar{\rho}_2$  if  $p = 2$ , and the  $U_{\mathfrak{p}}$ -eigenvalue of the  $p$ -stabilised new form is a lifting of  $\bar{\chi}_{\mathfrak{p},1}(\text{Frob}_{\mathfrak{p}})$ .

*Proof.* For  $p > 2$ , this is a result of Gee (Theorem 2.1 [13]). Let  $p = 2$ ; thus  $k_1 = k_2 = 2$ . For clarity, let  $\bar{\rho}$  denote  $\bar{\rho}_2 \otimes \bar{\epsilon}$  where  $\bar{\epsilon}$  is the mod 4 cyclotomic character. Clearly the twist of  $\rho_2$  by the Teichmüller lift of  $\bar{\epsilon}$  defines a modular lifting of  $\bar{\rho}$  potentially ordinary and potentially Barsotti-Tate at  $p$ . By class field theory, find a finite totally real soluble extension  $F_{\Sigma} \subset \bar{F}$  of  $F$  of even degree in which 2 remains split completely, and satisfies the following conditions:

- there exists a quaternion algebra  $D$  over  $F_{\Sigma}$  ramified exactly at a finite set  $\Sigma$  of finite primes of  $F_{\Sigma}$  not dividing 2;
- $\bar{\rho}|_{\text{Gal}(\bar{F}/F_{\Sigma})}$  is ramified exactly at  $\Sigma$  and the infinite places, and, in particular, for every prime  $\mathfrak{q} \in \Sigma$ ,  $\bar{\rho}|_{\text{Gal}(\bar{F}/F_{\Sigma})}$  at  $\mathfrak{q}$  is an extension of an unramified character by the twist of the character by  $\epsilon$  at  $\mathfrak{q}$ ;
- there exists a maximal open compact subgroup  $U \subset (D \otimes_{F_{\Sigma}} \mathbf{A}_{F_{\Sigma}}^{\infty})^{\times}$  such that  $U_{\mathfrak{q}} = \text{GL}_2(\mathcal{O}_{F_{\Sigma}\mathfrak{q}})$  for  $\mathfrak{q} \notin S^D$  and  $U_{\mathfrak{q}} = \mathcal{O}_{D\mathfrak{q}}^{\times}$  for  $\mathfrak{q} \in S^D$ , and a holomorphic cuspidal automorphic representation  $\pi_2$  of  $(D \otimes_{F_{\Sigma}} \mathbf{A}_{F_{\Sigma}})^{\times}$  with central character  $\psi$  such that  $\bar{\rho}|_{\text{Gal}(\bar{F}/F_{\Sigma})} \simeq \bar{\rho}_{\pi_2} : \text{Gal}(\bar{F}/F_{\Sigma}) \rightarrow \text{GL}_2(\bar{\mathbf{F}}_p)$  and  $\det \bar{\rho}|_{\text{Gal}(\bar{F}/F_{\Sigma})} = \bar{\psi}\bar{\epsilon}$  and such that  $\pi$  is unramified at every prime of  $F_{\Sigma}$  above 2.

It then follows from work of Khare-Wintenberger (See Corollary 4.7 and Theorem 10.1 in [18]) that there is a lifting  $\rho : \text{Gal}(\bar{F}/F_{\Sigma}) \rightarrow \text{GL}_2(\bar{\mathbf{Q}}_p)$  of  $\bar{\rho}|_{\text{Gal}(\bar{F}/F_{\Sigma})}$ , unramified outside  $S_{\Sigma,p} \amalg \Sigma \amalg S_{\Sigma,\infty}$  with  $\det \rho = \psi\epsilon$  such that, for every prime  $\mathfrak{p}$  of  $F_{\Sigma}$  above 2,  $\rho$  is ordinary at  $\mathfrak{p}$  and Barsotti-Tate and is of the form

$$\begin{pmatrix} \epsilon\tilde{\chi}_{\mathfrak{p},2} & * \\ 0 & \tilde{\chi}_{\mathfrak{p},1} \end{pmatrix}$$

where  $\tilde{\chi}_{\mathfrak{p},1}$  and  $\tilde{\chi}_{\mathfrak{p},2}$  are unramified liftings of  $\bar{\chi}_{\mathfrak{p},1}|_{\text{Gal}(\bar{F}/F_{\Sigma})}$  and  $\bar{\chi}_{\mathfrak{p},2}|_{\text{Gal}(\bar{F}/F_{\Sigma})}$  respectively. It then follows from the main theorem of Kisin [19] and Khare-Wintenberger [18], and by soluble descent that there exists a holomorphic cuspidal Hilbert eigenform  $f_1$  of weight  $k_1 = 2$  and of level prime to 2 such that  $\rho_{f_1}|_{\text{Gal}(\bar{F}/F_{\Sigma})} \simeq \rho$ .  $\square$



7  $\Lambda$ -ADIC COMPANION FORMS

**THEOREM 10** *Let  $p$  be a rational prime. Let  $F$  be a totally real field. Suppose that  $p$  splits completely in  $F$ . Let  $K$  be a finite extension of  $\mathbf{Q}_p$  with ring of integers  $\mathcal{O}$  and residue field  $k = \mathcal{O}/\mathfrak{m}$ . Suppose that*

$$\rho : \text{Gal}(\overline{F}/F) \rightarrow \text{GL}_2(\mathcal{O})$$

*is a continuous representation satisfying*

- $\rho$  ramifies at only finite many primes;
- $\overline{\rho} = (\rho \bmod \mathfrak{m})$  is absolutely irreducible when restricted to  $\text{Gal}(\overline{F}/F(\zeta_p))$ , and has a modular lifting which is potentially ordinary and potentially Barsotti-Tate at every prime of  $F$  above  $p$ ;
- for every prime ideal  $\mathfrak{p}$  of  $F$  above  $p$ , the restriction  $\rho|_{G_{\mathfrak{p}}}$  to the decomposition group  $G_{\mathfrak{p}}$  at  $\mathfrak{p}$  is the direct sum of characters  $\chi_{\mathfrak{p},1}$  and  $\chi_{\mathfrak{p},2} : G_{\mathfrak{p}} \rightarrow \mathcal{O}^{\times}$  such that the images of the inertia subgroup at  $\mathfrak{p}$  are finite and  $(\chi_{\mathfrak{p},1} \bmod \mathfrak{m}) \neq (\chi_{\mathfrak{p},2} \bmod \mathfrak{m})$ ;

*If  $p = 2$ , assume furthermore that*

- the image of the complex conjugation with respect to every embedding of  $F$  into  $\mathbf{R}$  is not the identity matrix;
- $\overline{\rho}$  has insoluble image;
- for every  $\mathfrak{p}$  of  $F$  above  $p$ ,  $\rho$  is unramified at  $\mathfrak{p}$ .

*Then there is a finite totally real soluble extension  $F_{\Sigma} \subset \overline{F}$  of  $F$  in which  $p$  splits completely; a finite set  $\Sigma$  of finite places of  $F_{\Sigma}$  (at which  $\rho|_{G_{\Sigma}}$ , where  $G_{\Sigma} \stackrel{\text{def}}{=} \text{Gal}(\overline{F}/F_{\Sigma})$  is ramified of conductor  $\mathfrak{n}_{\Sigma}$ ); an ideal  $\mathfrak{n}$  of  $\mathcal{O}_{F_{\Sigma}}$  coprime to  $p$  which  $\mathfrak{n}_{\Sigma}$  divides; and, for every subset  $P$  of the set  $S_{\Sigma,p}$  of places of  $F_{\Sigma}$  above  $p$ ,*

1. a character

$$\chi_P : G_{\Sigma} \rightarrow \mathcal{O}^{\times}$$

*of finite order, unramified outside a finite set of places containing  $S_{\Sigma,p}$ , such that the restriction to the inertia subgroup of  $G_{\Sigma}$  at  $\mathfrak{p}$  of  $\chi_P$  equals that of  $\chi_{\mathfrak{p},1}$  (resp.  $\chi_{\mathfrak{p},2}$ ) for all  $\mathfrak{p}$  in  $P$  (resp.  $S_{\Sigma,p} - P$ );*

2. a finite extension  $L$  of  $\text{Frac } \Lambda_K$  and a  $\Lambda$ -adic form

$$F_{\text{Hida},P} : h_{\mathcal{O}}^0(\mathfrak{n}_{\Sigma}) \rightarrow L;$$

3. a homomorphism  $f_P : h_{\mathcal{O}}^0(\mathfrak{n}) \rightarrow \mathcal{O}$  if  $p > 2$  while  $f_P : h_{\mathcal{O}}^0(\mathfrak{n})_{-} \rightarrow \mathcal{O}$  if  $p = 2$  satisfying

- $f_P(T_q) = \text{tr } \rho(\text{Frob}_q)/\chi_P(\text{Frob}_q)$  for all  $q$  not dividing  $\mathfrak{n}p$ ;
- $f_P(\mathbf{N}qS_q) = \det \rho(\text{Frob}_q)/\chi_P^2(\text{Frob}_q)$  for all  $q$  not dividing  $\mathfrak{n}p$ ;
- $f_P(U_q) = 0$  for  $q$  dividing  $\mathfrak{n}$  but not dividing  $p$ ;
- $f_P(U_{\mathfrak{p}}) = (\chi_{\mathfrak{p},1}/\chi_P)(\text{Frob}_{\mathfrak{p}})$  for every  $\mathfrak{p}$  in  $P$  and  $f_P(U_{\mathfrak{p}}) = (\chi_{\mathfrak{p},2}/\chi_P)(\text{Frob}_{\mathfrak{p}})$  for every  $\mathfrak{p}$  in  $S_{\Sigma,P} - P$ .

*Proof.* Choose a finite soluble totally real extension  $F_{\Sigma}$  of  $F$  in which  $p$  splits completely such that the restriction of  $\rho$  is absolutely irreducible when restricted to  $\text{Gal}(\overline{F}/F_{\Sigma}(\zeta_p))$ , unramified outside a finite set  $\Sigma \amalg S_{\Sigma,P} \amalg S_{\Sigma,\infty}$  of finite places  $q$  of  $F_{\Sigma}$  such that  $\rho|_{G_{\Sigma}}$  is of conductor 1 or  $q$  at  $q$ , and arises from—by Jacquet-Langlands—a cuspidal automorphic representation, *nearly* ordinary at every  $\mathfrak{p} \in S_{\Sigma,P}$  and special at  $q \in \Sigma$ , of the quaternion algebra  $D_{\Sigma}$  over  $F_{\Sigma}$  as in the previous section.

For every  $P \subseteq S_{\Sigma,P}$ , it follows from class field theory that one can choose  $\chi_P$ , of conductor 1 away from a finite set of places containing the set of places above  $p$ , as asserted in the theorem.

Let  $\rho_P$  denote  $\rho \otimes_{\text{Gal}(\overline{F}/F_{\Sigma})} \chi_P^{-1}$  and  $\overline{\rho}_P$  denote  $(\rho_P \bmod \mathfrak{m})$ . If we let  $\rho_{\Sigma}$  denote the modular lifting of  $\overline{\rho}$ , then  $\rho_{\Sigma} \otimes \chi_P^{-1}$  is a modular lifting of  $\overline{\rho}_P$ ; in fact it is ordinary at every  $\mathfrak{p} \in S_{\Sigma,P}$  by Jarvis’ level lowering results [15]—by which one shows  $\rho_{\Sigma} \otimes \chi_P^{-1}$  is crystalline at  $\mathfrak{p}$ —followed by Fontaine-Laffaille theory. Let  $\mathfrak{m}_P$  denote the corresponding maximal ideal of  $eh_2(\mathfrak{n}_{\Sigma}p)_{\mathcal{O}}$  if  $p > 2$  and  $eh_2(\mathfrak{n}_{\Sigma}4)_{\mathcal{O},-}$  if  $p = 2$ . It then follows from Hida theory [14] and results from preceding sections that there exists a finite extension  $L$  in an algebraic closure of  $\text{Frac } \Lambda_K$  which we fix; and, for every  $P \subseteq S_{\Sigma,P}$ , a  $\Lambda$ -adic eigenform  $F_{H,P} : h_{\mathcal{O}}^0(\mathfrak{n}_{\Sigma}) \rightarrow h_{\mathcal{O}}^0(\mathfrak{n}_{\Sigma})_{\mathfrak{m}_P} \rightarrow \mathcal{O}_L$ , and a height one prime  $\wp_P$  of  $\mathcal{O}_L$  such that

- $(\rho_{F_H} \bmod \wp_P) \sim \rho_P$
- for every distinct subsets  $P$  and  $Q$ ,  $(F_{H,P}, \wp_P)$  and  $(F_{H,Q}, \wp_Q)$  are in companion; more precisely, for every  $q$  not dividing  $\mathfrak{n}_{\Sigma}p$ ,  $(F_{H,Q}(T_q) \bmod \wp_Q) = (F_{H,P}(T_q S_{(P-(P \cap Q)) \cup (Q-(P \cap Q))}(q)^{-1}) \bmod \wp_P)$ ; for  $\mathfrak{p}$  in  $(P \cap Q) \cup ((S_{\Sigma,P} - P) \cap (S_{\Sigma,P} - Q))$ ,  $(F_{H,Q}(U_{\mathfrak{p}}) \bmod \wp_Q) = (F_{H,P}(U_{\mathfrak{p}}) \bmod \wp_P)$ , while for  $\mathfrak{p}$  in  $(P \cap (S_{\Sigma,P} - Q)) \cup ((S_{\Sigma,P} - P) \cap Q)$ ,  $(F_{H,Q}(U_{\mathfrak{p}}) \bmod \wp_Q) = (F_{H,P}(U_{\mathfrak{p}}^{-1} S^P(\mathfrak{p})) \bmod \wp_P)$ .

Let  $f_P$  be the composite  $h_{\mathcal{O}}^0(\mathfrak{n}_{\Sigma}) \rightarrow \mathcal{O}_L \rightarrow \mathcal{O}_L/\wp_P \simeq \mathcal{O}$ ; if the image of  $U_q$  for  $q$  dividing  $\mathfrak{n}_{\Sigma}$  but not dividing  $p$  is not zero, we may increase the level at  $q$  if necessary to assume the image of indeed zero (See [34] for example) .  $\square$

8 MODELS OF HILBERT MODULAR VARIETIES

Let  $F$  be a totally real field— $F_{\Sigma}$  in the preceding section—of degree  $d = [F : \mathbf{Q}]$  with ring of integers  $\mathcal{O}_F$ . Fix a rational prime  $p$  and an ideal  $\mathfrak{n}$  of  $\mathcal{O}_F$  prime to  $p$ . For every integer  $r \geq 1$ , fix a  $p^r$ -th primitive root  $\zeta_{p^r}$  of unity. For a prime

$\mathfrak{p}$  of  $F$  above  $p$ , let  $F_{\mathfrak{p}}$  denote the completion of  $F$  with respect to the absolute value corresponding to  $\mathfrak{p}$ ,  $k_{\mathfrak{p}}$  the residue field of  $F_{\mathfrak{p}}$ ,  $f_{\mathfrak{p}}$  the residue class degree, and  $e_{\mathfrak{p}}$  the ramification index.

Fix embeddings  $\mathbf{Q} \rightarrow \overline{\mathbf{Q}} \rightarrow \overline{\mathbf{Q}}_p$ . Let  $K$  denote a finite extension of  $\mathbf{Q}_p$  which contains the image of  $F$  by every embedding of  $F$  into  $\overline{\mathbf{Q}}_p$ ; and let  $\mathcal{O}$  denote its ring of integers and  $k$  denote the residue field.

For a fractional ideal  $I$  of  $F$  canonically ordered, let  $I^+$  denote the totally positive elements. Fix a set  $T$  of representatives in  $\mathbf{A}_F^{\times}$  of the strict ideal class group  $\mathbf{A}_F^{\times}/(F^{\times}(\mathcal{O}_F \otimes \mathbf{Z}^{\wedge})^{\times} F_{\infty}^{+\times})$ , and we shall let  $t$  also mean the fractional ideal  $t\mathfrak{d}$  corresponding to a representative  $t$  in  $T$ .

DEFINITION. A  $t$ -polarised Hilbert-Blumenthal abelian variety (henceforth abbreviated as HBAV) with level  $\Gamma_1(\mathfrak{n})$ -structure over a  $\mathcal{O}$ -scheme  $S$  is an abelian variety  $A$  over  $S$  of relative dimension  $d$  together with

- $i : \mathcal{O}_F \rightarrow \text{End}(A/S)$ ;
- a homomorphism  $\lambda : (t, t^+) \rightarrow (\text{Sym}(A/S), \text{Pol}(A/S))$  of ordered invertible  $\mathcal{O}_F$ -modules, where  $\text{Sym}(A/S)$  (resp.  $\text{Pol}(A/S)$ ) denotes the invertible  $\mathcal{O}_F$ -module (via  $i$ ) of symmetric homomorphisms (resp. polarisations), such that  $A \otimes_{\mathcal{O}_F} t \rightarrow A^{\vee}$ , induced by  $\lambda$ , is an isomorphism of HBAVs—it is shown in [41] that this is equivalent to the condition that there exists a prime-to- $p$  polarisation  $A \rightarrow A^{\vee}$ ; and to the ‘determinant condition’ on  $\text{Lie}(A)$  in the sense of Kottwitz;
- an  $\mathcal{O}_F/\mathfrak{n}$ -module morphism  $\eta : (\mathcal{O}_F/\mathfrak{n})^{\vee} = (GL_1 \otimes \mathfrak{d}_F^{-1})[\mathfrak{n}] \rightarrow A[\mathfrak{n}]$ .

DEFINITION. Let  $Y_{\Gamma_1(\mathfrak{n}, t)}$  (resp.  $Y_{\Gamma_1(\mathfrak{n}, t) \cap I_w}$ ) denote the  $\mathcal{O}$ -scheme representing the functor which sends an  $\mathcal{O}$ -scheme  $S$  to the set of isomorphism classes  $(A, i, \lambda, \eta)$  (resp.  $(A, i, \lambda, \eta, C)$ ) of  $t$ -polarised HBAVs with level  $\Gamma_1(\mathfrak{n})$ -level structure (resp. and a finite flat subgroup scheme  $C$  of  $A[p]$  with compatible  $\mathcal{O}_F$ -action locally free of rank  $\sum_{\mathfrak{p}} |\mathcal{O}_F/\mathfrak{p}|$ ).

It follows from [27] and [8] that if  $\mathfrak{n}$  does not divide 2, nor 3,  $Y_{\Gamma_1(\mathfrak{n}, t)}$  is a smooth scheme over  $\mathcal{O}$  of relative dimension  $[F : \mathbf{Q}]$ . If  $\mathfrak{n}$  does divide 2, or 3, we let  $Y_{\Gamma_1(\mathfrak{n}, t)}$  denote the  $\mathcal{O}$ -scheme

$$(\Gamma_1(\mathfrak{n}, t)/\Gamma_1(\mathfrak{m}, t)) \backslash Y_{\Gamma_1(\mathfrak{m}, t)}$$

for an auxiliary ideal  $\mathfrak{m}$  of  $\mathcal{O}_F$  such that  $\mathfrak{n}|\mathfrak{m}$  and  $\Gamma_1(\mathfrak{m})$  small enough, i.e., torsion-free.

Let  $\overline{Y}_{\Gamma_1(\mathfrak{n}, t)}$  denote the fibre over  $k$  of  $Y_{\Gamma_1(\mathfrak{n}, t)}$ ; and let  $\overline{Y}_{\Gamma_1(\mathfrak{n}, t) \cap I_w}$  denote the fibre over  $k$  of  $Y_{\Gamma_1(\mathfrak{n}, t) \cap I_w}$ .

It is a well-known result of Deligne-Ribet that the fibre  $\overline{Y}_{\Gamma_1(\mathfrak{n}, t)}$  is irreducible (Corollary 4.6 in [9]). It is a result of local model theory by Pappas that  $\overline{Y}_{\Gamma_1(\mathfrak{n}, t) \cap I_w}$  is normal (Corollary 2.2.3 in [25]).

Suppose that  $p$  splits completely in  $F$ . In which case, the  $p$ -divisible group of a HBAV over the ring of integers of a finite extension of  $\mathbf{Q}_p$  decomposes as the product of  $[F : \mathbf{Q}]$  one-dimensional  $p$ -divisible groups, one for each prime  $\mathfrak{p}$  of  $F$  above  $p$ , and this allows us to define ‘Katz-Mazur-Drinfeld’ higher level structures at  $p$  by defining level structures at  $\mathfrak{p}$  on the ‘ $\mathfrak{p}$ -divisible group’ for each  $\mathfrak{p}$ .

DEFINITION. Let  $r$  be an integer  $\geq 1$ . Define  $Y_{\Gamma_1(\mathbf{n},t)\cap\Gamma_1(p^r)}$  to be the  $\mathcal{O}$ -scheme representing the functor which sends an  $\mathcal{O}$ -scheme  $S$  to the set of isomorphism classes of the sextuples  $(A, i, \lambda, \eta, C, \eta_{\text{KM}})$  over  $S$  where  $(A, i, \lambda, \eta)$  is a  $t$ -polarised HBAV over  $S$  with  $\Gamma_1(\mathbf{n})$ -level structure,  $C$  is a finite flat subgroup scheme of  $A[p^r]$  locally free of finite rank  $|\mathcal{O}_F/p^r| = \sum_{\mathfrak{p}} |\mathcal{O}_F/\mathfrak{p}^r|$  with compatible action of  $\mathcal{O}_F$ , and an  $\mathcal{O}_F$ -linear group homomorphism  $\eta_{\text{KM}} : \mathcal{O}_F/p^r \rightarrow \text{Mor}(S, C) \subset \text{Mor}(S, A[p^r])$  such that the image of  $\eta_{\text{KM}}$  defines a ‘full set of sections’ in the sense of Katz-Mazur [17] (See 1.10.5 and 1.10.10 in [17]).

DEFINITION. For every prime  $\mathfrak{p}$  of  $F$  above  $p$ , let  $Y_{\Gamma_1(\mathbf{n},t)\cap\Gamma_1(p^r), I_{w_{\mathfrak{p}}}, K}$  denote the fine moduli space over  $K$  of the septuples  $(A, i, \lambda, \eta, C, \eta_{\text{KM}}, D_{\mathfrak{p}})$  where the sextuple  $(A, i, \lambda, \eta, C, \eta_{\text{KM}})$  defines a point of  $Y_{\Gamma_1(\mathbf{n},t)\cap\Gamma_1(p^r)} \times_{\text{Spec } \mathcal{O}_K} \text{Spec } K$ , and  $D_{\mathfrak{p}}$  is finite flat subgroup scheme of  $A[\mathfrak{p}]$  of rank  $|\mathcal{O}_F/\mathfrak{p}|$  which has trivial intersection with  $C$ .

9 COMPACTIFICATION

By an *unramified* cusp  $C$  of  $Y_{\Gamma_1(\mathbf{n},t)}$  over  $R$ , we shall mean a pair of fractional ideals  $M_1, M_2$  of  $F$  such that  $M_1 M_2^{-1} \simeq t$  which comes equipped with

- an  $\mathcal{O}_F \otimes_{\mathbf{Z}} R$ -linear isomorphism  $\lambda : M_1^{-1} \otimes_{\mathbf{Z}} R \simeq \mathcal{O}_F \otimes_{\mathbf{Z}} R$ ;
- an  $\mathcal{O}_F$ -linear embedding  $\eta : \mathcal{O}_F/\mathfrak{n} \rightarrow \mathfrak{n}^{-1} M_2^{-1} / M_2^{-1}$ .

For brevity, let  $M = M_1 M_2$ ,  $M^{\vee} = \text{Hom}_{\mathbf{Z}}(M, \mathbf{Z}) = \text{Hom}_{\mathcal{O}_F}(M, \mathfrak{d}_F^{-1}) \simeq \mathfrak{c} M_2^{-1} \mathfrak{d}_F^{-1}$ , and  $M^{\vee+} \subset M^{\vee}$  of the totally positive elements in  $(\mathfrak{c} M_2^{-1} \mathfrak{d}_F^{-1})^+$ . Choose a rational polyhedral cone decomposition  $\Sigma_C$  of  $(M^{\vee+} \otimes_{\mathbf{Z}} \mathbf{R}) \cup \{0\}$ . For a cone  $\sigma \subset (M^{\vee+} \otimes_{\mathbf{Z}} \mathbf{R})$ , we let  $\sigma^{\vee} \subset M \otimes_{\mathbf{Z}} \mathbf{R}$  denote the dual cone.

Let  $S_{\mathfrak{n}} = \text{Spec } \mathbf{Z}[q^{\mathfrak{n}^{-1}M}]$  and  $S_{\mathfrak{n}} \hookrightarrow S_{\mathfrak{n},\sigma}$  denote the affine torus embedding (see Theorem 2.5 in [6]) corresponding to the cone  $\sigma$  and let  $S_{\mathfrak{n},\sigma}^{\wedge} = \text{Spf } \mathbf{Z}[[q^{\mathfrak{n}^{-1}M \cap \sigma^{\vee}}]]$  denote the formal completion of  $S_{\mathfrak{n},\sigma}$  along the boundary  $S_{\mathfrak{n},\sigma}^{\infty} \stackrel{\text{def}}{=} S_{\mathfrak{n},\sigma} - S_{\mathfrak{n}}$ .

Let  $T_{\mathfrak{n},\sigma} = \text{Spec } \mathbf{Z}[[q^{\mathfrak{n}^{-1}M \cap \sigma^{\vee}}]]$  and  $T_{\mathfrak{n},\sigma}^0 = T_{\mathfrak{n},\sigma} - S_{\mathfrak{n},\sigma}^{\infty} = \text{Spec } \mathbf{Z}[[q^{\mathfrak{n}^{-1}M \cap \sigma^{\vee}}, q^{-\mathfrak{n}^{-1}M \cap \sigma^{\vee}}]]$ . The henselisation of  $(S_{\mathfrak{n},\sigma}, S_{\mathfrak{n},\sigma}^{\infty})$  projects onto an affine étale scheme  $U_{\mathfrak{n},\sigma}$  over  $S_{\mathfrak{n},\sigma}$  which approximates  $S_{\mathfrak{n},\sigma}^{\wedge}$  in the sense of Artin, and let  $U_{\mathfrak{n},\sigma}^0 = U_{\mathfrak{n},\sigma} \times_{T_{\mathfrak{n},\sigma}} T_{\mathfrak{n},\sigma}^0$ .

The Mumford construction applied to the  $\mathcal{O}_F$ -linear ‘period’ map  $q : M_2 \rightarrow GL_1(U_{\mathfrak{n},\sigma}) \otimes_{\mathbf{Z}} \mathfrak{d}_F^{-1} M_1^{-1}$  gives rise to a semi-abelian scheme

$$\text{Tate}_{M_1, M_2}(q) \stackrel{\text{def}}{=} (GL_1 \otimes_{\mathbf{Z}} \mathfrak{d}^{-1} M_1^{-1})/q^{M_2}$$

over the complete ring  $U_{\mathfrak{n},\sigma}$  with action of  $\mathcal{O}_F$ , whose pull-back, which we shall denote by  $\text{Tate}_{M_1, M_2}^0(q)$  to  $U_{\mathfrak{n},\sigma}^0$ , is naturally a HBAV,  $t$ -polarised

$$\begin{aligned} \text{Tate}_{M_1, M_2}(q) \otimes_{\mathcal{O}_F} M_1 M_2^{-1} &\simeq (GL_1 \otimes_{\mathbf{Z}} \mathfrak{d}^{-1} M_2^{-1})/q^{M_1} \\ &\parallel \\ &\text{Tate}_{M_2, M_1}(q) \simeq \text{Tate}_{M_1, M_2}(q)^\vee, \end{aligned}$$

with level  $\Gamma_1(\mathfrak{n})$ -structure, and which gives rise to a map

$$U_{\mathfrak{n},\sigma}^0 \times_{\text{Spec } \mathbf{Z}} \text{Spec } \mathcal{O} \rightarrow Y_{\Gamma_1(\mathfrak{n},t)}.$$

We glue  $\coprod_{T/\simeq} \coprod_{\sigma \in \Sigma_C} U_{\mathfrak{n},\sigma} \times_{\text{Spec } \mathbf{Z}} \text{Spec } \mathcal{O}$  along the map to get a toroidal compactification  $X_{\Gamma_1(\mathfrak{n},t)}$  over  $\mathcal{O}$  of  $Y_{\Gamma_1(\mathfrak{n},t)}$  ([26]). Similarly, one can define a compactification  $X_{\Gamma_1(\mathfrak{n},t) \cap I_w}$  over  $\mathcal{O}$  of  $Y_{\Gamma_1(\mathfrak{n},t) \cap I_w}$  with its choice of a rational cone decomposition compatible with that of  $X_{\Gamma_1(\mathfrak{n},t)}$ .

Let

$$\text{Tate}_{M_1, M_2, S}^0(q) \stackrel{\text{def}}{=} \text{Tate}_{M_1, M_2}(q) \times_{\text{Spec } \mathbf{Z}[[q^M, q^{-M}]]} S$$

for a  $\mathbf{Z}[[q^M, q^{-M}]]$ -scheme  $S$ ; it is  $t$ -polarised. Let  $S$  be a  $\mathcal{O} \otimes_{\mathbf{Z}} \mathbf{Z}[[q^M, q^{-M}]]$ -scheme. Then there is a ‘connected-étale’ exact sequence

$$0 \rightarrow (GL_1 \otimes_{\mathbf{Z}} \mathfrak{d}^{-1} M_1^{-1})[p^r] \rightarrow \text{Tate}_{M_1, M_2, S}(q)[p^r] \rightarrow (1/p^r)M_2/M_2 \rightarrow 0$$

of  $(\mathcal{O}_F/p^r)$ -modules schemes over  $S$ .

LEMMA 11 *Fix an integer  $r \geq 1$ . Let  $S$  be a connected  $\mathcal{O} \otimes_{\mathbf{Z}} \mathbf{Z}[[q^M, q^{-M}]]$ -scheme. Suppose that  $C$  is an  $\mathcal{O}_F$ -stable finite flat subgroup scheme of  $\text{Tate}_{M_1, M_2, S}^0(q)[p^r]$  of order  $|\mathcal{O}_F/p^r|$ . Then for every  $\tau = \tau_{\mathfrak{p}}$ , there exists a unique pair of non-negative integers  $\rho_{\tau,1}, \rho_{\tau,2}$  such that  $\rho_{\tau,1} + \rho_{\tau,2} = r$  and such that*

$$C_{\mathfrak{p}} \cap (GL_1 \otimes_{\mathbf{Z}} \mathfrak{d}^{-1} M_1^{-1})[p^r] \simeq (GL_1 \otimes_{\mathbf{Z}} \mathfrak{d}^{-1} M_1^{-1})[\mathfrak{p}^{\rho_{\tau,1}}]$$

and the image of  $C_{\mathfrak{p}}$  in  $(1/p^r)M_2/M_2$  is isomorphic to  $\mathfrak{p}^{-\rho_{\tau,2}}M_2/M_2$ .

*Proof.* This is essentially Proposition 13.6.2 in [17].  $\square$

By a cusp of  $C$  of  $Y_{\Gamma_1(\mathfrak{n},t) \cap \Gamma_1(p^r)}$  over  $R$ , we shall mean a pair of fractional ideals  $M_1, M_2$  of  $F$  such that  $M_1 M_2^{-1} \simeq t$  which comes equipped with

- an  $\mathcal{O}_F \otimes_{\mathbf{Z}} R$ -linear isomorphism  $\lambda : M_1^{-1} \otimes_{\mathbf{Z}} R \simeq \mathcal{O}_F \otimes_{\mathbf{Z}} R$ ;
- an  $\mathcal{O}_F$ -linear embedding  $\eta : \mathcal{O}_F/\mathfrak{n} \rightarrow \mathfrak{n}^{-1} M_2^{-1}/M_2^{-1}$ .

- an  $\mathcal{O}_F$ -linear isomorphism  $\eta_{\text{KM}} : \mathcal{O}_F/p^r \simeq p^{-r}M_2/M_2$ .

Let  $M = M_1M_2$  as above. Fix an integer  $r \geq 1$ . Suppose that  $S$  is an  $\mathcal{O} \otimes_{\mathbf{Z}} \mathbf{Z}[\zeta_{p^r}][[q^{(1/p^r)M}, -(1/p^r)M]]$ -scheme.

DEFINITION Let  $\zeta_r$  denote the image of 1 by

$$\zeta_{\text{KM},r} : (\mathcal{O}_F/p^r) \simeq \mathfrak{d}^{-1}/p^r\mathfrak{d}^{-1} \simeq GL_1[p^r] \otimes_{\mathbf{Z}} \mathfrak{d}^{-1} \simeq (GL_1 \otimes_{\mathbf{Z}} \mathfrak{d}^{-1})[p^r]$$

and  $\zeta_{r,\tau}$  denote its  $\tau = \tau_{\mathfrak{p}}$  component. We often allow  $\zeta_r$  and  $\zeta_{r,\tau}$  to mean their images in  $(GL_1 \otimes_{\mathbf{Z}} \mathfrak{d}^{-1})(S)$  and  $\text{Tate}_{M_1,M_2,S}(q)(S)$ .

Let  $\eta_r^{\text{ét}}$  denote the image of 1 by

$$\eta_{\text{KM},r}^{\text{ét}} : (\mathcal{O}_F/p^r) \xrightarrow{\eta_{\text{KM}}} p^{-r}M_2/M_2 \xrightarrow{q} q^{p^{-r}M_2/M_2}$$

defining a point of  $\text{Tate}_{M_1,M_2}(q)(S)$  of exact order  $|\mathcal{O}_F/p^r|$ . Let  $\eta_{r,\tau}^{\text{ét}}$  denote its  $\tau = \tau_{\mathfrak{p}}$  component.

LEMMA 12 *Fix an integer  $r \geq 1$ . Let  $S$  be a connected  $\mathcal{O} \otimes_{\mathbf{Z}} \mathbf{Z}[[q^M, q^{-M}]]$ -scheme. Suppose that  $C$  is an  $\mathcal{O}_F$ -stable finite flat subgroup scheme of  $\text{Tate}_{M_1,M_2,S}^0(q)[p^r]$  of order  $|\mathcal{O}_F/p^r|$ . Suppose that  $C$  is of type  $\rho = (\rho_{\tau,1}, \rho_{\tau,2})_{\tau}$ . Let  $P_{\text{KM}} \in C(S)$  denote a point of exact order  $|\mathcal{O}_F/p^r|$ . Then for every  $\tau = \tau_{\mathfrak{p}}$ ,  $P_{\text{KM},\tau}$  is of the form  $\zeta_{r,\tau}^{\sigma_{\tau,1}} \eta_{r,\tau}^{\text{ét},\sigma_{\tau,2}}$  for a pair of integers  $0 \leq \sigma_{\tau,1} \leq \rho_{\tau,1}$  and  $0 \leq \sigma_{\tau,2} \leq \rho_{\tau,2}$  such that both  $\sigma_{\tau,1}$  and  $\sigma_{\tau,2}$  are coprime to  $p$ .*

*proof.* This is essentially 13.6.3 in [17].  $\square$

10 GENERIC FIBRES

With  $\mathfrak{n}$  fixed, for every integer  $r \geq 1$ , let  $\overline{\mathbf{U}}_r$  denote the quotient group of the totally positive units of  $F$  by the subgroup of elements which are squares of elements in  $\mathcal{O}_F$  which are congruent to 1 mod  $\mathfrak{n}p^r$ . If  $r = 0$ , we simply write  $\overline{\mathbf{U}}$ .

Let  $Y_{\Gamma_1(\mathfrak{n})}, X_{\Gamma_1(\mathfrak{n})}, Y_{\Gamma_1(\mathfrak{n}) \cap \text{Iw}}, X_{\Gamma_1(\mathfrak{n}) \cap \text{Iw}}, Y_{\Gamma_1(\mathfrak{n}) \cap \Gamma_1(p^r)}, Y_{\Gamma_1(\mathfrak{n}) \cap \Gamma_1(p^r), \text{Iw}_{\mathfrak{p}}, K}$  respectively denote the disjoint unions,  $t$  ranging over  $T$ , of  $Y_{\Gamma_1(\mathfrak{n},t)}, X_{\Gamma_1(\mathfrak{n},t)}, Y_{\Gamma_1(\mathfrak{n},t) \cap \text{Iw}}, X_{\Gamma_1(\mathfrak{n},t) \cap \text{Iw}}, Y_{\Gamma_1(\mathfrak{n},t) \cap \Gamma_1(p^r)}, Y_{\Gamma_1(\mathfrak{n},t) \cap \Gamma_1(p^r), \text{Iw}_{\mathfrak{p}}, K}$ .

Let  $X_{\Gamma_1(\mathfrak{n}),K}, X_{\Gamma_1(\mathfrak{n}) \cap \text{Iw},K}$  respectively denote the generic fibres over  $K$  of the  $\mathcal{O}_K$ -schemes  $X_{\Gamma_1(\mathfrak{n})}, X_{\Gamma_1(\mathfrak{n}) \cap \text{Iw}}$ .

Let  $X_{\Gamma_1(\mathfrak{n}) \cap \Gamma_1(p^r),K}, X_{\Gamma_1(\mathfrak{n}) \cap \Gamma_1(p^r), \text{Iw}_{\mathfrak{p}},K}$  respectively denote the toroidal compactifications of the  $K$ -schemes  $Y_{\Gamma_1(\mathfrak{n}) \cap \Gamma_1(p^r),K}, Y_{\Gamma_1(\mathfrak{n}) \cap \Gamma_1(p^r), \text{Iw}_{\mathfrak{p}},K}$ .

Let  $Y_{\Gamma_1(\mathfrak{n},t)}^{\wedge}, X_{\Gamma_1(\mathfrak{n},t)}^{\wedge}, Y_{\Gamma_1(\mathfrak{n},t) \cap \text{Iw}}^{\wedge}, X_{\Gamma_1(\mathfrak{n},t) \cap \text{Iw}}^{\wedge}$  respectively denote the formal completions of  $Y_{\Gamma_1(\mathfrak{n},t)}, X_{\Gamma_1(\mathfrak{n},t)}, Y_{\Gamma_1(\mathfrak{n},t) \cap \text{Iw}}, X_{\Gamma_1(\mathfrak{n},t) \cap \text{Iw}}$  along their closed  $k$ -fibres  $\overline{Y}_{\Gamma_1(\mathfrak{n},t)}, \overline{X}_{\Gamma_1(\mathfrak{n},t)}, \overline{Y}_{\Gamma_1(\mathfrak{n},t) \cap \text{Iw}}, \overline{X}_{\Gamma_1(\mathfrak{n},t) \cap \text{Iw}}$ . Let

$Y_{\Gamma_1(\mathfrak{n})}^\wedge, X_{\Gamma_1(\mathfrak{n})}^\wedge, Y_{\Gamma_1(\mathfrak{n})\cap I_w}^\wedge, X_{\Gamma_1(\mathfrak{n})\cap I_w}^\wedge$  denote their disjoint unions over  $T$ .

Finally, let  $Y_{\Gamma_1(\mathfrak{n})}^{\text{rig}}, X_{\Gamma_1(\mathfrak{n})}^{\text{rig}}, Y_{\Gamma_1(\mathfrak{n})\cap I_w}^{\text{rig}}, X_{\Gamma_1(\mathfrak{n})\cap I_w}^{\text{rig}}$  respectively denote the Raynaud rigid generic fibres of  $Y_{\Gamma_1(\mathfrak{n})}^\wedge, X_{\Gamma_1(\mathfrak{n})}^\wedge, Y_{\Gamma_1(\mathfrak{n})\cap I_w}^\wedge, X_{\Gamma_1(\mathfrak{n})\cap I_w}^\wedge$ .

11  $p$ -ADIC CLASSICAL HILBERT MODULAR FORMS

Suppose that  $(k = \sum_{\tau \in \text{Hom}(F,K)} k_\tau \tau, w = \sum_{\tau \in \text{Hom}(F,K)} w_\tau \tau) \in \mathbf{Z}^{\text{Hom}(F,K)} \times \mathbf{Z}^{\text{Hom}(F,K)}$  is such that  $w = 2w_\tau - k_\tau$  is independent of  $\tau$  (this is Taylor’s  $\mu$  in [38]).

For  $S \in \{Y_{\Gamma_1(\mathfrak{n}),K}, Y_{\Gamma_1(\mathfrak{n})\cap I_w,K}, Y_{\Gamma_1(\mathfrak{n})\cap \Gamma_1(p^r),K}\}$ , let  $\text{Lie}^\vee(A/S)$  (resp.  $H_{\text{dR}}^1(A/S)$ ) denote the pull-back by the identity section of the sheaf of relative differentials of the universal HBAV  $A$  over  $S$  (resp. the higher direct image of the relative de Rham complex). By the decomposition,

$$\mathcal{O}_F \otimes_{\mathbf{Z}} \mathcal{O} \simeq \prod_{\tau \in \text{Hom}(F,K)} \mathcal{O}_\tau$$

where  $\mathcal{O}_\tau$  is  $\mathcal{O}$  into which  $F$  embeds by  $\tau$ , we have

$$\text{Lie}^\vee(A/S) = \bigoplus_{\tau \in \text{Hom}(F,K)} \text{Lie}^\vee(A/S)_\tau, \quad H_{\text{dR}}^1(A/S) = \bigoplus_{\tau \in \text{Hom}(F,K)} H_{\text{dR}}^1(A/S)_\tau$$

where  $\text{Lie}^\vee(A/S)$  and  $H_{\text{dR}}^1(A/S)$  are locally free sheaves of  $\mathcal{O}_S$ -modules of rank 1 and 2 respectively. Following Hida [14], let

$$L_{(k,w)} = \bigotimes_{\tau \in \text{Hom}(F,K)} \left( \bigwedge H_{\text{dR}}^1(A/S)_\tau \right)^{\otimes w/2} \otimes_{\mathcal{O}_S} (\text{Lie}^\vee(A/S))_\tau^{\otimes k_\tau}$$

If  $k$  is parallel, more precisely, if  $(k, w) = ((k, \dots, k), (k/2, \dots, k/2))$ , we will often write  $L_k$  for  $L_{(k,w)}$ . We shall also let  $L_{(k,w)}$  denote its extension to the compactification.

Let  $\pi_1$  (resp.  $\pi_{2,\mathfrak{p}}$ ) denote the degeneracy map

$$X_{\Gamma_1(\mathfrak{n})\cap \Gamma_1(p^r), I_w, K} \longrightarrow X_{\Gamma_1(\mathfrak{n})\cap \Gamma_1(p^r), K}$$

defined, on the non-cuspdail points, by

$$(A, i, \lambda, \eta, C, \eta_{\text{KM}}, D_{\mathfrak{p}}) \mapsto (A, i, \lambda, \eta, C, \eta_{\text{KM}})$$

(resp.  $(A/D_{\mathfrak{p}}, (i \bmod D_{\mathfrak{p}}), (\lambda \bmod D_{\mathfrak{p}}), (\eta \bmod D_{\mathfrak{p}}), (\eta_{\text{KM}} \bmod D_{\mathfrak{p}}))$ ).

## 12 CANONICAL SUBGROUPS FOR ONE-DIMENSIONAL FORMAL GROUPS

Let  $L$  be a finite extension of  $K$ , and let  $\text{val}_L$  be a valuation on  $L$  normalised so that  $\text{val}_L(p) = 1$ . Let  $G$  be a one-dimensional principally polarised  $p$ -divisible/Barsotti-Tate group over  $\mathcal{O}_L$ .

DEFINITION. The identity component  $G^\wedge$  of  $G$  is a one-dimensional formal group, and define  $\text{Ha}(G)$  to be  $\text{val}_L(a)$  for  $a$  as defined in Proposition 3.6.6, [16] (see also [29]).

By definition,  $G$  is ordinary if and only if  $\text{Ha}(G) = 0$ .

Let  $C$  be a finite flat subgroup scheme of  $G[p]$  of order  $p$ .

DEFINITION. Define  $\deg(G, C)$  to be  $1 - \text{val}_L(\text{Ann}(\text{coker}(\text{Lie}^\vee(G/C) \rightarrow \text{Lie}^\vee(G))))$ .

It follows immediately from the definition that  $\deg(G, C) + \deg(G/C, G[p]/C) = 1$ .

Suppose that  $\deg(G, C) < p/(p+1)$ . Then there exists a canonical subgroup  $H(G)$  of  $G$ . If  $C = H(G)$ , then  $\deg(G, C) = \text{Ha}(G)$ . To see this, note that  $H(G)(L)$  consists of 0 and  $p-1$  points  $P$  of the formal group  $G^\wedge$  of valuation  $(1 - \text{Ha}(G))/(p-1)$  (Theorem 3.10.7, [16]). Since  $\deg(G, C) = 1 - \prod_P \text{val}(P)$  (Lemma 1.3 [24]),  $\deg(G, C) = \text{Ha}(G)$ .

LEMMA 13 *Let  $r$  be a rational number  $< p/(p+1)$ . Suppose that  $G$  is not ordinary. Then*

$$\{(G, C) \mid \text{Ha}(G) \leq r\}$$

*divides into two disjoint subsets, namely*

$$\{(G, C) \mid C = H(G) \text{ and } \deg(G, C) \in (0, r]\}$$

*and*

$$\{(G, C) \mid C \neq H(G) \text{ and } \deg(G, C) \in [1 - r/p, 1)\}.$$

*On the other hand,*

$$\{(G, C) \mid \text{Ha}(G/C) \leq r\}$$

*divides into two disjoint subsets, namely*

$$\begin{aligned} & \{(G, C) \mid \deg(G, C) \in (0, r/p], C = H(G), \text{ and } \text{Ha}(G) < 1/(p+1)\} \\ \cup & \{(G, C) \mid \deg(G, C) \in (0, r/p], C \neq H(G), \text{ and } \text{Ha}(G) < p/(p+1)\} \end{aligned}$$

*and*

$$\begin{aligned} & \{(G, C) \mid \deg(G, C) \in [1 - r, 1), C = H(G), 1/(p+1) < \text{Ha}(G) < p/(p+1)\} \\ \cup & \{(G, C) \mid \deg(G, C) \in [1 - r, 1), C \neq H(G), \text{Ha}(G) \geq p/(p+1)\}. \end{aligned}$$



*Proof.* This follows from canonical subgroup theorem in [29].  $\square$

Fix an integer  $n \geq 1$  and suppose furthermore that  $\deg(G, C) \leq p^{1-n}/(p+1) < p/(p+1)$ . Then define subgroup  $H_n = H_n(G)$  of  $G$  order  $p^n$  inductively as follows: If  $n = 1$ , set  $H_1 = D$ . If  $n > 1$ , then let  $H_n$  to be the pre-image by the map  $G \rightarrow G/H(G)$  of  $H_{n-1}(G/H(G)) \subset G/H(G)$ .

PROPOSITION 14 *Suppose that one-dimensional principally polarised  $p$ -divisible group  $G$  over  $\mathcal{O}_L$  has a subgroup  $H_n(G)$  as defined above. Suppose that  $m \geq 1$  is an integer. Suppose that  $C_m$  is a subgroup of  $G$  of order  $p^m$  such that  $H_n(G) \cap C_m = \{0\}$ , and suppose that  $D_{m+n}$  is a cyclic subgroup of  $G$  of order  $p^{m+n}$  such that  $H_n(G) \subseteq D_{m+n}$ . Then  $\deg(G/C_m) < p^{1-(m+n)}/(p+1)$  and  $G/C_m$  has the subgroup  $H_{m+n}(G/C_m)$ . Indeed,  $H_{m+n}(G/C_m) = (D_{m+n} + C_m)/C_m$ .*

*Proof.* This can be proved as in Proposition 3.5 in [3].  $\square$

### 13 $p$ -ADIC OVERCONVERGENT HILBERT MODULAR FORMS

Let  $X_{\Gamma_1(n),K}^{\text{an}}, X_{\Gamma_1(n) \cap \text{Iw},K}^{\text{an}}, X_{\Gamma_1(n) \cap \Gamma_1(p^r),K}^{\text{an}}$  respectively denote the rigid analytic spaces in the sense of Tate ([2]) associated to the  $K$ -schemes  $X_{\Gamma_1(n),K}, X_{\Gamma_1(n) \cap \text{Iw},K}, X_{\Gamma_1(n) \cap \Gamma_1(p^r),K}$ .

Given a closed point of  $Y_{\Gamma_1(n)}^{\text{rig}}$ , it corresponds to a point  $(A, \lambda, \eta)$  defined over the integer  $\mathcal{O}_L$  of a finite extension  $L$  of  $K$ . We then define  $\deg_{\tau}(A)$ , for  $\tau = \tau_{\mathfrak{p}}$  for a place  $\mathfrak{p}$  of  $F$  above  $p$ , to be ‘deg’ as in the previous section with the (one-dimension) Barsotti-Tate group of  $\mathfrak{p}$ -power torsions of  $A$  in place of ‘ $G$ ’.

The  $\mathcal{O}$ -scheme  $X_{\Gamma_1(n)}$  is of finite type, hence  $X_{\Gamma_1(n)}^{\text{rig}}$  is quasi-compact. There exists a finitely many sufficiently small affine formal schemes  $U^{\wedge}$  such that their generic fibres  $U^{\text{rig}}$  form an admissible covering of  $X_{\Gamma_1(n)}^{\text{rig}}$ . Let  $U_{\text{good}}^{\wedge}$  denote the smooth formal scheme  $U^{\wedge} \cap Y_{\Gamma_1(n)}^{\wedge}$  and let  $i : U_{\text{good}}^{\wedge} \hookrightarrow U^{\wedge}$ . On each  $U_{\text{good}}^{\wedge}$ , there is a function whose corresponding rigid function has its valuation  $\deg_{\mathfrak{p}}$ ; indeed, apply the construction to the formal completion of the ‘universal’ semi-abelian scheme over  $X_{\Gamma_1(n)}$  along the underlying scheme of  $U_{\text{good}}^{\wedge}$ . We may think of the function on  $U_{\text{good}}^{\wedge}$  as a lift of the Hasse invariant at  $\mathfrak{p}$ , and it follows from Kocher’s principle that  $i_* \mathcal{O}_{U_{\text{good}}^{\wedge}} = \mathcal{O}_{U^{\wedge}}$ , i.e., the function extends to  $U^{\wedge}$ . The valuation of its induced function on the generic fibre  $U^{\text{rig}}$  extends the function on  $U_{\text{good}}^{\text{rig}}$ . Glue these functions on  $U^{\text{rig}}$ ’s, there is a rigid function on  $X_{\Gamma_1(n)}^{\text{rig}} \simeq X_{\Gamma_1(n)}^{\text{an}}$  that defines deg.

DEFINITION. If  $I \subset [0, 1)$  is a closed, open, or half open interval with endpoint in  $\mathbf{Q}$ , define the rigid space  $X_{\Gamma_1(n),K}^{\text{an}} I = \coprod_t X_{\Gamma_1(n,t),K}^{\text{an}} I$  to be the admissible open set of points whose degrees are all in the range  $I$ .

For every  $t$ ,  $X_{\Gamma_1(n,t),K}^{\text{an}}$  is connected; this follows from the fact that  $X_{\Gamma_1(n,t),K}^{\text{rig}}$  is connected (since  $\overline{X}_{\Gamma_1(n,t)}$  is irreducible) and its ordinary locus is open, dense, and connected.

Similarly, given a closed point of  $Y_{\Gamma_1(n) \cap I_w}^{\text{rig}}$ , it corresponds to a point  $(A, \lambda, \eta, C)$  defined over the integer  $\mathcal{O}_L$  of a finite extension  $L$  of  $K$ . Let  $B = A/C$  and  $S = \text{Spec } \mathcal{O}_L$ ; let  $\text{val}_S$  denote the valuation on  $L$  normalised such that  $\text{val}_S(p) = 1$ . Then the  $\mathcal{O}_F$ -equivariant map of  $\mathcal{O}_S$ -modules

$$\text{Lie}^\vee(B/S) \longrightarrow \text{Lie}^\vee(A/S)$$

decomposes into

$$\text{Lie}^\vee(B/S)_\tau \longrightarrow \text{Lie}^\vee(A/S)_\tau$$

for every  $\tau \in \text{Hom}(F, K)$ , and, for the unique prime  $\mathfrak{p}$  of  $F$  above  $p$  corresponding to  $\tau$ , let  $\text{deg}_\mathfrak{p}((A, C))$  denote  $1 - \text{val}_S(\text{Ann}(\text{Coker}(\text{Lie}^\vee(B/S)_\tau \rightarrow \text{Lie}^\vee(A/S)_\tau)))$ . Applying the construction to the universal HBAV over  $Y_{\Gamma_1(n) \cap I_w}^{\text{rig}}$ , we locally have functions on  $Y_{\Gamma_1(n) \cap I_w}^{\text{rig}}$  whose valuations define the degrees. As for  $\text{deg}(A)$ , Kocher's principle allows us to extend the function to  $X_{\Gamma_1(n) \cap I_w}^{\text{rig}} \simeq X_{\Gamma_1(n) \cap I_w}^{\text{an}}$

DEFINITION. If  $S_1$  and  $S_2$  are disjoint subsets of  $\text{Hom}(F, K)$  and if  $I, I_1, I_2 \subseteq [0, 1]$  are closed, open, or half open intervals with endpoints in  $\mathbf{Q}$ , define the rigid space  $(X_{\Gamma_1(n) \cap I_w, K}^{\text{an}}, I_{1, S_1} I_{2, S_2})$  to be the admissible open set of points whose degree at  $\tau \in \text{Hom}(F, K) - S_1 - S_2$  (resp.  $S_1$ , resp.  $S_2$ ) is in the range  $I$  (resp.  $I_1$ , resp.  $I_2$ ).

DEFINITION. Let  $\pi_1$  (resp.  $\pi_{2, \mathfrak{p}}$ ) denote the degeneracy map

$$X_{\Gamma_1(n) \cap I_w, K}^{\text{an}} \longrightarrow X_{\Gamma_1(n), K}^{\text{an}}$$

which, on the non-cuspidal points, is defined by

$$(A, i, \lambda, \eta, C) \mapsto (A, i, \lambda, \eta)$$

$$\text{(resp. } (A, i, \lambda, \eta, C) \mapsto (A/C_\mathfrak{p}, (i \bmod C_\mathfrak{p}), (\lambda \bmod C_\mathfrak{p}), (\eta \bmod C_\mathfrak{p}))\text{)}$$

DEFINITION. Let  $\pi$  denote the degeneracy map

$$X_{\Gamma_1(n) \cap \Gamma_1(p^r), K}^{\text{an}} \longrightarrow X_{\Gamma_1(n) \cap I_w, K}^{\text{an}}$$

which, on the non-cuspidal points, is defined by

$$(A, i, \lambda, \eta, \eta_{\text{KM}}) \mapsto (A/\langle pP_{\eta_{\text{KM}}} \rangle, (i \bmod \langle pP_{\eta_{\text{KM}}} \rangle), (\lambda \bmod \langle pP_{\eta_{\text{KM}}} \rangle), (\eta \bmod \langle pP_{\eta_{\text{KM}}} \rangle)).$$

where by  $P_{\eta_{\text{KM}}}$ , we mean the image of 1 by  $\eta_{\text{KM}}$ .

DEFINITION. Define  $(X_{\Gamma_1(\mathfrak{n}) \cap \Gamma_1(p^r), K}^{\text{an}} I) I_1 S_1 I_2 S_2$  to be the preimage by  $\pi$  of  $(X_{\Gamma_1(\mathfrak{n}) \cap \text{Iw}, K}^{\text{an}} I, ) I_1 S_1 I_2 S_2$ .

For  $0 \leq r \leq p/(p + 1)$ , it follows from the previous section that

$$\pi_1^{-1}(X_{\Gamma_1(\mathfrak{n}), K}^{\text{an}}[0, r]) \simeq X_{\Gamma_1(\mathfrak{n}) \cap \text{Iw}, K}^{\text{an}}[0, r] \coprod X_{\Gamma_1(\mathfrak{n}) \cap \text{Iw}, K}^{\text{an}}[1 - r/p, 1];$$

and for  $\tau = \tau_{\mathfrak{p}}$

$$\begin{aligned} & \pi_{2, \mathfrak{p}}^{-1}(X_{\Gamma_1(\mathfrak{n}), K}^{\text{an}}[0, r]) \\ & \simeq (X_{\Gamma_1(\mathfrak{n}) \cap \text{Iw}, K}^{\text{an}}[0, r])_{\tau}[0, r/p] \coprod (X_{\Gamma_1(\mathfrak{n}) \cap \text{Iw}, K}^{\text{an}}[0, r])_{\tau}[1 - r, 1]. \end{aligned}$$

The theory of canonical subgroups provides rigid sections:

$$\pi_1 : X_{\Gamma_1(\mathfrak{n}) \cap \text{Iw}, K}^{\text{an}}[0, r] \xrightarrow{\simeq} X_{\Gamma_1(\mathfrak{n}), K}^{\text{an}}[0, r]$$

and

$$\pi_{2, \mathfrak{p}} : X_{\Gamma_1(\mathfrak{n}) \cap \text{Iw}, K}^{\text{an}}[1 - r, 1] \xrightarrow{\simeq} X_{\Gamma_1(\mathfrak{n}), K}^{\text{an}}[0, r].$$

On the other hand,

$$\pi_1 : X_{\Gamma_1(\mathfrak{n}) \cap \text{Iw}, K}^{\text{an}}[1 - r/p, 1] \longrightarrow X_{\Gamma_1(\mathfrak{n}), K}^{\text{an}}[0, r]$$

is finite flat of degree  $|\mathcal{O}_F/p|$ , and

$$\pi_{2, \mathfrak{p}} : X_{\Gamma_1(\mathfrak{n}) \cap \text{Iw}, K}^{\text{an}}[0, r/p] \longrightarrow X_{\Gamma_1(\mathfrak{n}), K}^{\text{an}}[0, r]$$

is finite flat of degree  $|\mathcal{O}_F/\mathfrak{p}|$ .

Hida [14] proves (Theorem 5.6 in [14]) that, for a character  $\psi : \text{Fr}_{\mathfrak{np}^\infty} \rightarrow K^\times$  which factors through  $I_{\mathfrak{np}^r}$  and  $k \geq 2$ , an element  $F_H : h_{\mathcal{O}}^0(\mathfrak{n}) \rightarrow L$  of  $\mathcal{S}_{\mathcal{O}}^0(\mathfrak{n}) \otimes L$  defines, modulo  $(\ker(\epsilon \circ \text{Art})^{k-2}\psi)$ , a cusp eigenform of weight  $k$  and level  $\Gamma_1(\mathfrak{np}^r)$  which is an eigenform with its  $T_{\mathfrak{m}}$ -eigenvalue  $F_H(T_{\mathfrak{m}}) \bmod (\ker(\epsilon \circ \text{Art})^{k-2}\psi)$  and  $S$  acting by  $(\epsilon \circ \text{Art})^{k-2}\psi$ . Indeed  $I_{\mathfrak{np}^\infty}$ -action defines the character of  $F_H \bmod (\ker(\epsilon \circ \text{Art})^{k-2}\psi)$ , i.e.,

$$\begin{aligned} & (F_H \bmod (\ker(\epsilon \circ \text{Art})^{k-2}\psi))(\langle \rangle) \\ & = \psi(\psi_{F_H} \bmod (\ker(\epsilon \circ \text{Art})^{k-2}\psi))(\psi_T \circ \epsilon^{2-k}) \end{aligned}$$

where  $\psi_{F_H}$  is the composite  $\text{Tor}_{\mathfrak{np}^\infty} \hookrightarrow I_{\mathfrak{np}^\infty} \xrightarrow{\langle \rangle} h_{\mathcal{O}}^0(\mathfrak{n})$  followed by  $F_H : h_{\mathcal{O}}^0(\mathfrak{n}) \rightarrow L$ ; and  $\psi_T$  is the ‘Teichmüller character’, the projection from  $\mathbf{Z}_p^\times$  to its torsion subgroup of finite order. We shall prove that the specialisation  $F_H \bmod \ker(\epsilon \circ \text{Art})^{k-2}\psi$  defines a  $p$ -ordinary overconvergent eigenform of weight  $k$  and of level  $\Gamma_1(\mathfrak{np}^r)$  for any  $k = 1$ .

For  $\epsilon$  such that  $0 \leq \epsilon < 1/(p^{r-2}(p+1))$ , the theory of canonical subgroups in [29] (see also Proposition 2.3.1 and 2.4.1 in [21]) shows that  $U_p \stackrel{\text{def}}{=} \prod_{\mathfrak{p}} U_{\mathfrak{p}}$  defines a completely continuous endomorphism on  $H^0(X_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p^r),K}^{\text{an}}[0, \epsilon], L_k(\text{cusps}))^{\overline{\mathbf{U}}}$ , where  $X_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p^r),K}^{\text{an}}[0, \epsilon]$  is the pre-image by the forgetful morphism of  $X_{\Gamma_1(\mathfrak{n}),K}^{\text{an}}[0, \epsilon]$ . We remark that, when  $F = \mathbf{Q}$ , this is proved in [4] Lemma 2.3 as a result of calculations with  $q$ -expansions.

By Serre’s theory [31], there is an idempotent  $\mathbf{e}$  commuting with  $U_p$  by which we may write

$$\begin{aligned} & H^0(X_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p^r),K}^{\text{an}}[0, \epsilon], L_k(\text{cusps}))^{\overline{\mathbf{U}}} \\ &= \mathbf{e}H^0(X_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p^r),K}^{\text{an}}[0, \epsilon], L_k(\text{cusps}))^{\overline{\mathbf{U}}} \\ &+ (1 - \mathbf{e})H^0(X_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p^r),K}^{\text{an}}[0, \epsilon], L_k(\text{cusps}))^{\overline{\mathbf{U}}} \end{aligned}$$

where  $\mathbf{e}H^0(X_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p^r),K}^{\text{an}}[0, \epsilon], L_k(\text{cusps}))^{\overline{\mathbf{U}}}$  is finite-dimensional  $K$ -vector space and all the generalised eigenvalues of  $U_p$  are units, while  $U_p$  is topologically nilpotent on the complement. It is well-known that  $\mathbf{e} = e|H^0(X_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p^r),K}^{\text{an}}[0, \epsilon], L_k(\text{cusps}))^{\overline{\mathbf{U}}}$ .

LEMMA 15 *For any integer  $k$ , the  $p$ -adic eigenform  $F_H \bmod (\ker(\epsilon \circ \text{Art})^{k-2}\psi)$  as above is overconvergent of weight  $k$  and of level  $\Gamma_1(\mathfrak{np}^r)$ .*

*Proof.* This can be proved as in Lemma 1 in [5]; replace the Eisenstein series ‘ $E$ ’ of weight  $(p-1)$  therein by the pull-back to  $X_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p^r),K}^{\text{an}}$  of a characteristic zero lifting of a sufficiently large power of the Hasse invariant.  $\square$

It follows from the theorem in the previous section that, given a  $p$ -adic representation

$$\rho : \text{Gal}(\overline{F}/F) \rightarrow GL_2(\mathcal{O})$$

as in the main theorem, there are

1. a finite soluble totally real field extension  $F_{\Sigma} \subset \overline{F}$  of  $F$  in which  $p$  splits completely,
2. a finite set  $S = \Sigma \amalg S_{\Sigma, \mathfrak{p}} \amalg S_{\Sigma, \infty}$  of places in  $F_{\Sigma}$ , where  $S_{\Sigma, \mathfrak{p}}$  denotes the set of places of  $F$  above  $p$  and  $S_{\Sigma, \infty}$  denotes the set of infinite places of  $F_{\Sigma}$ ,
3. an ideal  $\mathfrak{n}$  of  $\mathcal{O}_F$  divisible by  $\mathfrak{n}_{\Sigma} = \prod_{\mathfrak{q} \in \Sigma} \mathfrak{q}$ ,
4.  $2^{|S_{\Sigma, \mathfrak{p}}|}$  characters  $\chi_P : \text{Gal}(\overline{F}/F_{\Sigma}) \rightarrow \mathcal{O}^{\times}$  of finite order and  $2^{|S_{\Sigma, \mathfrak{p}}|}$  weight one  $p$ -ordinary overconvergent cuspidal Hilbert modular eigenforms  $f_P$  of ‘tame level’  $\mathfrak{n}$ , one for every subset  $P$  of  $S_{\Sigma, \mathfrak{p}}$ , such that:

- $f_P$  is the weight one specialisation of the  $\Lambda$ -adic companion form  $F_{\text{Hida},P} : h_{\mathcal{O}}^0(\mathfrak{n}) \rightarrow K$ , with character  $\psi_P = \psi_P^{S_{\Sigma,P}} \psi_{P,S_{\Sigma,P}}$  of  $(\mathcal{O}_{F_{\Sigma}}/\mathfrak{n})^{\times} \times (\mathcal{O}_{F_{\Sigma}}/p)^{\times}$
- the Galois representation  $\rho_P$  associated to  $f_P$  is  $\rho|_{\text{Gal}(\overline{F}/F_{\Sigma})} \otimes \chi_P^{-1}$ ,
- $\rho_P$  is unramified outside  $S$  and ordinary at every place in  $S_{\Sigma,P}$ ,

and the  $f_P$ 's are 'in companion' in the sense that

- $c(\mathcal{O}_{F_{\Sigma}}, f_P) = 1$ , and  $c(\mathfrak{m}, f_P) = 0$  if  $\mathfrak{m}$  is not coprime to  $\mathfrak{n}$ ;
- $c(\mathfrak{q}, f_P) = \text{tr } \rho(\text{Frob}_{\mathfrak{q}})/\chi_P(\text{Frob}_{\mathfrak{q}})$  for every prime ideal  $\mathfrak{q}$  not dividing  $\mathfrak{n}p$ ;
- for  $\mathfrak{p}$  in  $P$ ,  $c(\mathfrak{m}, f_P)(\chi_P \circ \text{Art})(\mathfrak{m}) = c(\mathfrak{m}, f_{P-\{\mathfrak{p}\}})(\chi_{P-\{\mathfrak{p}\}} \circ \text{Art})(\mathfrak{m})$  for every ideal  $\mathfrak{m}$  coprime to  $\mathfrak{n}p$ ;
- for  $\mathfrak{p}$  in  $P$ , the character of  $f_P$  at  $\mathfrak{p}$  is  $\chi_{P-\{\mathfrak{p}\}}\chi_P^{-1}$  while for  $\mathfrak{p} \in S_{\Sigma,P} - P$ , the character of  $f_P$  at  $\mathfrak{p}$  is  $\chi_{P \cup \{\mathfrak{p}\}}\chi_P^{-1}$ ;
- for  $\mathfrak{p}$  in  $P$ ,  $(\chi_P^{S_{\Sigma,P}} \circ \text{Art})(\mathfrak{p}) = (\psi_{P-\{\mathfrak{p}\}}^{S_{\Sigma,P}} \circ \text{Art})(\mathfrak{p})$ ;
- for a place  $\mathfrak{p}$  of  $P$ , the  $U_{\mathfrak{p}}$ -eigenvalue of  $f_P$  is  $(\chi_{\mathfrak{p},1}\chi_P^{-1})(\text{Frob}_{\mathfrak{p}})$  while for  $\mathfrak{p}$  in  $S_{\Sigma,P} - P$ , the  $U_{\mathfrak{p}}$ -eigenvalue of  $f_P$  is  $(\chi_{\mathfrak{p},2}\chi_P^{-1})(\text{Frob}_{\mathfrak{p}})$ .

14 ANALYTIC CONTINUATION OF OVERCONVERGENT EIGENFORMS

Fix  $\tau = \tau_{\mathfrak{p}}$  throughout the section (except the last two assertions).

DEFINITION. Fix  $t$ . For brevity, let  $X_{\Gamma_1(\mathfrak{n},t) \cap \text{Iw},\tau}$  denote  $(X_{\Gamma_1(\mathfrak{n},t) \cap \text{Iw},K}[0,r])[0,1]_{\tau}$ ; and for an integer  $n \geq 0$ , let  $X_{\Gamma_1(\mathfrak{n},t) \cap \text{Iw},\tau,n}$  denote  $(X_{\Gamma_1(\mathfrak{n},t) \cap \text{Iw},K}[0,r])[0,1 - 1/p^n(p+1)]_{\tau}$ . Let  $X_{\Gamma_1(\mathfrak{n}) \cap \text{Iw},\tau}$  (resp.  $X_{\Gamma_1(\mathfrak{n}) \cap \text{Iw},\tau,n}$ ) denote the disjoint union over  $T$  of  $X_{\Gamma_1(\mathfrak{n},t) \cap \text{Iw},\tau}$  (resp.  $X_{\Gamma_1(\mathfrak{n},t) \cap \text{Iw},\tau,n}$ ).

PROPOSITION 16 For every integer  $n \geq 0$ ,  $X_{\Gamma_1(\mathfrak{n}) \cap \text{Iw},\tau,n}$  is an admissible open subset of  $X_{\Gamma_1(\mathfrak{n}) \cap \text{Iw},\tau}$ , and the  $X_{\Gamma_1(\mathfrak{n}) \cap \text{Iw},\tau,n}$  form an admissible covering of  $X_{\Gamma_1(\mathfrak{n}) \cap \text{Iw},\tau}$ . For every  $t$  and every  $n \in \mathbf{Z}_{\geq 0}$ ,  $X_{\Gamma_1(\mathfrak{n},t) \cap \text{Iw},\tau,n}$  is connected.

Proof. Clear.  $\square$

DEFINITION. Let  $X_{\Gamma_1(\mathfrak{n},t) \cap \Gamma_1(p^r),\tau}$  (resp.  $X_{\Gamma_1(\mathfrak{n},t) \cap \Gamma_1(p^r),\tau,n}$ ) denote the pre-image by the degeneracy morphism

$$\pi : X_{\Gamma_1(\mathfrak{n},t) \cap \Gamma_1(p^r),K}^{\text{an}} \rightarrow X_{\Gamma_1(\mathfrak{n},t) \cap \text{Iw},K}^{\text{an}}$$

of  $X_{\Gamma_1(\mathfrak{n},t) \cap \text{Iw},\tau}$  (resp.  $X_{\Gamma_1(\mathfrak{n},t) \cap \text{Iw},\tau,n}$ ).

Let  $X_{\Gamma_1(\mathfrak{n}) \cap \Gamma_1(p^r),\tau}$  (resp.  $X_{\Gamma_1(\mathfrak{n}) \cap \Gamma_1(p^r),\tau,n}$ ) denote the disjoint union over  $T$  of  $X_{\Gamma_1(\mathfrak{n},t) \cap \Gamma_1(p^r),\tau}$  (resp.  $X_{\Gamma_1(\mathfrak{n},t) \cap \Gamma_1(p^r),\tau,n}$ ).

PROPOSITION 17 *For every integer  $n \geq 0$ ,  $X_{\Gamma_1(n) \cap \Gamma_1(p^r), \tau, n}$  is an admissible open subset of  $X_{\Gamma_1(n) \cap \Gamma_1(p^r), \tau}$ , and the  $X_{\Gamma_1(n) \cap \Gamma_1(p^r), \tau, n}$  form an admissible covering of  $X_{\Gamma_1(n) \cap \Gamma_1(p^r), \tau}$ . For every  $t$  and an integer  $n \geq 0$ ,  $X_{\Gamma_1(n, t) \cap \Gamma_1(p^r), \tau, n}$  is connected.*

*Proof.* Analogous to the proposition above.  $\square$

COROLLARY 18 *We have  $\pi_1^{-1}(X_{\Gamma_1(n) \cap \Gamma_1(p^r), \tau, n+1}) \subset \pi_{2, \mathfrak{p}}^{-1}(X_{\Gamma_1(n) \cap \Gamma_1(p^r), \tau, n})$ .*

*Proof.* This follows from [29].  $\square$

Let  $(\text{Tate}_{M_1, M_2}(q) = (GL_1 \otimes_{\mathbf{Z}} \mathfrak{d}^{-1}M_1^{-1})/q^{M_2}, i, \lambda, \eta, \eta_{\text{KM}} : 1 \mapsto \zeta_r)$  over  $\mathcal{O} \otimes \mathbf{Z}((q^{M_1}M_2^{-1}))$  for the pair  $M_1, M_2$  of the fractional ideals such that  $M_1M_2^{-1} \simeq t$  be a family of HBAVs around a cusp of  $X_{\Gamma_1(n, t) \cap \Gamma_1(p^r), n}$ . Choose (non-canonically) once for all a basis of the pull-back by  $\text{Max}(\mathcal{O} \otimes_{\mathbf{Z}} \mathbf{Z}((q^M))) \rightarrow X_{\Gamma_1(n) \cap \Gamma_1(p^r), K}^{\text{an}}$  of the line bundle  $L_k$ , since a subgroup of  $\text{Tate}_{M_1, M_2}(q)[\mathfrak{p}]$  of order  $|\mathcal{O}_F/\mathfrak{p}|$ , disjoint from  $\eta_r$ , is of the form  $\zeta\eta + q^{M_2}$  where  $\zeta$  ranges over the  $|\mathcal{O}_F/\mathfrak{p}|$  points of  $(GL_1 \otimes_{\mathbf{Z}} \mathfrak{d}^{-1}M_1^{-1})(S)[\mathfrak{p}]$  and  $\eta = \eta_{1, \mathfrak{p}}^{\text{ét}} \in q^{\mathfrak{p}^{-1}M_2/M_2}$ ,  $U_{\mathfrak{p}}(f)(\text{Tate}_{M_1, M_2}(q), i, \lambda, \eta, \eta_{\text{KM}})$  is:

$$\begin{aligned} & |\mathcal{O}_F/\mathfrak{p}|^{-1} \sum_{\zeta} f(((GL_1 \otimes_{\mathbf{Z}} \mathfrak{d}^{-1}M_1^{-1})/q^{M_2})/(\zeta\eta)) \\ = & |\mathcal{O}_F/\mathfrak{p}|^{-1} \sum_{\zeta} f((GL_1 \otimes_{\mathbf{Z}} \mathfrak{d}^{-1}M_1^{-1})/(\zeta q_{\eta})^{\mathfrak{p}^{-1}M_2}) \\ = & |\mathcal{O}_F/\mathfrak{p}|^{-1} \sum_{\zeta} |\mathcal{O}_F/t_1|^{-1} \sum_{\nu \in (\mathfrak{p}^{-1}M)^+} c(\mathfrak{p}M^{-1}\nu, f)(\zeta q_{\eta})^{\nu} \\ = & |\mathcal{O}_F/\mathfrak{p}|^{-1} |\mathcal{O}_F/t_{\mathfrak{p}}|^{-1} \sum_{\nu \in (\mathfrak{p}^{-1}M)^+} \left( \sum_{\zeta} \zeta^{\nu} \right) c(\mathfrak{p}M^{-1}\nu, f) q_{\eta}^{\nu} \\ = & |\mathcal{O}_F/\mathfrak{p}|^{-1} |\mathcal{O}_F/t_{\mathfrak{p}}|^{-1} \sum_{\nu \in M^+} |\mathcal{O}_F/\mathfrak{p}| c(\mathfrak{p}M^{-1}\nu, f) q^{\nu} \end{aligned}$$

where  $q_{\eta}$  denotes a representative in  $q^{\mathfrak{p}^{-1}M_2}$  of the class  $\eta \in q^{\mathfrak{p}^{-1}M_2/M_2} = q^{\mathfrak{p}^{-1}M_2}/q^{M_2}$  defined earlier; and  $t_{\mathfrak{p}}$  represents the class of  $\mathfrak{p}t \simeq \mathfrak{p}M_1M_2^{-1}$ .

THEOREM 19 *Suppose that  $f \in H^0((X_{\Gamma_1(n) \cap \Gamma_1(p^r), K})_{\tau}[0, \epsilon], L_k)$  is an eigenform for  $U_{\mathfrak{p}}$  with non-zero eigenvalue, then  $f$  extends to  $X_{\Gamma_1(n) \cap \Gamma_1(p^r), \tau} = (X_{\Gamma_1(n) \cap \Gamma_1(p^r)})_{\tau}[0, 1)$ .*

DEFINITION. Let

$$\begin{array}{c} X_{\Gamma_1(n, t) \cap \Gamma_1(p^r), \tau}^{[0]} \subset X_{\Gamma_1(n, t) \cap \Gamma_1(p^r), \tau}^{[1]} \subset \cdots \subset X_{\Gamma_1(n, t) \cap \Gamma_1(p^r), \tau}^{[r-1]} \\ \parallel \\ X_{\Gamma_1(n, t) \cap \Gamma_1(p^r), \tau} \end{array}$$

denote the admissible open subsets of  $X_{\Gamma_1(n, t) \cap \Gamma_1(p^r)}^{\text{an}}$  defined in such a way that the non-cuspidal  $S$ -points of  $X_{\Gamma_1(n, t) \cap \Gamma_1(p^r), \tau}^{[s]}$  parameterises  $(A/S, i, \lambda, \eta)$

equipped with a point  $P_{\eta_{\text{KM}}}$  of exact order  $\sum_{\mathfrak{p}} |\mathcal{O}_F/\mathfrak{p}^r|$  where  $A/S$  is either  $\mathfrak{p}$ -non-ordinary, or it is  $\mathfrak{p}$ -ordinary and  $H_{r-s}(A[\mathfrak{p}])$  equals the subgroup generated by  $|\mathcal{O}_F/\mathfrak{p}|^s P_{\eta_{\text{KM},\mathfrak{p}}}$ .

For every  $0 \leq s \leq r - 1$ ,  $X_{\Gamma_1(n,t) \cap \Gamma_1(p^r),\tau}^{[s]}$  is connected since it is the pre-image of a closed subset of the union of irreducible components intersecting precisely at the  $\mathfrak{p}$ -non-ordinary locus of  $X_{\Gamma_1(n,t) \cap \Gamma_1(p^r),\tau}$ .

**THEOREM 20** *If  $r$  is an integer  $\geq 2$  and suppose that  $f \in H^0(X_{\Gamma_1(n) \cap \Gamma_1(p^r),\tau}, L_k)$  is an eigenform for  $U_{\mathfrak{p}}$  with non-zero eigenvalue. Then  $f$  extends to  $X_{\Gamma_1(n,t) \cap \Gamma_1(p^r),\tau}^{[r-1]}$ .*

*Proof.* This can be proved as Lemma 6.1 in [3]  $\square$

**COROLLARY 21** *If  $f \in H^0(X_{\Gamma_1(n) \cap \text{Iw},K}^{\text{an}}[0, \epsilon], L_k)$  for some  $0 < \epsilon < 1$  is an eigenform for every  $U_{\mathfrak{p}}$ ,  $\mathfrak{p}|p$ , with non-zero eigenvalue, then  $f$  extends to  $H^0(X_{\Gamma_1(n) \cap \text{Iw},K}[0, 1], L_k)$ . Similarly, if  $f \in H^0(X_{\Gamma_1(n) \cap \Gamma_1(p^r),K}^{\text{an}}[0, \epsilon], L_k)$ , for some  $0 < \epsilon < 1$  is an eigenform for every  $U_{\mathfrak{p}}$ ,  $\mathfrak{p}|p$ , with non-zero eigenvalue, then  $g$  extends to  $H^0(X_{\Gamma_1(n) \cap \Gamma_1(p^r),K}[0, 1], L_k)$ .*

## 15 GLUING EIGENFORMS

### 15.1 THE IWAHORI CASE

**DEFINITION.** For every subset  $P$  of the set of places of  $F$  above  $p$ , let  $w_P$  denote the automorphism of  $X_{\Gamma_1(n) \cap \text{Iw},K}^{\text{an}}$  defined by a composite (independent of ordering) of the  $w_{\mathfrak{p}}$  for all  $\mathfrak{p}$  in  $P$ .

**THEOREM 22** *For every subset  $P$  of the set  $S = S_{\mathfrak{p}}$  of places of  $F$  above  $p$ , suppose  $f_P \in H^0(X_{\Gamma_1(n),K}^{\text{an}}, L_k)$  is an overconvergent modular form of parallel weight  $k = \sum_{\tau \in \text{Hom}(F,K)} k\tau \in \mathbf{Z}$  and of level  $\Gamma_1(n)$ . Assume furthermore that*

- the Fourier coefficient  $c(f_P, \mathcal{O}_F) = 1$ ;
- for every place  $\mathfrak{p}$  of  $F$  above  $p$ , there exist  $\alpha_{\mathfrak{p}}, \beta_{\mathfrak{p}} \in K$  such that  $\alpha_{\mathfrak{p}} \neq \beta_{\mathfrak{p}}$  and such that, for every  $P$ ,  $f_P$  is an eigenform for  $U_{\mathfrak{p}}$  with eigenvalue  $\alpha_{\mathfrak{p}}$  if  $\mathfrak{p} \in P$  whilst with eigenvalue  $\beta_{\mathfrak{p}}$  if  $\mathfrak{p} \notin P$ ;
- for all ideal  $\mathfrak{m}$  of  $\mathcal{O}_F$  coprime to  $p$ ,  $c(\mathfrak{m}, f_P)$  are equal for every  $P$ .

Then every  $f_P$  is a classical Hilbert modular eigenform of weight  $k$  and of level  $\Gamma_1(n) \cap \text{Iw}$ .

*Proof.* By the isomorphism

$$\pi_1 : X_{\Gamma_1(n) \cap \text{Iw},K}^{\text{an}}[0, r] \xrightarrow{\cong} X_{\Gamma_1(n),K}^{\text{an}}[0, r]$$

for  $r < p/(p + 1)$  given by the canonical subgroups theorem [29], we may think of  $f_P$  as an element of  $H^0(X_{\Gamma_1(n) \cap \text{Iw},K}^{\text{an}}[0, r], L_k)$ . It follows from results in [29]

that  $\pi_1^* f_P$  extends to a section over  $X_{\Gamma_1(n) \cap Iw, K}^{\text{an}}[0, 1]$ . For brevity, we shall only show that  $f_P$ , with  $P$  the (full) set  $S$  of places of  $F$  above  $p$ , is classical; the general case follows by changing the roles of  $\alpha_p$  and  $\beta_p$ .

Choose a rational number  $r \in \mathbf{Q}$  with  $1/2 < r < p/(p + 1)$ . Suppose that  $f_S$  extends to a section of  $L_k$  over  $(X_{\Gamma_1(n) \cap Iw, K}[0, r])[0, 1]_{S-P}$  for some  $P \subseteq S$ . Fix a prime  $\mathfrak{p} \in P$ . It suffices to show that  $f_S$  extends to  $(X_{\Gamma_1(n) \cap Iw, K}[0, r])[0, 1]_{S-(P-\{\mathfrak{p}\})}$ .

For  $f \in H^0(X_{\Gamma_1(n), K}[0, r], L_k)$  and for every subset  $Q \subseteq S - P$ , let  $f^Q$  denote the restriction of  $f$  to  $(X_{\Gamma_1(n) \cap Iw, K}[0, r])[1 - r, 1]_Q[0, r]_{(S-P)-Q}$  by the map  $\pi_1 \circ w_Q$  which defines an isomorphism

$$\begin{aligned} (X_{\Gamma_1(n) \cap Iw, K}[0, r])[1 - r, 1]_Q, [0, r]_{(S-P)-Q} &\simeq X_{\Gamma_1(n), K}[0, r] \\ &\parallel \\ (X_{\Gamma_1(n) \cap Iw, K}[0, r])[1 - r, 1]_Q &\quad . \end{aligned}$$

The pre-image by  $\pi_{2, \mathfrak{p}} \circ w_Q$  of  $(X_{\Gamma_1(n), K}[0, r])(0, r)_{\mathfrak{p}}$  is the union of two components

$$(X_{\Gamma_1(n) \cap Iw, K}[0, r])[1 - r, 1]_Q(1 - r, 1)_{\mathfrak{p}} \coprod (X_{\Gamma_1(n) \cap Iw, K}[0, r])[1 - r, 1]_Q(0, r/p)_{\mathfrak{p}}$$

and it induces an isomorphism

$$(X_{\Gamma_1(n) \cap Iw, K}[0, r])[1 - r, 1]_Q(1 - r, 1)_{\mathfrak{p}} \simeq (X_{\Gamma_1(n), K}[0, r])(0, r)_{\mathfrak{p}}$$

on the one component and a finite flat morphism of degree  $|\mathcal{O}_F/\mathfrak{p}|$

$$(X_{\Gamma_1(n) \cap Iw, K}[0, r])[1 - r, 1]_Q(0, r/p)_{\mathfrak{p}} \longrightarrow (X_{\Gamma_1(n), K}[0, r])(0, r)_{\mathfrak{p}}$$

on the other.

We are going to glue  $f_S$  and  $f_{S-\{\mathfrak{p}\}}$ ; more precisely glue  $f_S^Q$  and  $f_{S-\{\mathfrak{p}\}}^Q$ .

Let  $F$  denote the section

$$(\alpha_{\mathfrak{p}} f_S^Q - \beta_{\mathfrak{p}} f_{S-\{\mathfrak{p}\}}^Q) / (\alpha_{\mathfrak{p}} - \beta_{\mathfrak{p}}) \in H^0((X_{\Gamma_1(n), K}[0, r])(0, r)_{\mathfrak{p}}, L_k)$$

and  $G$  denote the section

$$|\mathcal{O}_F/\mathfrak{p}| \langle \mathfrak{p} \rangle (f_S^Q - f_{S-\{\mathfrak{p}\}}^Q) / (\alpha_{\mathfrak{p}} - \beta_{\mathfrak{p}}) \in H^0((X_{\Gamma_1(n) \cap Iw, K}[0, r])[1 - r, 1]_Q(0, r)_{\mathfrak{p}}, L_k)$$

Since one can show readily the  $q$ -expansions of  $\pi_{2, \mathfrak{p}}^* F$  and  $G$  are equal at around  $C = (\text{Tate}_{M_1, M_2}(q), \dots, \langle \zeta_1 \rangle)$ , we shall glue  $\pi_{2, \mathfrak{p}}^* F$  and  $G$  at  $(X_{\Gamma_1(n) \cap Iw, K}[0, r])[1 - r]_Q(0, r/p)_{\mathfrak{p}}$  to construct an extension  $F'$  of  $F$  to a section over  $(X_{\Gamma_1(n), K}[0, r])[0, 1]_{\mathfrak{p}}$ ; this extension constructs an extension of  $f_S^Q$  (and  $f_{S-\{\mathfrak{p}\}}^Q$ ) to  $(X_{\Gamma_1(n) \cap Iw, K}[0, r])[1 - r, 1]_Q[0, r]_{(S-P)-Q}[0, 1]_{\mathfrak{p}}$  and therefore to  $(X_{\Gamma_1(n) \cap Iw, K}[0, r])[0, 1]_{S-(P-\{\mathfrak{p}\})}$  (by assumption, there is an extension ‘over  $[0, 1]$ ’ at  $S - P$ )



Gluing of  $\pi_{2,\mathfrak{p}}^*F$  and  $G$  is analogous to [3] since we have a commutative diagram

$$\begin{array}{ccc} (X_{\Gamma_1(\mathfrak{n})\cap\text{Iw},K}[0,r])[1-r]_{\mathcal{O}}(0,r/p)_{\mathfrak{p}} & \longrightarrow & (X_{\Gamma_1(\mathfrak{n})\cap\text{Iw},K}[0,r])[1-r]_{\mathcal{O}}(0,1)_{\mathfrak{p}} \\ \downarrow & & \downarrow \\ (X_{\Gamma_1(\mathfrak{n}),K}[0,r])(0,r)_{\mathfrak{p}} & \longrightarrow & (X_{\Gamma_1(\mathfrak{n}),K}[0,r])(0,1)_{\mathfrak{p}} \end{array}$$

where the vertical arrows are  $\pi_{2,\mathfrak{p}} \circ w_{\mathcal{O}}$  but of degree  $|\mathcal{O}_F/\mathfrak{p}|$  on the left and  $1 + |\mathcal{O}_F/\mathfrak{p}|$  on the right.  $\square$

15.2 THE  $\Gamma_1(p)$  CASE

For every  $t$  in  $T$  and for every subset  $P$  of the set  $S = S_{\mathfrak{p}}$  of places of  $F$  above  $p$ , we will let

$$\stackrel{\text{def}}{=} \pi_1^{-1}(X_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p),\text{Iw}_{\mathfrak{p}},K}[0,1])(0,1)_{S-P} \subset X_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p),\text{Iw}_{\mathfrak{p}},K}^{\text{an}}$$

Let  $w_P$  denote the composite of the  $w_{\zeta_{\mathfrak{p}}}$  for all  $\mathfrak{p} \in P$ . Note that

$$(X_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p),\text{Iw}_{\mathfrak{p}},K}[0,1])(0,1)_{S-P} \xrightarrow{\cong} w_{S-P}^{-1}X_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p),\text{Iw}_{\mathfrak{p}},K}[0,1]$$

and each  $(X_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p),\text{Iw}_{\mathfrak{p}},K}[0,1])(0,1)_{S-P}$  is connected since it is isomorphic to  $X_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p),\mathcal{O}_K}[0,1]$  and the latter is connected since it is the pre-image of a connected component in the Zariski topology of the closed fibre. Let  $(X_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p),\text{Iw}_{\mathfrak{p}},K}[0,1])(0,1)_{S-P}$  denote the disjoint union over  $T$  of  $(X_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p),\text{Iw}_{\mathfrak{p}},K}[0,1])(0,1)_{S-P}$ .

DEFINITION. If  $f$  is a section of  $L_k$  over  $X_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p),K}[0,1]$ , then  $w_{S-P}^*f$  is a section of  $w_{S-P}^*L_k$  over  $w_{S-P}^{-1}(X_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p),K}[0,1]) = (X_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p),K}[0,1])(0,1)_{S-P}$ . Because  $p$  is inverted, the natural morphism of invertible sheaves

$$L_k|(X_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p),K}[0,1])(0,1)_{S-P} \xrightarrow{\cong} w_{S-P}^*(L_k|X_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p),K}[0,1])$$

is an isomorphism and we let  $f|_{w_P}$  denote the section of  $L_k$  over  $(X_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p),K}[0,1])(0,1)_{P-S}$  corresponding to  $w_{S-P}^*f$  by the isomorphism.

THEOREM 23 For every subset  $P$  of  $S = S_{\mathfrak{p}}$ , let  $f_P \in H^0(X_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p),K}[0,1], L_k)^{\overline{\mathbf{U}}_1}$  be an overconvergent Hilbert modular form of parallel weight  $k = \sum_{\tau \in \text{Hom}(F,K)} k\tau \in \mathbf{Z}$  and of level  $\Gamma_1(\mathfrak{np})$ . For every subset  $P$  of  $S$ , suppose that  $f_P$  has a Hecke character

$$\psi_P \stackrel{\text{def}}{=} \psi_P^S \psi_{S,P} : (\mathcal{O}_F/\mathfrak{np})^{\times} \simeq (\mathcal{O}_F/\mathfrak{n})^{\times} \times (\mathcal{O}_F/p)^{\times} \longrightarrow \mathcal{O}^{\times};$$

and that  $\psi_P^S(\mathfrak{p}) = \psi_{P-\{\mathfrak{p}\}}^S(\mathfrak{p})$  for every  $\mathfrak{p}$  in  $P$ . Suppose that

- the Fourier coefficients  $c(\mathcal{O}_F, f_P) = 1$  and  $c(\mathfrak{m}, f_P) = 0$  if  $\mathfrak{m}$  and  $\mathfrak{n}$  are not coprime,
- for every  $\mathfrak{p} \in P$ ,  $c(\mathfrak{m}, f_P) = \psi_{P,\mathfrak{p}}(\mathfrak{m})c(\mathfrak{m}, f_{P-\{\mathfrak{p}\}})$  for every ideal  $\mathfrak{m}$  coprime to  $\mathfrak{n}\mathfrak{p}$ , where by  $\psi_{P,\mathfrak{p}}$  we mean the  $\mathfrak{p}$ -component of  $\psi_{P,S}$  which we assume non-trivial,
- for every  $\mathfrak{p}$  in  $S$ ,  $f_P$  is an  $U_{\mathfrak{p}}$ -eigenform with non-zero eigenvalue  $\alpha(\mathfrak{p}, f_P)$ , and for every  $\mathfrak{p} \in P$ ,  $\alpha(\mathfrak{p}, f_P)\alpha(\mathfrak{p}, f_{P-\{\mathfrak{p}\}}) = \psi_P^S(\mathfrak{p})|\mathcal{O}_F/\mathfrak{p}|^{k-1} = \psi_{P-\{\mathfrak{p}\}}^S(\mathfrak{p})|\mathcal{O}_F/\mathfrak{p}|^{k-1}$ .

Then  $f_P$  is a section of  $L_k$  over  $X_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p),\mathcal{O}_K}^{\text{an}}$ .

*Proof.* For every subset  $P$  of  $S$ , let  $g_P$  denote  $f_P|_{w_P} \in H^0((X_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p),K}[0,1])(0,1]_{S-P}, L_k)$ . Clearly  $g_S = f_S$ . We shall prove that  $f_S$  is classical.

Fix an integer  $0 \leq n \leq |S|$  and suppose that the  $g_P$  with  $P \subseteq S$  such that  $|P| \geq n$  glue together to define sections, which will again be denoted by  $g_P$ , over

$$\bigcup_{P \subseteq S, |P| \geq n} (X_{\Gamma_1(\mathfrak{n})\cap\Gamma_1(p),K}[0,1])(0,1]_{P-S}.$$

Fix a subset  $P \subseteq S$  with  $\#P = n$  and fix  $\mathfrak{p} \in P$ . It suffice to show that  $g_P (\simeq w_{S-P}^* f_P)$  and (a constant multiple of)  $g_{P-\{\mathfrak{p}\}} (\simeq w_{S-(P-\{\mathfrak{p}\})}^* f_{P-\{\mathfrak{p}\}} = w_{\{\mathfrak{p}\}}^* w_{S-P}^* f_{P-\{\mathfrak{p}\}})$  glue.

Let  $\alpha_{\mathfrak{p}}$  (resp.  $\beta_{\mathfrak{p}}$ ) denote the  $U_{\mathfrak{p}}$ -eigenvalue  $\alpha(\mathfrak{p}, f_P)$  (resp.  $\alpha(\mathfrak{p}, f_{P-\{\mathfrak{p}\}})$ ). Fix a  $p$ -th root  $\zeta_1$  of unity. Let  $(\text{Tate}_{M_1, M_2}(q), \dots, \eta_{\text{KM}} : 1 \mapsto \zeta_1)$  be a point around a cusp  $C$ . By abuse of notation, we call it  $C$ .

There is a morphism

$$\pi_1 : X_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p),Iw_{\mathfrak{p}},K}^{\text{an}} \longrightarrow X_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p),K}^{\text{an}}$$

defined, on the non-cuspidal points, by

$$(A, i, \lambda, \eta, \eta_{\text{KM}}, D_{\mathfrak{p}}) \mapsto (A, i, \lambda, \eta, \eta_{\text{KM}})$$

and, for  $\gamma \in (\mathcal{O}_F/\mathfrak{p})^\times$ ,

$$\pi_{2,\gamma} : X_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p),Iw_{\mathfrak{p}},K}^{\text{an}} \longrightarrow X_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p),K}^{\text{an}}$$

defined, on the non-cuspidal points, by

$$(A, i, \lambda, \eta, \eta_{\text{KM}}, D_{\mathfrak{p}}) \mapsto (A/(\gamma\eta_{\text{KM}}(1)_{\mathfrak{p}} + D_{\mathfrak{p}}), \dots, (\eta_{\text{KM}} \bmod (\gamma\eta_{\text{KM}}(1)_{\mathfrak{p}} + D_{\mathfrak{p}}))).$$

To single out, let

$$\pi_{2,\mathfrak{p}} : X_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p),Iw_{\mathfrak{p}},K}^{\text{an}} \longrightarrow X_{\Gamma_1(\mathfrak{n},t)\cap\Gamma_1(p),K}^{\text{an}}$$

denote the morphism ‘ $\gamma = 0$  in  $\mathcal{O}_F/\mathfrak{p}$ ’ which takes  $(A, i, \lambda, \eta, \eta_{\text{KM}}, D_{\mathfrak{p}})$  to  $(A/D_{\mathfrak{p}}, \dots, (\eta_{\text{KM}} \bmod D_{\mathfrak{p}}))$ .

By abuse of notation, let  $C$  also denote the pre-image

$$(\text{Tate}_{M_1, M_2}(q), \dots, \eta_{\text{KM}} : 1 \mapsto \zeta_1, \langle \eta_1^{\text{et}} \rangle) \in X_{\Gamma_1(n, t) \cap \Gamma_1(p), I_{w_{\mathfrak{p}}}, K}^{\text{an}}$$

by  $\pi_1$  above of  $C = (\text{Tate}_{M_1, M_2}(q) \dots, \zeta_1)$  for  $(M_1, M_2) = (\mathcal{O}_F, t^{-1})$  and  $M = M_1 M_2 = t^{-1}$ ; and let  $C_P \in (X_{\Gamma_1(n, t) \cap \Gamma_1(p), I_{w_{\mathfrak{p}}}, K}[0, 1])(0, 1]_{S-P}$  denote the cups  $w_{S-P}^* C$ . Then

$$(g_P | \pi_1)(C_P) = pr^* f(\text{Tate}_{M_1, M_2}(q), \dots, \zeta_1) = |\mathcal{O}_F/t|^{-1} \sum_{\nu \in M^+} c(\nu M^{-1}, f_P) q^\nu$$

On the other hand, for  $\gamma \in (\mathcal{O}_F/\mathfrak{p})^\times$ ,

$$\begin{aligned} & (g_P | \pi_{2, \gamma})(C_P) \\ &= (f_P | \pi_{2, \gamma})(\text{Tate}_{M_1, M_2}(q), \dots, \zeta_1) \\ &= pr^* f_P(\text{Tate}_{M_1, M_2}(q)/(\zeta^\gamma \eta), \dots) \text{ where } \eta := \eta_{1, \mathfrak{p}}^{\text{et}} \text{ and } \zeta := \zeta_{1, \mathfrak{p}} \\ &= pr^* f_P((GL_1 \otimes_{\mathbf{Z}} \mathfrak{d}^{-1} M_1^{-1})/(\zeta^\gamma q_\eta)^{\mathfrak{p}^{-1} M_2}, \dots) \\ &= |\mathcal{O}_F/t_{\mathfrak{p}}|^{-1} \sum_{\nu \in (\mathfrak{p}^{-1} M)^+} c(\nu \mathfrak{p} M^{-1}, f_P) \zeta^{\gamma \nu} q_\eta^\nu \end{aligned}$$

where  $t_{\mathfrak{p}}$  is one of the (fixed) representatives of the narrow class group of  $F$  representing the class of  $t_{\mathfrak{p}}$ , and where  $q_\eta$  denote a representative in  $q^{\mathfrak{p}^{-1} M_2}$  of the class  $\eta \in q^{\mathfrak{p}^{-1} M_2}/q^{M_2}$ . Finally

$$\begin{aligned} (g_P | \pi_{2, \mathfrak{p}})(C_P) &= pr^*(f_P | \pi_{2, \mathfrak{p}})(\text{Tate}_{M_1, M_2}(q), \dots, \zeta_{\mathfrak{p}}) \\ &= |\mathcal{O}_F/t_{\mathfrak{p}}|^{-1} \sum_{\nu \in (\mathfrak{p}^{-1} M)^+} c(\nu \mathfrak{p} M^{-1}, f_P) q_\eta^\nu \end{aligned}$$

For brevity, let  $S$  denote the ‘Gauss sum’

$$S \stackrel{\text{def}}{=} \sum_{\gamma \in (\mathcal{O}_F/\mathfrak{p})^\times} \zeta^\gamma \psi_{P, \mathfrak{p}}(\gamma)$$

for  $(M_1, M_2) = (\mathcal{O}_F, t^{-1})$ . Then for  $\nu \in (\mathfrak{p}^{-1} M)^+$  such that  $\nu \mathfrak{p} M^{-1} \subset \mathcal{O}_F^+$  is not divisible by  $\mathfrak{p}$ ,

$$S = \sum_{\gamma \in (\mathcal{O}_F/\mathfrak{p})^\times} \zeta^{\gamma \nu \mathfrak{p} M^{-1}} \psi_{P, \mathfrak{p}}(\gamma \nu \mathfrak{p} M^{-1}) = \psi_{P, \mathfrak{p}}(\nu \mathfrak{p} M^{-1}) \sum_{\gamma \in (\mathcal{O}_F/\mathfrak{p})^\times} \zeta^{\gamma \nu} \psi_{P, \mathfrak{p}}(\gamma).$$

It then follows that

$$\begin{aligned}
 & \sum_{\gamma \in (\mathcal{O}_F/\mathfrak{p})^\times} \psi_{P,\mathfrak{p}}(\gamma)(g_P|\pi_{2,\gamma})(C_P) \\
 = & \sum_{\gamma \in (\mathcal{O}_F/\mathfrak{p})^\times} \psi_{P,\mathfrak{p}}(\gamma)|\mathcal{O}_F/t_{\mathfrak{p}}|^{-1} \sum_{\nu \in (\mathfrak{p}^{-1}M)^+} c(\nu \mathfrak{p}M^{-1}, f_P)\zeta^{\gamma\nu} q_\eta^\nu \\
 = & |\mathcal{O}_F/t_{\mathfrak{p}}|^{-1} \sum_{\nu \in (\mathfrak{p}^{-1}M)^+} c(\nu \mathfrak{p}M^{-1}, f_P)q_\eta^\nu \sum_{\gamma \in (\mathcal{O}_F/\mathfrak{p})} \psi_{P,\mathfrak{p}}(\gamma)\zeta^{\gamma\nu} \\
 = & S|\mathcal{O}_F/t_{\mathfrak{p}}|^{-1} \sum_{\nu \in (\mathfrak{p}^{-1}M)^+, \mathfrak{p} \nmid \nu \mathfrak{p}M^{-1}} c(\nu \mathfrak{p}M^{-1}, f_P)\psi_{P,\mathfrak{p}}^{-1}(\nu \mathfrak{p}M^{-1})q_\eta^\nu \\
 = & S|\mathcal{O}_F/t_{\mathfrak{p}}|^{-1} \sum_{\nu \in (\mathfrak{p}^{-1}M)^+, \mathfrak{p} \nmid \nu \mathfrak{p}M^{-1}} c(\nu \mathfrak{p}M^{-1}, f_{P-\{\mathfrak{p}\}})q_\eta^\nu \\
 = & S|\mathcal{O}_F/t_{\mathfrak{p}}|^{-1} \sum_{\nu \in (\mathfrak{p}^{-1}M)^+} c(\nu \mathfrak{p}M^{-1}, f_{P-\{\mathfrak{p}\}})q_\eta^\nu \\
 & - S|\mathcal{O}_F/t_{\mathfrak{p}}|^{-1} \sum_{\nu \in (\mathfrak{p}^{-1}M)^+, \mathfrak{p} \mid \nu \mathfrak{p}M^{-1}} c(\nu \mathfrak{p}M^{-1}, f_{P-\{\mathfrak{p}\}})q_\eta^\nu \\
 = & S|\mathcal{O}_F/t_{\mathfrak{p}}|^{-1} \sum_{\nu \in (\mathfrak{p}^{-1}M)^+} c(\nu \mathfrak{p}M^{-1}, f_{P-\{\mathfrak{p}\}})q_\eta^\nu \\
 & - S|\mathcal{O}_F/t_{\mathfrak{p}}|^{-1} \sum_{\nu \in M^+} c(\nu M, f_{P-\{\mathfrak{p}\}})q_\eta^\nu \\
 = & S(|\mathcal{O}_F/t_{\mathfrak{p}}|^{-1} \sum_{\nu \in (\mathfrak{p}^{-1}M)^+} c(\nu \mathfrak{p}M^{-1}, f_{P-\{\mathfrak{p}\}})q_\eta^\nu - (U_{\mathfrak{p}}g_{P-\{\mathfrak{p}\}}|\pi_1)(C_P)) \\
 = & S(g_{P-\{\mathfrak{p}\}}|\pi_{2,\mathfrak{p}} - \beta_{\mathfrak{p}}g_{P-\{\mathfrak{p}\}}|\pi_1)(C_P)
 \end{aligned}$$

By the connectedness of  $(X_{\Gamma_1(n,t) \cap \Gamma_1(p),K}[0,1])(0,1]_{S-P}$ ,

$$\sum_{\gamma \in (\mathcal{O}_F/\mathfrak{p})^\times} \psi_{P,\mathfrak{p}}(\gamma)(g_P|\pi_{2,\gamma}) = S(g_{P-\{\mathfrak{p}\}}|\pi_{2,\mathfrak{p}} - \beta_{\mathfrak{p}}g_{P-\{\mathfrak{p}\}}|\pi_1)$$

on  $(X_{\Gamma_1(n,t) \cap \Gamma_1(p),K}[0,1])(0,1]_{S-P}$ .

Let  $(A, i, \lambda, \eta, \eta_{KM}, D_{\mathfrak{p}} = \langle Q_{\mathfrak{p}} \rangle)$  be a non-cuspidal point of  $X_{\Gamma_1(n,t) \cap \Gamma_1(p),Iw_{\mathfrak{p}},K}^{\text{an}}$  and let  $P = \eta_{KM}(1) = P^{\mathfrak{p}} \times P_{\mathfrak{p}}$  and  $Q = P^{\mathfrak{p}} \times Q_{\mathfrak{p}}$ . Then

$$|\mathcal{O}_F/\mathfrak{p}|_{\alpha_{\mathfrak{p}}}g_P(A, i, \lambda, \eta, Q) - pr^*g_P(A/\langle P_{\mathfrak{p}} \rangle, \dots, \overline{\eta}, \overline{Q})$$

where  $\overline{\eta} := \eta \bmod \langle P_{\mathfrak{p}} \rangle$ , and  $\overline{Q} := Q \bmod \langle P_{\mathfrak{p}} \rangle$  is:

$$\begin{aligned}
 & = |\mathcal{O}_F/\mathfrak{p}|U_{\mathfrak{p}}g_P(A, i, \lambda, \eta, Q) - pr^*g_P(A/\langle P_{\mathfrak{p}} \rangle, \dots, \overline{\eta}, \overline{Q}) \\
 & = \sum_{C_{\mathfrak{p}} \subset A[\mathfrak{p}], C_{\mathfrak{p}} \neq \langle P_{\mathfrak{p}} \rangle, \langle Q_{\mathfrak{p}} \rangle} pr^*g_P(A/C_{\mathfrak{p}}, \dots, (Q \bmod C_{\mathfrak{p}})) \\
 & = \sum_{C_{\mathfrak{p}} = \langle \gamma P_{\mathfrak{p}} + Q_{\mathfrak{p}} \rangle, \gamma \in (\mathcal{O}_F/\mathfrak{p})^\times} pr^*g_P(A/\langle \gamma P_{\mathfrak{p}} + Q_{\mathfrak{p}} \rangle, \dots, (Q \bmod \langle \gamma P_{\mathfrak{p}} + Q_{\mathfrak{p}} \rangle)) \\
 & = \sum_{\gamma \in (\mathcal{O}_F/\mathfrak{p})^\times} \psi_{P,\mathfrak{p}}(-\gamma)pr^*g_P(A/\langle \gamma P_{\mathfrak{p}} + Q_{\mathfrak{p}} \rangle, \dots, (P^{\mathfrak{p}} \times P_{\mathfrak{p}} \bmod \langle \gamma P_{\mathfrak{p}} + Q_{\mathfrak{p}} \rangle)) \\
 & = \psi_{P,\mathfrak{p}}(-1) \sum_{\gamma \in (\mathcal{O}_F/\mathfrak{p})^\times} \psi_{P,P}(\gamma)(g_P|\pi_{2,\gamma})(A, \dots, \eta, P) \\
 & = \psi_{P,\mathfrak{p}}(-1)S(g_{P-\{\mathfrak{p}\}}|\pi_{2,\mathfrak{p}} - \beta_{\mathfrak{p}}g_{P-\{\mathfrak{p}\}}|\pi_1)(A, \dots, \eta, P) \\
 & = \psi_{P,\mathfrak{p}}(-1)S(pr^*g_{P-\{\mathfrak{p}\}}(A/\langle Q_{\mathfrak{p}} \rangle, \dots, P \bmod \langle Q_{\mathfrak{p}} \rangle) - \beta_{\mathfrak{p}}g_{P-\{\mathfrak{p}\}}(A, \dots, P) \\
 & = \psi_{P,\mathfrak{p}}(-1)S((g_{P-\{\mathfrak{p}\}}|w_{\zeta_{\mathfrak{p}}})(A, \dots, P^{\mathfrak{p}} \times (-Q_{\mathfrak{p}})) \\
 & \quad - \alpha_{\mathfrak{p}}^{-1}\psi_{\mathfrak{p}}^S(\mathfrak{p})|\mathcal{O}_F/\mathfrak{p}|pr^*(g_{P-\{\mathfrak{p}\}}|w_{\zeta_{\mathfrak{p}}})(A/\langle P_{\mathfrak{p}} \rangle, \dots, \mathfrak{p}^{-1}\overline{\eta}, (P^{\mathfrak{p}} \times (-Q_{\mathfrak{p}}) \bmod \langle P_{\mathfrak{p}} \rangle))) \\
 & = S((g_{P-\{\mathfrak{p}\}}|w_{\zeta_{\mathfrak{p}}})(A, \dots, Q) - |\mathcal{O}_F/\mathfrak{p}|^{-1}\alpha_{\mathfrak{p}}^{-1}pr^*(g_{P-\{\mathfrak{p}\}}|w_{\zeta_{\mathfrak{p}}})(A/\langle P_{\mathfrak{p}} \rangle, \dots, \overline{Q})).
 \end{aligned}$$

Therefore

$$\begin{aligned}
 & (|\mathcal{O}_F/\mathfrak{p}|_{\alpha_{\mathfrak{p}}}g_P - S(g_{P-\{\mathfrak{p}\}}|w_{\zeta_{\mathfrak{p}}})) (A, \dots, Q) \\
 = & (|\mathcal{O}_F/\mathfrak{p}|_{\alpha_{\mathfrak{p}}}^{-1}pr^*(|\mathcal{O}_F/\mathfrak{p}|_{\alpha_{\mathfrak{p}}}g_P - S(g_{P-\{\mathfrak{p}\}}|w_{\zeta_{\mathfrak{p}}})) (A/\langle P_{\mathfrak{p}} \rangle, \dots, (Q \bmod \langle P_{\mathfrak{p}} \rangle)))
 \end{aligned}$$

It suffices to show that  $|\mathcal{O}_F/\mathfrak{p}|_{\alpha_{\mathfrak{p}}}g_P - S(g_{P-\{\mathfrak{p}\}}|w_{\zeta_{\mathfrak{p}}})$  is identically zero; in which case, one can glue  $g_P$  and  $(|\mathcal{O}_F/\mathfrak{p}|_{\alpha_{\mathfrak{p}}})^{-1}S(g_{P-\{\mathfrak{p}\}}|w_{\zeta_{\mathfrak{p}}})$  as desired. Showing that it is identically zero is exactly as in [3].  $\square$

15.3 THE  $\Gamma_1(p^r)$ ,  $r \geq 2$ , CASE

THEOREM 24 *Let  $S$  denote the set  $S_{\mathfrak{p}}$  of places of  $F$  above  $p$ . For any set  $P \subseteq S$ , let  $f_P \in H^0(X_{\Gamma_1(\mathfrak{n}) \cap \Gamma_1(p^r), K}[0, 1), L_k)_{\overline{\mathbf{U}}_r}$  be an overconvergent modular form of weight  $k = \sum_{\tau \in \text{Hom}(F, K)} k_{\tau} \in \mathbf{Z}$  and of level  $\Gamma_1(\mathfrak{n}p^r)$ .*

*Suppose that, for every  $P \subseteq S$ ,  $f_P$  has a character  $\psi_P^S \psi_{S, P}$  of  $(\mathcal{O}_F/\mathfrak{n}p^r)^{\times} \simeq (\mathcal{O}_F/\mathfrak{n})^{\times} \times (\mathcal{O}_F/p^r)^{\times}$ . Suppose furthermore that  $f_P$  is an eigenform for  $U_{\mathfrak{p}}$  with non-zero eigenvalue for every  $\mathfrak{p} \in S$ . Suppose finally that, for every  $P \subseteq S$ ,*

- $c(\mathcal{O}_F, f_P) = 1$ ;
- $c(\mathfrak{m}, f_P) = 0$  if  $\mathfrak{m}$  and  $\mathfrak{n}$  are not coprime;
- for every  $\mathfrak{p} \in P$ ,  $c(\mathfrak{m}, f_P) = \psi_{\mathfrak{p}, P}(\mathfrak{m})c(\mathfrak{m}, f_{P - \{\mathfrak{p}\}})$  for every ideal  $\mathfrak{m}$  coprime to  $\mathfrak{n}p$ , where  $\psi_{\mathfrak{p}, P}$  is the  $\mathfrak{p}$ -component of  $\psi_{S, P}$ .

*Then the  $f_P$  are classical Hilbert modular forms in  $H^0(X_{\Gamma_1(\mathfrak{n}) \cap \Gamma_1(p^r), K}^{\text{an}}, L_k)_{\overline{\mathbf{U}}_r}$ .*

*Proof.* As in the previous subsection, we shall prove the theorem by induction. For every subset  $P$  of  $S$ , let  $g_P$  denote  $f_P|w_P \in H^0(X_{\Gamma_1(\mathfrak{n}) \cap \Gamma_1(p^r), K}[0, 1)(0, 1]_{S-P}, L_k)$ . We shall prove that  $f_S$  is classical. Fix an integer  $0 \leq n \leq |S|$  and suppose that the  $g_P$  with  $P \subseteq S$  such that  $|P| \geq n$  glue together to define sections, which will again be denoted by  $g_P$ , over

$$\bigcup_{P \subseteq S, |P| \geq n} X_{\Gamma_1(\mathfrak{n}) \cap \Gamma_1(p^r), K}[0, 1)(0, 1]_{S-P}.$$

Fix a subset  $P \subseteq S$  with  $\#P = n$ , and fix  $\mathfrak{p} \in P$ . It suffices to show that  $g_P$  and (a constant multiple of)  $g_{P - \{\mathfrak{p}\}}$  glue.

Let  $C$  denote a point  $(\text{Tate}_{M_1, M_2}(q), i, \lambda, \eta, P)$  around a cusp  $(M_1, M_2) = (\mathcal{O}_F, t^{-1})$  where

$$P = \eta_{\text{KM}}(1) = P^{\mathfrak{p}} \times P_{\mathfrak{p}} \in \text{Tate}_{M_1, M_2}(q)(\mathcal{O} \otimes_{\mathbf{Z}} \mathbf{Z}((q^M)))$$

where  $P^{\mathfrak{p}} \stackrel{\text{def}}{=} \prod_{\mathfrak{q}|p, \mathfrak{q} \neq \mathfrak{p}} \zeta_{1, \mathfrak{q}}$  and  $P_{\mathfrak{p}} \stackrel{\text{def}}{=} \zeta_{1, \mathfrak{p}} \eta_{1, \mathfrak{p}}^{\text{et}}$ .

For brevity, let  $\mu$  denote  $\zeta_r^{p^{r-1}}$ , and  $\mu_{\mathfrak{p}}$  its  $\mathfrak{p}$ -component.

We shall compute  $q$ -expansions of  $g_P$  and  $g_{P - \{\mathfrak{p}\}}$  at the cusp  $C_P \stackrel{\text{def}}{=} w_{S-P}^* C$ . Let  $\alpha_{\mathfrak{p}}$  denote the  $U_{\mathfrak{p}}$ -eigenvalue of  $f_P$ .

$$\begin{aligned} & |\mathcal{O}_F/\mathfrak{p}| \alpha_{\mathfrak{p}} g_P(C_P) \\ &= |\mathcal{O}_F/\mathfrak{p}| U_{\mathfrak{p}} g_P(C_P) \\ &= \sum_{C_{\mathfrak{p}} \subset \text{Tate}(q)[\mathfrak{p}], C_{\mathfrak{p}} \neq \langle \mu_{\mathfrak{p}} \rangle} pr^* f_P(\text{Tate}_{M_1, M_2}(q)/C_{\mathfrak{p}}, \dots, (P \bmod C_{\mathfrak{p}})) \\ &= \sum_{\gamma \in (\mathcal{O}_F/\mathfrak{p})^{\times}} pr^* f_P((GL_1 \otimes_{\mathbf{Z}} \mathfrak{d}^{-1} M_1^{-1})/(\mu_{\mathfrak{p}}^{\gamma} q_{\eta})^{p^{-1} M_2}, \dots, P) \\ &= \sum_{\gamma} pr^* f_P((GL_1 \otimes_{\mathbf{Z}} \mathfrak{d}^{-1} M_1^{-1})/(\mu_{\mathfrak{p}}^{\gamma} q_{\eta})^{p^{-1} M_2}, \dots, P^{\mathfrak{p}} \times \{\zeta_{r, \mathfrak{p}} \mu_{\mathfrak{p}}^{-\gamma}\}) \\ &= \sum_{\gamma} pr^* f_P((GL_1 \otimes_{\mathbf{Z}} \mathfrak{d}^{-1} M_1^{-1})/(\mu_{\mathfrak{p}}^{\gamma} q_{\eta})^{p^{-1} M_2}, \dots, P^{\mathfrak{p}} \times \{\zeta_{r, \mathfrak{p}}^{1-p^{r-1}\gamma}\}) \\ &= \sum_{\gamma} \psi_{\mathfrak{p}, P}((1 - p^{r-1}\gamma)\mathfrak{p}M^{-1}) pr^* f_P((GL_1 \otimes_{\mathbf{Z}} \mathfrak{d}^{-1} M_1^{-1})/(\mu_{\mathfrak{p}}^{\gamma} q_{\eta})^{p^{-1} M_2}, \dots, \zeta_r) \\ &= \sum_{\gamma} \psi_{\mathfrak{p}, P}((1 - p^{r-1}\gamma)\mathfrak{p}M^{-1}) \sum_{\nu \in (\mathfrak{p}^{-1}M)^+} c(\nu \mathfrak{p}M^{-1}, f_P)(\mu_{\mathfrak{p}}^{\gamma} q_{\eta})^{\nu} \\ &= \sum_{\nu \in (\mathfrak{p}^{-1}M)^+} c(\nu \mathfrak{p}M^{-1}, f_P) q_{\eta}^{\nu} \sum_{\gamma} \psi_{\mathfrak{p}, P}((1 - p^{r-1}\gamma)\mathfrak{p}M^{-1}) \mu_{\mathfrak{p}}^{\gamma \nu}. \end{aligned}$$

We know that  $\psi_{S,P}$  has a conductor  $p^r$ , and hence  $\psi_{S,P}((1+p^{r-1})\mathfrak{p}M^{-1}) = \mu^{\nu_1}$  for some integer  $0 < \nu_1 < p$  (not that  $1 + p^{r-1}$  is thought of as an element of  $\mathfrak{p}^{-1}M_2 \xrightarrow{\mathfrak{p}M_2^{-1}M_1} M_1 \rightarrow M_1/\mathfrak{p}^r M_1 \simeq \mathcal{O}_F/\mathfrak{p}^r$ ); it therefore follows that  $\psi_{\mathfrak{p},P}((1 + p^{r-1})\mathfrak{p}M^{-1}) = \mu_{\mathfrak{p}}^{\nu_1}$ . In particular,  $\psi_{\mathfrak{p},P}((1 - p^{r-1}\gamma)\mathfrak{p}M^{-1}) = \mu_{\mathfrak{p}}^{-\gamma\nu_1}$ . Hence

$$\sum_{\gamma} \psi_{\mathfrak{p},P}((1 - p^{r-1}\gamma)\mathfrak{p}M^{-1})\mu_{\mathfrak{p}}^{\gamma\nu} = \sum_{\gamma} \mu_{\mathfrak{p}}^{\gamma(\nu-\nu_1)} = \begin{cases} |\mathcal{O}_F/\mathfrak{p}| & \text{if } \mathfrak{p} | (\nu - \nu_1) \\ 0 & \text{otherwise} \end{cases}$$

Therefore,

$$g_P(C_P) = (\alpha_{\mathfrak{p}}|\mathcal{O}_F/\mathfrak{p}|)^{-1}|\mathcal{O}_F/\mathfrak{p}| \sum_{\nu \in (\mathfrak{p}^{-1}M)^+, \mathfrak{p} | (\nu - \nu_1)} c(\nu\mathfrak{p}M^{-1}, f_P)q_{\eta}^{\nu}.$$

We now calculate the  $q$ -expansion of  $g_{P-\{\mathfrak{p}\}}|w_{\zeta_{\mathfrak{p}}}$  at  $C_P$ . Firstly, note that

$$(g_{P-\{\mathfrak{p}\}}|w_{\zeta_{\mathfrak{p}}})(C_P) = pr^*g_{P-\{\mathfrak{p}\}}(\text{Tate}_{M_1, M_2}(q)/(\zeta_{r,\mathfrak{p}}\eta_{1,\mathfrak{p}}^{\text{et}}, \dots, P^{\mathfrak{p}} \times Q_{\mathfrak{p}}))$$

where  $Q_{\mathfrak{p}}$  is defined by  $\langle \zeta_{r,\mathfrak{p}}\eta_{1,\mathfrak{p}}^{\text{et}}, Q_{\mathfrak{p}} \rangle = \zeta_{\mathfrak{p}}$ . Tensoring over  $\mathcal{O}_F$  with  $\mathfrak{p}^{r-1}$  on  $GL_1 \otimes \mathfrak{d}^{-1}M_1^{-1}$  induces an isomorphism

$$\begin{aligned} & (GL_1 \otimes \mathfrak{d}^{-1}M_1^{-1})/\langle q^{M_2}, \zeta_{r,\mathfrak{p}}q \rangle \\ \simeq & (GL_1 \otimes \mathfrak{d}^{-1}M_1^{-1})/\langle q^{\mathfrak{p}^{r-1}M_2}, \zeta_{r-(r-1),\mathfrak{p}}q^{\mathfrak{p}^{r-2}M_2} \rangle \\ \simeq & (GL_1 \otimes \mathfrak{d}^{-1}M_1^{-1})/\langle (\zeta_{1,\mathfrak{p}}q)^{\mathfrak{p}^{r-2}M_2} \rangle \end{aligned}$$

The HBAV

$$\text{Tate}_{M_1, \mathfrak{p}^{r-2}M_2}(\zeta_{1,\mathfrak{p}}q) \stackrel{\text{def}}{=} (GL_1 \otimes \mathfrak{d}^{-1}M_1^{-1})/\langle (\zeta_{1,\mathfrak{p}}q)^{\mathfrak{p}^{r-2}M_2} \rangle$$

is naturally  $t\mathfrak{p}^{2-r} (\simeq (\mathfrak{p}^{r-2}M_2)^{-1}M_1)$ -polarised, and comes equipped with the level structure  $\langle \mathfrak{p} \rangle^{r-1}\eta$  and the point  $P^{\mathfrak{p}} \times \{\eta_{1,\mathfrak{p}}^{\text{et}}\}$  of order  $|\mathcal{O}_F/\mathfrak{p}^r|$ ; it defines a point of  $X_{\Gamma_1(\mathfrak{n}) \cap \Gamma_1(p^r), \tau}^{[r-1]}$ . For a point  $(A, i, \lambda, \eta, P)$  of  $X_{\Gamma_1(\mathfrak{n}, t) \cap \Gamma_1(p^r), \tau}^{[r-1]}$ ,

$$\begin{aligned} & (|\mathcal{O}_F/\mathfrak{p}|\beta_{\mathfrak{p}})^{r-1}g_{P-\{\mathfrak{p}\}}(A, i, \lambda, \eta, P) \\ = & (|\mathcal{O}_F/\mathfrak{p}|U_{\mathfrak{p}})^{r-1}g_{P-\{\mathfrak{p}\}}(A, i, \lambda, \eta, P) \\ = & \sum_{C_{\mathfrak{p}} \subset A[\mathfrak{p}], |C_{\mathfrak{p}}| = |\mathcal{O}_F/\mathfrak{p}^{r-1}|, C_{\mathfrak{p}} \cap P = \{1\}} pr^*g_{P-\{\mathfrak{p}\}}(A/C_{\mathfrak{p}}, \dots, (P \bmod C_{\mathfrak{p}})) \end{aligned}$$

In which case, observe that  $(A/C_{\mathfrak{p}}, \dots, (P \bmod C_{\mathfrak{p}})) \in X_{\Gamma_1(\mathfrak{n}, t) \cap \Gamma_1(p^r), \tau}^{[0]}$  and that it allows one to extend finite slope  $U_{\mathfrak{p}}$ -eigenforms over  $X_{\Gamma_1(\mathfrak{n}, t) \cap \Gamma_1(p^r), \tau}^{[0]}$  to  $U_{\mathfrak{p}}$ -eigenforms over  $X_{\Gamma_1(\mathfrak{n}, t) \cap \Gamma_1(p^r), \tau}^{[r-1]}$  by ‘analytic continuation’. If we let

$$(A, i, \lambda, \eta, P) = (\text{Tate}_{M_1, \mathfrak{p}^{r-2}M_2}(\zeta_{1,\mathfrak{p}}q), \dots, \langle \mathfrak{p} \rangle^{r-1}\eta, P^{\mathfrak{p}} \times \{\eta_{1,\mathfrak{p}}^{\text{et}}\})$$

then the cyclic subgroups  $C_{\mathfrak{p}}$  of order  $|\mathcal{O}_F/\mathfrak{p}^{r-1}|$ , disjoint from the subgroup of order  $|\mathcal{O}_F/\mathfrak{p}|$  generated by  $\eta_{1,\mathfrak{p}}^{\text{et}}$ , are of the form

$(\zeta_{r,\mathfrak{p}} \zeta_{r-1,\mathfrak{p}}^{\nu_{r-1}} \eta_{1,\mathfrak{p}}^{\text{et},\mathfrak{p}^{-1}M_2}) / (\zeta_{1,\mathfrak{p}} q^{\mathfrak{p}^{r-2}M_2})$  for  $\nu_{r-1} \in M_1/\mathfrak{p}^{r-1}M_1 \simeq \mathcal{O}_F/\mathfrak{p}^{r-1}$ . Then

$$\begin{aligned} & ((GL_1 \otimes \mathfrak{d}^{-1}M_1^{-1}) / (\zeta_{1,\mathfrak{p}} q^{\mathfrak{p}^{r-2}M_2})) / ((\zeta_{r,\mathfrak{p}} \zeta_{r-1,\mathfrak{p}}^{\nu_{r-1}} \eta_{1,\mathfrak{p}}^{\text{et},\mathfrak{p}^{-1}M_2}) / (\zeta_{1,\mathfrak{p}} q^{\mathfrak{p}^{r-2}M_2})) \\ \simeq & (GL_1 \otimes_{\mathbf{Z}} \mathfrak{d}^{-1}M_1) / (\zeta_{r,\mathfrak{p}} \zeta_{r-1,\mathfrak{p}}^{\nu_{r-1}} \eta_{1,\mathfrak{p}}^{\text{et},\mathfrak{p}^{-1}M_2}) \\ \simeq & (GL_1 \otimes_{\mathbf{Z}} \mathfrak{d}^{-1}M_1) / (\zeta_{r,\mathfrak{p}}^{1+p\nu_{r-1}} q_{\eta})^{\mathfrak{p}^{-1}M_2}, \end{aligned}$$

where  $q_{\eta}$  is a representative in  $q^{\mathfrak{p}^{-1}M_2}$  of  $\eta_{1,\mathfrak{p}}^{\text{et}} \in q^{\mathfrak{p}^{-1}M_2}/q^{M_2}$ , is naturally  $(\mathfrak{p}^{-1}M_2)^{-1}M_1 \simeq \mathfrak{p}t$ -polarised. Then there exists a non-zero constant  $\kappa_1$  such that

$$\begin{aligned} & (g_{P-\{\mathfrak{p}\}} | w_{\zeta_{\mathfrak{p}}}) (C_P) \\ &= \sum_{\nu_{r-1}} p^{r^*} g_{P-\{\mathfrak{p}\}} ((GL_1 \otimes_{\mathbf{Z}} \mathfrak{d}^{-1}M_1) / \zeta_{r,\mathfrak{p}} \zeta_{r-1,\mathfrak{p}}^{\nu_{r-1}} \eta_{1,\mathfrak{p}}^{\text{et},\mathfrak{p}^{-1}M_2}), \dots, (\mathfrak{p})^{r-1} \eta, P^{\mathfrak{p}} \times \{\eta_{1,\mathfrak{p}}^{\text{et}}\}) \\ &= \psi^S(\mathfrak{p}^{r-1}) \sum_{\nu_{r-1}} p^{r^*} g_{P-\{\mathfrak{p}\}} ((GL_1 \otimes_{\mathbf{Z}} \mathfrak{d}^{-1}M_1) / (\zeta_{r,\mathfrak{p}}^{1+p\nu_{r-1}} \eta_{1,\mathfrak{p}}^{\text{et},\mathfrak{p}^{-1}M_2}), \dots, \eta, P^{\mathfrak{p}} \times \{q_{1,\mathfrak{p}}^{\text{et},-1}\}) \\ &= \psi^S(-\mathfrak{p}^{r-1}) \sum_{\nu_{r-1}} p^{r^*} g_{P-\{\mathfrak{p}\}} ((GL_1 \otimes_{\mathbf{Z}} \mathfrak{d}^{-1}M_1) / (\zeta_{r,\mathfrak{p}}^{1+p\nu_{r-1}} \eta_{1,\mathfrak{p}}^{\text{et},\mathfrak{p}^{-1}M_2}), \dots, \eta, P^{\mathfrak{p}} \times \{\zeta_{r,\mathfrak{p}}^{1+p\nu_{r-1}}\}) \\ &= \kappa_1 \sum_{\nu_{r-1}} \psi_{\mathfrak{p},P}((1+p\nu_{r-1})\mathfrak{p}M^{-1})^{-1} p^{r^*} g_{P-\{\mathfrak{p}\}} ((GL_1 \otimes_{\mathbf{Z}} \mathfrak{d}^{-1}M_1) / (\zeta_{r,\mathfrak{p}}^{1+p\nu_{r-1}} \eta_{1,\mathfrak{p}}^{\text{et},\mathfrak{p}^{-1}M_2}), \dots, \zeta_r) \\ &= \kappa_1 \sum_{\nu_{r-1}} \psi_{\mathfrak{p},P}((1+p\nu_{r-1})\mathfrak{p}M^{-1})^{-1} \sum_{\nu \in (\mathfrak{p}^{-1}M)^+} c(\nu \mathfrak{p}M^{-1}, f_{P-\{\mathfrak{p}\}}) (\zeta_{r,\mathfrak{p}}^{1+p\nu_{r-1}} q_{\eta})^{\nu} \\ &= \kappa_1 \sum_{\nu} c(\nu \mathfrak{p}M^{-1}, f_{P-\{\mathfrak{p}\}}) q_{\eta}^{\nu} \sum_{\nu_{r-1}} \psi_{\mathfrak{p},P}((1+p\nu_{r-1})\mathfrak{p}M^{-1})^{-1} \zeta_{r,\mathfrak{p}}^{(1+p\nu_{r-1})\nu} \end{aligned}$$

where  $\nu_{r-1}$  ranges over  $\in (\mathcal{O}_F/\mathfrak{p}^{r-1})$ . For brevity, let  $S_{\nu} \stackrel{\text{def}}{=} \sum_{\nu_{r-1} \in (\mathcal{O}_F/\mathfrak{p}^{r-1})} \psi_{\mathfrak{p},P}((1+p\nu_{r-1})\mathfrak{p}M^{-1})^{-1} \zeta_{r,\mathfrak{p}}^{(1+p\nu_{r-1})\nu}$ . As in the proof of Theorem 11.1 in [3], one can deduce that

$$S_{\nu} = \mu_{\mathfrak{p}}^{\nu-\nu_1} S_{\nu}$$

where  $\nu_1$  is defined by  $\psi_{\mathfrak{p},P}((1+p^{r-1})\mathfrak{p}M^{-1}) = \mu_{\mathfrak{p}}^{\nu_1}$ , and therefore  $S_{\nu} = 0$  unless  $\mathfrak{p} | (\nu - \nu_1)$ ; one also deduces that, for  $\nu \in (\mathfrak{p}^{-1}M)^+$  such that  $\mathfrak{p} | (\nu - \nu_1)$ ,

$$S_{\nu\nu'} = \psi_{\mathfrak{p},P}(\nu'\mathfrak{p}M^{-1}) S_{\nu}$$

for  $\nu' \in (\mathfrak{p}^{-1}M)^+$  such that  $\nu'\mathfrak{p}M^{-1} \equiv 1 \pmod{\mathfrak{p}}$  and therefore  $S_{\nu\nu'} / \psi_{\mathfrak{p},P}((\nu\mathfrak{p}M^{-1})(\nu'\mathfrak{p}M^{-1})) = S_{\nu} / \psi_{\mathfrak{p},P}(\nu\mathfrak{p}M^{-1})$ . Consequently, there is a non-zero constant  $\kappa_2$  such that

$$\begin{aligned} (g_{P-\{\mathfrak{p}\}} | w_{\zeta_{\mathfrak{p}}}) (C_P) &= \kappa_1 \sum_{\nu \in (\mathfrak{p}^{-1}M)^+, \mathfrak{p} | (\nu - \nu_2)} c(\nu \mathfrak{p}^{-1}M, f_{P-\{\mathfrak{p}\}}) S_{\nu} q_{\eta}^{\nu} \\ &= \kappa_2 \sum_{\nu, \mathfrak{p} | (\nu - \nu_2)} c(\nu \mathfrak{p}^{-1}M, f_{P-\{\mathfrak{p}\}}) \psi_{\mathfrak{p},P}(\nu \mathfrak{p}M^{-1}) q_{\eta}^{\nu} \\ &= \kappa_2 \sum_{\nu, \mathfrak{p} | (\nu - \nu_2)} c(\nu \mathfrak{p}^{-1}M, f_P) q_{\eta}^{\nu}. \end{aligned}$$

Therefore  $\alpha_{\mathfrak{p}} g_P$  and  $\kappa_2^{-1} g_{P-\{\mathfrak{p}\}}$  agree at  $C_P$  and hence  $f_P$  and  $(\alpha_{\mathfrak{p}} \kappa_2)^{-1} g_{P-\{\mathfrak{p}\}}$  glue together.  $\square$

REFERENCES

[1] T. Barnet-Lamb, T. Gee & D. Geraghty, *congruences between Hilbert modular forms: constructing ordinary lifts, II*, to appear from Math. Res. Lett.

- [2] S.Bosch, U.Güntzer, R.Remmert, *Non-archimedean analysis*, Springer-Verlag (1984)
- [3] K.Buzzard, *Analytic continuation of overconvergent eigenforms*, J. Amer. Math. Soc. 16 (2002), 29-55.
- [4] K.Buzzard, M.Dickinson, N.Shepherd-Barron & R.Taylor, *On icosahedral Artin representations*, Duke Math. J. 109 (2001), 283-318.
- [5] K.Buzzard & R.Taylor, *Companion forms and weight one forms*, Annals of Math. 149 (1999), 905-919.
- [6] C.-L. Chai, & G.Faltings, *Degeneration of abelian varieties*, Springer-Verlag (1990).
- [7] G.Cornell, J.H.Silverman, & G.Stevens eds., *Modular forms and Fermat's Last Theorem*, Springer-Verlag (1998).
- [8] P.Deligne & G.Pappas, *Singularités des espaces de modules de Hilbert, en les caractéristiques divisant le discriminant*, Compositio Math. 90 (1994), 59-79.
- [9] P.Deligne & K.Ribet, *Values of abelian L-functions at negative integers over totally real fields*, Invent. Math. 59 (1980), 227-286.
- [10] M.Dickinson, *On modularity of certain 2-adic Galois representations*, Duke Math. J. 109 (2001), 319-382.
- [11] T.Ekedahl, *An effective version of Hilbert's irreducibility theorem*, in "Séminaire de Théorie des Nombres, Paris 1988-1989", Progress in Mathematics 91, Birkhäuser.
- [12] T.Gee, *A modular lifting theorem for weight 2 Hilbert modular forms*, Math. Res. Lett 13 (2006), 275-284.
- [13] T.Gee, *Companion forms over totally real fields, II*, Duke Math. J. 136 (2007), 275-284.
- [14] H.Hida, *On p-adic Hecke algebras for  $GL_2$  over totally real fields*, Annals of Math. 128 (1988), 295-384.
- [15] F.Jarvis, *Correspondences on Shimura curves and Mazur's principle above p*, Pacific J. Math. 213 (2004) 267-280.
- [16] N.Katz, *p-adic properties of modular schemes and modular forms*, LNM 350, Springer-Verlag (1973), 69-170.
- [17] N.Katz & B.Mazur, *Arithmetic moduli of elliptic curves*, Annals of Maths Studies 108, PUP (1985).



- [18] C.Khare & J.P.Wintenberger, *Serre's modularity conjecture (II)*, Invent. Math. 178 (2009), 505-586.
- [19] M.Kisin, *Modularity of 2-adic Barsotti-Tate representations*, Invent. Math. 178 (2009), 587-634.
- [20] M.Kisin, *Moduli of finite flat group schemes and modularity*, Annals of Math. 170 (2009), 1085-1180.
- [21] M.Kisin & K.F.Lai, *Overconvergent Hilbert modular forms*, Amer. J. Math 127 (2005), 735-783.
- [22] A.W.Knapp, *Introduction to the Langlands program*, Proceedings of symposia in pure mathematics, 61 (1997), 245-302.
- [23] R.Langlands, *Base change for  $GL(2)$* , Annals of Math. Studies 96, PUP (1980)
- [24] J.Lubin, *finite subgroups and isogenies of one-parameter formal Lie groups*, Annals of Math. 85 (1967), 296-302.
- [25] G.Pappas, *Arithmetic models for Hilbert modular varieties*, Compos. Maths. 98 (1995), 43-76.
- [26] M.Rapoport, *Compactification de l'espace de modules de Hilbert-Blumethal*, Compositio Math. 36 (1978), 255-335.
- [27] K.Rubin, *Modularity of mod 5 representations*, in [7], p.463-474.
- [28] J.Rogawski & J.Tunnell, *On Artin L-functions associated to Hilbert modular forms of weight one*, Invent. Math. 74 (1983), 1-42
- [29] S.Sasaki, *Analytic continuation of overconvergent Hilbert eigenforms*, Compositio Math. 146 (2010), 541-560.
- [30] S.Sasaki, *Integral models of Hilbert modular varieties in the ramified case, deformations of modular Galois representations, and weight one forms*, pre-print.
- [31] J.P.Serre, *Endomorphismes complètement continus des espaces de Banach  $p$ -adiques*, Publ. Math. IHES 12 (1962), 69-85.
- [32] J.P.Serre, *Topics in Galois theory*, Research Notes in Mathematics 1, Jones and Bartlett Publishers (1992).
- [33] N.Shepherd-Barron & R.Taylor, *Mod 2 and mod 5 icosahedral representations*, J. Amer. Math. Soc. 10 (1997), 283-298.
- [34] G.Shimura, *The special values of the zeta functions associated with Hilbert modular forms*, Duke Math. J. 45 (1978), 637-679.

- [35] C.Skinner & A.Wiles, *Nearly ordinary deformations of irreducible residual representations*, Annales de la Faculté des Sciences de Toulouse X (2001), 185-215.
- [36] R.Taylor, *Galois representations*, Annales de la Faculte des Sciences de Toulouse 13 (2004), 73-119.
- [37] R.Taylor, *Icosahedral Artin representations*, Pacific Journal of Math., Olga Taussky-Todd memorial issue (1997) 337-347.
- [38] R.Taylor, *On Galois representations associated to Hilbert modular forms*, Invent. Math. 98 (1989), 265-280.
- [39] R.Taylor, *On icosahedral Artin representations II*, Amer. J. Math 125 (2003), 549-566.
- [40] R.Taylor, *On the meromorphic continuation of degree two  $L$ -functions*, Documenta Mathematica, Extra Volume: John Coates' Sixtieth Birthday (2006), 729-779.
- [41] I.Vollaard, *On the Hilbert-Blumenthal moduli problem*, Journal of the Institute of Mathematics of Jussieu 4 (2005), 653-683.
- [42] A.Wiles, *Modular elliptic curves and Fermat's Last Theorem*, Annals of Math. 142 (1995), 443-551.
- [43] A.Wiles, *On ordinary  $\lambda$ -adic representations associated to modular forms*, Invent. Math. 94 (1988), 529-573.

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