

EXTENDING SELF-MAPS  
TO PROJECTIVE SPACE OVER FINITE FIELDS

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ABSTRACT. Using the closed point sieve, we extend to finite fields the following theorem proved by A. Bhatnagar and L. Szpiro over infinite fields: if  $X$  is a closed subscheme of  $\mathbb{P}^n$  over a field, and  $\phi: X \rightarrow X$  satisfies  $\phi^*\mathcal{O}_X(1) \simeq \mathcal{O}_X(d)$  for some  $d \geq 2$ , then there exists  $r \geq 1$  such that  $\phi^r$  extends to a morphism  $\mathbb{P}^n \rightarrow \mathbb{P}^n$ .

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## 1 INTRODUCTION

Let  $k$  be a field. Given a closed subscheme  $X \subseteq \mathbb{P}^n$  over  $k$ , and given a self-map (i.e.,  $k$ -scheme endomorphism)  $\phi: X \rightarrow X$ , does  $\phi$  extend to a self-map  $\psi: \mathbb{P}^n \rightarrow \mathbb{P}^n$ ? Such questions have applications in arithmetic dynamics: for instance, [Fak03, Corollary 2.4] uses a positive answer to a variant of this to show that the Morton–Silverman uniform boundedness conjecture for preperiodic points of a self-map of projective space over a number field [MS94, p. 100] implies the uniform boundedness conjecture for torsion points on abelian varieties over a number field.

If the extension  $\psi$  exists, then  $\psi^*\mathcal{O}(1) \simeq \mathcal{O}(d)$  for some integer  $d$ , and then  $\phi^*\mathcal{O}_X(1) \simeq \mathcal{O}_X(d)$ . But A. Bhatnagar and L. Szpiro [BS12, Proposition 2.3] gave an example showing that the existence of  $d$  such that  $\phi^*\mathcal{O}_X(1) \simeq \mathcal{O}_X(d)$  is not sufficient for the extension  $\psi$  to exist.

To obtain an extension theorem, one can relax the requirements. Two ways of doing this lead to the following questions:

QUESTION 1.1 (Changing the embedding). Let  $X$  be a projective  $k$ -scheme. Let  $\mathcal{L}$  be an ample line bundle on  $X$ . Let  $\phi: X \rightarrow X$  be a morphism such that  $\phi^*\mathcal{L} \simeq \mathcal{L}^{\otimes d}$  for some  $d \geq 1$ . Does there exist a closed immersion  $X \hookrightarrow \mathbb{P}^n$  such that  $\phi$  extends to a morphism  $\mathbb{P}^n \rightarrow \mathbb{P}^n$ ?

QUESTION 1.2 (Replacing the self-map by a power). Let  $X$  be a closed subscheme of  $\mathbb{P}^n$  over  $k$ . Let  $\phi: X \rightarrow X$  be a morphism such that  $\phi^*\mathcal{O}_X(1) \simeq \mathcal{O}_X(d)$  for some  $d \geq 2$ . Then there exists  $r \geq 1$  such that  $\phi^r$  extends to a morphism  $\mathbb{P}^n \rightarrow \mathbb{P}^n$ .

*Remark 1.3.* Section 4 explains why we cannot allow  $d = 1$  in Question 1.2.

Suppose that  $k$  is infinite. Then the answer to both questions is yes: see [Fak03, Corollary 2.3] and [BS12, Theorem 2.1], respectively (in the proof of the latter, one should replace the prime avoidance lemma there by the lemma used in [Fak03], that a finite union of proper subspaces in a vector space over an infinite field cannot cover the whole space). A positive answer to Question 1.2 is also an immediate consequence of [Fak03, Proposition 2.1] if one notices that the statement and proof there remain valid if hypothesis (1) is imposed only for  $n = d$  instead of all  $n \geq 0$ . (The word “variety” in [Fak03] and [BS12] may be read as “scheme of finite type”, so there is no difference between “projective variety” and “projective scheme”.)

Our main result is the following:

THEOREM 1.4. *Question 1.2 has a positive answer over any field  $k$ .*

In the case where  $k$  is finite, the general position arguments in [Fak03] and [BS12] fail, so a new idea is needed. To prove Theorem 1.4, we use the closed point sieve introduced in [Poo04] to show that a *random* choice leads to an extension of  $\phi$ , even though we cannot exhibit one explicitly. As far as we know, this is the first time that sieve techniques have been applied to a problem in dynamics.

*Remark 1.5.* See [MZMS13, Theorem 3] for an analogous statement on self-maps of equicharacteristic complete local rings.

*Remark 1.6.* We still do not know if Question 1.1 has a positive answer when  $k$  is finite.

## 2 EXTENDING MORPHISMS TO PROJECTIVE SPACE

The finite field case of Theorem 1.4 will be proved with the aid of the following quantitative theorem, involving a zeta function  $\zeta_U(s)$  defined as in [Poo04]:

THEOREM 2.1. *Let  $k$  be a finite field  $\mathbb{F}_q$ . Fix a closed subscheme  $X$  of  $\mathbb{P}^n = \text{Proj } S$  over  $k$ . Let  $U := \mathbb{P}^n - X$ . Let  $I = \bigoplus_{d \geq 0} I_d \subseteq S = \bigoplus_{d \geq 0} S_d$  be the homogeneous ideal of  $X \subseteq \mathbb{P}^n$ . Let  $N \geq n$ . Fix  $f_0, \dots, f_N \in S_d$ . Then if*

$g_0, \dots, g_N$  are chosen independently and uniformly at random from the finite set  $I_d$ ,

$$\text{Prob}(f_0 + g_0, \dots, f_N + g_N \text{ have no common zeros on } U) = \zeta_U(N+1)^{-1} + o(1),$$

where the  $o(1)$  is bounded by a function of  $k, X, n, N$ , and  $d$  that tends to 0 as  $d \rightarrow \infty$  while  $k, X, n$ , and  $N$  are fixed.

Theorem 2.1 will be proved in Section 3. For now, we show how it implies Theorem 1.4, through the following:

**THEOREM 2.2.** *Fix a closed subscheme  $X$  of  $\mathbb{P}^n$  over a field  $k$ . If  $d$  is sufficiently large and  $N \geq n$ , then any morphism  $\phi: X \rightarrow \mathbb{P}^N$  such that  $\phi^* \mathcal{O}(1) \simeq \mathcal{O}_X(d)$  extends to a morphism  $\mathbb{P}^n \rightarrow \mathbb{P}^N$ .*

*Proof.* Let  $z_0, \dots, z_N$  be the homogeneous coordinates on  $\mathbb{P}^N$ . For sufficiently large  $d$ , the restriction map  $S_d = \Gamma(\mathbb{P}^n, \mathcal{O}(d)) \rightarrow \Gamma(X, \mathcal{O}_X(d))$  is surjective. So each  $\phi^*(z_i)$  is the restriction of some  $f_i \in S_d$ .

If  $k$  is infinite, the proof of [Fak03, Proposition 2.1] applies for any  $d$  that is moreover large enough that  $X$  is cut out in  $\mathbb{P}^n$  by homogeneous polynomials of degree at most  $d$ .

If  $k$  is finite, Theorem 2.1 implies that for sufficiently large  $d$ , there exist  $g_0, \dots, g_N \in I_d$  such that  $f_0 + g_0, \dots, f_N + g_N$  have no common zeros in  $\mathbb{P}^n - X$ . On the other hand, restricted to  $X$ , they define the same map  $\phi$  as  $f_0, \dots, f_N$  do, so they have no common zeros on  $X$  either. Thus  $f_0 + g_0, \dots, f_N + g_N$  define a morphism  $\mathbb{P}^n \rightarrow \mathbb{P}^N$  extending  $\phi$ .  $\square$

*Proof of Theorem 1.4.* Apply Theorem 2.2 with  $N = n$  and with  $\phi$  equal to a sufficiently large power of the  $\phi$  given in Theorem 1.4.  $\square$

### 3 PROOF OF THEOREM 2.1

The idea of the proof of Theorem 2.1, borrowed from [Poo04], is to sieve out, for each closed point  $P \in U$ , the  $(g_0, \dots, g_N)$  for which  $f_0 + g_0, \dots, f_N + g_N$  have a common zero at  $P$ . Heuristically, the probability that a given  $f_i + g_i$  vanishes at  $P$  is  $q^{-\deg P}$ , so, assuming independence, the probability that  $f_0 + g_0, \dots, f_N + g_N$  have no common zeros on  $U$  should be

$$\prod_{\text{closed } P \in U} \left(1 - q^{-(N+1) \deg P}\right) = \zeta_U(N+1)^{-1}.$$

But independence holds only for finitely many  $P$ , so to make this rigorous, we impose the conditions only for  $P$  of degree up to some bound  $\rho$ , and then prove that the number of  $(g_0, \dots, g_N)$  sieved out by higher-degree closed points is negligible.

## 3.1 POINTS OF LOW DEGREE

Let  $f = (f_0, \dots, f_N)$  and  $g = (g_0, \dots, g_N)$ . Let  $V(f + g)$  be the common zero locus of the  $f_i + g_i$ . Given  $\rho \in \mathbb{Z}_{>0}$  and a  $k$ -scheme  $Z$ , let  $Z_{<\rho}$  be the set of closed points of  $Z$  of degree less than  $\rho$ , and define  $Z_{>\rho}$  similarly.

LEMMA 3.1 (Points of low degree). *For fixed  $\rho$ , if  $d$  is sufficiently large, then*

$$\text{Prob}(V(f + g) \cap U_{<\rho} = \emptyset) = \prod_{P \in U_{<\rho}} \left(1 - q^{-(N+1) \deg P}\right).$$

*Proof.* Let  $\mathcal{I}$  be the ideal sheaf of  $X \subseteq \mathbb{P}^n$ . View  $U_{<\rho}$  as a 0-dimensional closed subscheme of  $\mathbb{P}^n$ . By [Poo08, Lemma 2.1], if  $d$  is sufficiently large, then the restriction map  $I_d \rightarrow \Gamma(U_{<\rho}, \mathcal{I} \cdot \mathcal{O}_{U_{<\rho}}(d))$  is surjective. In particular, for each  $i$ , the tuple of “values”  $((f_i + g_i)(P))_{P \in U_{<\rho}}$  is equidistributed. The residue field at  $P$  has size  $q^{\deg P}$ , so the probability that  $f + g$  vanishes at  $P$  is  $q^{-(N+1) \deg P}$ , and the probability that  $f + g$  is nonvanishing at all  $P \in U_{<\rho}$  is

$$\prod_{P \in U_{<\rho}} \left(1 - q^{-(N+1) \deg P}\right). \quad \square$$

## 3.2 POINTS OF MEDIUM DEGREE

Let  $U_{a \leq ? \leq b}$  be the set of closed points of  $U$  of degree between  $a$  and  $b$ . As in [Poo08, Section 2], fix  $c$  so that  $S_1 I_m = I_{m+1}$  for all  $m \geq c$ .

LEMMA 3.2 (Points of medium degree). *If  $d$  is sufficiently large, then*

$$\text{Prob}(V(f + g) \cap U_{\rho \leq ? \leq d-c} = \emptyset) = O(q^{-\rho}).$$

*Proof.* By [Poo08, Lemma 2.2], the fraction of  $h \in I_d$  vanishing at a closed point  $P$  of degree  $e \in [\rho, d - c]$  is at most  $q^{-\min(d-c, e)} = q^{-e}$ . The set of  $g_i \in I_d$  such that  $f_i + g_i$  vanishes at  $P$  is either empty or a coset of this set of polynomials  $h$ , so  $\text{Prob}(f_i + g_i \text{ vanishes at } P) \leq q^{-e}$ . Hence  $\text{Prob}(f + g \text{ vanishes at } P) \leq q^{-(N+1)e}$ . Summing over all  $P \in U_{\rho \leq ? \leq d-c}$  and using the trivial bound that  $U$  contains  $O(q^{Ne})$  closed points of degree  $e$  yields

$$\sum_{e=\rho}^{d-c} O(q^{Ne}) q^{-(N+1)e} = O(q^{-\rho}). \quad \square$$

## 3.3 POINTS OF HIGH DEGREE

LEMMA 3.3. *Given a closed subvariety  $Z \subset \mathbb{P}^n$  such that  $\dim Z \cap U > 0$ , the probability that a random  $h \in I_d$  vanishes identically on  $Z$  is at most  $q^{-(d-c)}$ .*

*Proof.* Choose  $P \in (Z \cap U)_{>d-c}$ . If  $h$  vanishes on  $Z$ , it vanishes at  $P$ . By [Poo08, Lemma 4.1],  $\text{Prob}(h(P) = 0) \leq q^{-(d-c)}$ .  $\square$

LEMMA 3.4 (Points of high degree). *We have*

$$\text{Prob}(V(f + g) \cap U_{>d-c} = \emptyset) = 1 - o(1)$$

as  $d \rightarrow \infty$ .

*Proof.* Let  $W_{-1} = \mathbb{P}^n$ . For  $i = 0, \dots, N$ , let  $W_i$  be the common zero locus of  $f_0 + g_0, \dots, f_i + g_i$ . We pick  $g_0, \dots, g_N$  randomly one at a time.

*Claim 1:* For  $i = -1, \dots, n - 2$ , conditioned on a choice of  $g_0, \dots, g_i$  for which  $\dim W_i \cap U = n - i - 1$ , the probability that  $\dim W_{i+1} \cap U = n - i - 2$  is  $1 - o(1)$  as  $d \rightarrow \infty$ .

*Proof of Claim 1:* We have  $\dim W_{i+1} \cap U = n - i - 2$  if  $f_{i+1} + g_{i+1}$  does not vanish identically on any irreducible component of  $W_i \cap U$ . The number of such components is at most the number of components of  $W_i$ , which, by Bézout’s theorem as in [Ful84, p. 10], is at most  $O(d^{i+1})$ . For each component  $Z$  meeting  $U$ , the set of  $g_{i+1}$  such that  $f_{i+1} + g_{i+1}$  vanishes identically on  $Z$  is either empty or a coset of the subspace of  $h \in I_d$  vanishing identically on  $Z$ , and the probability that  $h$  vanishes on  $Z$  is at most  $q^{-(d-c)}$ , by Lemma 3.3. Thus the desired probability is at least  $1 - O(d^{i+1})q^{-(d-c)} = 1 - o(1)$ .

*Claim 2:* Conditioned on a choice of  $g_0, \dots, g_{n-1}$  for which  $\dim W_{n-1} \cap U$  is finite,  $\text{Prob}(W_n \cap U_{>d-c} = \emptyset) = 1 - o(1)$  as  $d \rightarrow \infty$ .

*Proof of Claim 2:* By Bézout’s theorem again,  $\#(W_{n-1} \cap U) = O(d^n)$ . For each  $P \in W_{n-1} \cap U$ , the set of  $g_n \in I_d$  such that  $f_n + g_n$  vanishes at  $P$  is either empty or a coset of the subspace of  $h \in I_d$  vanishing at  $P$ . If, moreover,  $\deg P > d - c$ , then  $\text{Prob}(h(P) = 0) \leq q^{-(d-c)}$  by [Poo08, Lemma 4.1]. Thus the desired probability is at least  $1 - O(d^n)q^{-(d-c)} = 1 - o(1)$  as  $d \rightarrow \infty$ .

Applying Claim 1 inductively and finally Claim 2 shows that with probability  $1 - o(1)$ , we have  $W_n \cap U_{>d-c} = \emptyset$  and hence also  $V(f + g) \cap U_{>d-c} = \emptyset$  since  $V(f + g) \subseteq W_n$ .  $\square$

### 3.4 END OF PROOF

Combining Lemmas 3.1, 3.2, and 3.4 shows that for any  $\rho \in \mathbb{Z}_{>0}$ ,

$$\text{Prob}(V(f + g) \cap U = \emptyset) = \prod_{P \in U_{<\rho}} (1 - q^{-(N+1)\deg P}) - O(q^{-\rho}) - o(1)$$

as  $d \rightarrow \infty$ . Applying this to larger and larger  $\rho$  completes the proof of Theorem 2.1.

## 4 A COUNTEREXAMPLE

Here we show that Question 1.2 has a negative answer if we allow  $d = 1$ , even for projective integral varieties over  $k = \mathbb{C}$ . Our counterexample is inspired by [BS12, Proposition 2.3].

Let  $k = \mathbb{C}$ . Let  $X$  be the image of the morphism  $\mathbb{P}^1 \rightarrow \mathbb{P}^3$  given by  $(x : y) \mapsto (x^4 : x^3y : xy^3 : y^4)$ . Let  $\phi: X \rightarrow X$  correspond under  $X \simeq \mathbb{P}^1$  to the automorphism of  $\mathbb{P}^1$  given by  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ . For  $r \geq 1$ , the self-map  $\phi^r$  corresponds to  $\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$ . But this does not preserve the span of  $\{x^4, x^3y, xy^3, y^4\}$ , since the coefficient of  $x^2y^2$  in  $(x + ry)^4$  is nonzero. Thus  $\phi^r$  cannot be the restriction of an automorphism of  $\mathbb{P}^3$ .

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