

TORELLI THEOREM FOR THE
DELIGNE–HITCHIN MODULI SPACE, II

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ABSTRACT. Let X and X' be compact Riemann surfaces of genus at least three. Let G and G' be nontrivial connected semisimple linear algebraic groups over \mathbb{C} . If some components $\mathcal{M}_{\text{DH}}^d(X, G)$ and $\mathcal{M}_{\text{DH}}^{d'}(X', G')$ of the associated Deligne–Hitchin moduli spaces are biholomorphic, then X' is isomorphic to X or to the conjugate Riemann surface \bar{X} .

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1. INTRODUCTION

Let X be a compact connected Riemann surface of genus $g \geq 3$. Let \bar{X} denote the conjugate Riemann surface; by definition, it consists of the real manifold underlying X and the almost complex structure $J_{\bar{X}} := -J_X$. Let G be a nontrivial connected semisimple linear algebraic group over \mathbb{C} . The topological types of holomorphic principal G -bundles E over X correspond to elements of $\pi_1(G)$. Let $\mathcal{M}_{\text{Higgs}}^d(X, G)$ denote the moduli space of semistable Higgs G -bundles (E, θ) over X with E of topological type $d \in \pi_1(G)$.

The *Deligne–Hitchin moduli space* [Si3] is a complex analytic space $\mathcal{M}_{\text{DH}}^d(X, G)$ associated to X , G and d . It is the twistor space for the hyper-Kähler structure on $\mathcal{M}_{\text{Higgs}}^d(X, G)$; see [Hi2, §9]. Deligne [De] has constructed it together with a surjective holomorphic map

$$\mathcal{M}_{\text{DH}}^d(X, G) \longrightarrow \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}.$$

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The inverse image of $\mathbb{C} \subseteq \mathbb{CP}^1$ is the moduli space $\mathcal{M}_{\text{Hod}}^d(X, G)$ of holomorphic principal G -bundles over X endowed with a λ -connection. In particular, every fiber over $\mathbb{C}^* \subset \mathbb{CP}^1$ is isomorphic to the moduli space of holomorphic G -connections over X . The fiber over $0 \in \mathbb{CP}^1$ is $\mathcal{M}_{\text{Higgs}}^d(X, G)$, and the fiber over $\infty \in \mathbb{CP}^1$ is $\mathcal{M}_{\text{Higgs}}^{-d}(\overline{X}, G)$.

In this paper, we study the dependence of these moduli spaces on X . Our main result, Theorem 5.3, states that the complex analytic space $\mathcal{M}_{\text{DH}}^d(X, G)$ determines the unordered pair $\{X, \overline{X}\}$ up to isomorphism. We also prove that $\mathcal{M}_{\text{Higgs}}^d(X, G)$ and $\mathcal{M}_{\text{Hod}}^d(X, G)$ each determine X up to isomorphism; see Theorem 5.1 and Theorem 5.2.

The key technical result is Proposition 3.1, which says the following: Let Z be an irreducible component of the fixed point locus for the natural \mathbb{C}^* -action on a moduli space $\mathcal{M}_{\text{Higgs}}^d(X, G)$ of Higgs G -bundles. Then,

$$\dim Z \leq (g - 1) \cdot \dim_{\mathbb{C}} G,$$

with equality holding only if Z is the moduli space $\mathcal{M}^d(X, G)$ of holomorphic principal G -bundles of topological type d over X .

In [BGHL], the case of $G = \text{SL}(r, \mathbb{C})$ was considered.

If $g = 2$ and $G = \text{SL}(2, \mathbb{C})$, then the above theorems are no longer valid. So we have assumed that $g \geq 3$.

2. SOME MODULI SPACES ASSOCIATED TO A COMPACT RIEMANN SURFACE

Let X be a compact connected Riemann surface of genus $g \geq 3$. Let G be a nontrivial connected semisimple linear algebraic group defined over \mathbb{C} , with Lie algebra \mathfrak{g} .

2.1. PRINCIPAL G -BUNDLES. We consider holomorphic principal G -bundles E over X . Recall that the topological type of E is given by an element $d \in \pi_1(G)$ [Ra]; this is a finite abelian group. The *adjoint vector bundle* of E is the holomorphic vector bundle

$$\text{ad}(E) := E \times^G \mathfrak{g}$$

over X , using the adjoint action of G on \mathfrak{g} . E is called *stable* (respectively, *semistable*) if

$$(1) \quad \text{degree}(\text{ad}(E_P)) < 0 \quad (\text{respectively, } \leq 0)$$

for every maximal parabolic subgroup $P \subsetneq G$ and every holomorphic reduction of structure group E_P of E to P ; here $\text{ad}(E_P) \subset \text{ad}(E)$ is the adjoint vector bundle of E_P .

Let $\mathcal{M}^d(X, G)$ denote the moduli space of semistable holomorphic principal G -bundles E over X of topological type $d \in \pi_1(G)$. It is known that $\mathcal{M}^d(X, G)$ is an irreducible normal projective variety of dimension $(g - 1) \cdot \dim_{\mathbb{C}} G$ over \mathbb{C} .

2.2. HIGGS G -BUNDLES. The holomorphic cotangent bundle of X will be denoted by K_X .

A *Higgs G -bundle* over X is a pair (E, θ) consisting of a holomorphic principal G -bundle E over X and a holomorphic section

$$\theta \in H^0(X, \text{ad}(E) \otimes K_X),$$

the so-called *Higgs field* [Hi1, Si1]. The pair (E, θ) is called *stable* (respectively, *semistable*) if the inequality (1) holds for every holomorphic reduction of structure group E_P of E to a maximal parabolic subgroup $P \subsetneq G$ such that $\theta \in H^0(X, \text{ad}(E_P) \otimes K_X)$.

Let $\mathcal{M}_{\text{Higgs}}^d(X, G)$ denote the moduli space of semistable Higgs G -bundles (E, θ) over X such that E is of topological type $d \in \pi_1(G)$. It is known that $\mathcal{M}_{\text{Higgs}}^d(X, G)$ is an irreducible normal quasiprojective variety of dimension $2(g-1) \cdot \dim_{\mathbb{C}} G$ over \mathbb{C} [Si2]. We regard $\mathcal{M}^d(X, G)$ as a closed subvariety of $\mathcal{M}_{\text{Higgs}}^d(X, G)$ by means of the embedding

$$\mathcal{M}^d(X, G) \hookrightarrow \mathcal{M}_{\text{Higgs}}^d(X, G), \quad E \longmapsto (E, 0).$$

There is a natural algebraic symplectic structure on $\mathcal{M}_{\text{Higgs}}^d(X, G)$; see [Hi1, BR].

2.3. REPRESENTATIONS OF THE SURFACE GROUP IN G . Fix a base point $x_0 \in X$. The fundamental group of X admits a standard presentation

$$\pi_1(X, x_0) \cong \langle a_1, \dots, a_g, b_1, \dots, b_g \mid \prod_{i=1}^g a_i b_i a_i^{-1} b_i^{-1} = 1 \rangle$$

which we choose in such a way that it is compatible with the orientation of X . We identify the fundamental group of G with the kernel of the universal covering $\tilde{G} \rightarrow G$. The *type* $d \in \pi_1(G)$ of a homomorphism $\rho : \pi_1(X, x_0) \rightarrow G$ is defined by

$$d := \prod_{i=1}^g \alpha_i \beta_i \alpha_i^{-1} \beta_i^{-1} \in \pi_1(G) \subset \tilde{G}$$

for any choice of lifts $\alpha_i, \beta_i \in \tilde{G}$ of $\rho(a_i), \rho(b_i) \in G$. This is also the topological type of the principal G -bundle E_ρ over X given by ρ . The space $\text{Hom}^d(\pi_1(X, x_0), G)$ of all homomorphisms $\rho : \pi_1(X, x_0) \rightarrow G$ of type $d \in \pi_1(G)$ is an irreducible affine variety over \mathbb{C} , and G acts on it by conjugation. The GIT quotient

$$\mathcal{M}_{\text{Rep}}^d(X, G) := \text{Hom}^d(\pi_1(X, x_0), G) // G$$

doesn't depend on x_0 . It is an affine variety of dimension $2(g-1) \cdot \dim_{\mathbb{C}} G$ over \mathbb{C} , which carries a natural symplectic form [AB, Go]. Its points represent equivalence classes of completely reducible homomorphisms ρ . There is a natural bijective map

$$\mathcal{M}_{\text{Rep}}^d(X, G) \longrightarrow \mathcal{M}_{\text{Higgs}}^d(X, G)$$

given by a variant of the Kobayashi–Hitchin correspondence [Si1]. This bijective map is not holomorphic.

2.4. HOLOMORPHIC G –CONNECTIONS. Let $p : E \rightarrow X$ be a holomorphic principal G –bundle. Because the vertical tangent space at every point of the total space E is canonically isomorphic to \mathfrak{g} , there is a natural exact sequence

$$0 \rightarrow E \times \mathfrak{g} \rightarrow TE \xrightarrow{dp} p^*TX \rightarrow 0$$

of G –equivariant holomorphic vector bundles over E . Taking the G –invariant direct image under p , it follows that the *Atiyah bundle* for E

$$\text{At}(E) := p_*(TE)^G \subset p_*(TE)$$

sits in a natural exact sequence of holomorphic vector bundles

$$(2) \quad 0 \rightarrow \text{ad}(E) \rightarrow \text{At}(E) \xrightarrow{dp} TX \rightarrow 0$$

over X . This exact sequence is called the *Atiyah sequence*. A *holomorphic connection* on E is a splitting of the Atiyah sequence, or in other words a holomorphic homomorphism

$$D : TX \rightarrow \text{At}(E)$$

such that $dp \circ D = \text{id}_{TX} [\text{At}]$. It always exists if E is semistable [AzBi, p. 342, Theorem 4.1], [BG, p. 20, Theorem 1.1]. The curvature of D is a holomorphic 2–form with values in $\text{ad}(E)$, so D is automatically flat.

A *holomorphic G –connection* is a pair (E, D) where E is a holomorphic principal G –bundle over X , and D is a holomorphic connection on E . Such a pair is automatically semistable, because the degree of a flat vector bundle is zero. Let $\mathcal{M}_{\text{conn}}^d(X, G)$ denote the moduli space of holomorphic G –connections (E, D) over X such that E is of topological type $d \in \pi_1(G)$. It is known that $\mathcal{M}_{\text{conn}}^d(X, G)$ is an irreducible quasiprojective variety of dimension $2(g - 1) \cdot \dim_{\mathbb{C}} G$ over \mathbb{C} .

Sending each holomorphic G –connection to its monodromy defines a map

$$(3) \quad \mathcal{M}_{\text{conn}}^d(X, G) \rightarrow \mathcal{M}_{\text{Rep}}^d(X, G)$$

which is biholomorphic, but not algebraic; it is called Riemann–Hilbert correspondence. The inverse map sends a homomorphism $\rho : \pi_1(X, x_0) \rightarrow G$ to the associated principal G –bundle E_ρ , endowed with the induced holomorphic connection D_ρ .

2.5. λ –CONNECTIONS. Let $p : E \rightarrow X$ be a holomorphic principal G –bundle. For any $\lambda \in \mathbb{C}$, a λ –connection on E is a holomorphic homomorphism of vector bundles

$$D : TX \rightarrow \text{At}(E)$$

such that $dp \circ D = \lambda \cdot \text{id}_{TX}$ for the epimorphism dp in the Atiyah sequence (2). Therefore, a 0–connection is a Higgs field, and a 1–connection is a holomorphic connection.

If D is a λ –connection on E with $\lambda \neq 0$, then $\lambda^{-1}D$ is a holomorphic connection on E . In particular, the pair (E, D) is automatically semistable in this case.

Let $\mathcal{M}_{\text{Hod}}^d(X, G)$ denote the moduli space of triples (λ, E, D) , where $\lambda \in \mathbb{C}$, E is a holomorphic principal G -bundle over X of topological type $d \in \pi_1(G)$, and D is a semistable λ -connection on E ; see [Si2]. There is a canonical algebraic map

$$(4) \quad \text{pr} = \text{pr}_X : \mathcal{M}_{\text{Hod}}^d(X, G) \longrightarrow \mathbb{C}, \quad (\lambda, E, D) \longmapsto \lambda.$$

Its fibers over $\lambda = 0$ and $\lambda = 1$ are $\mathcal{M}_{\text{Higgs}}^d(X, G)$ and $\mathcal{M}_{\text{conn}}^d(X, G)$, respectively. The Riemann–Hilbert correspondence (3) allows to define a holomorphic open embedding

$$j = j_X : \mathbb{C}^* \times \mathcal{M}_{\text{Rep}}^d(X, G) \hookrightarrow \mathcal{M}_{\text{Hod}}^d(X, G), \quad (\lambda, \rho) \longmapsto (\lambda, E_\rho, \lambda D_\rho)$$

with image $\text{pr}^{-1}(\mathbb{C}^*)$. This map commutes with the projections onto \mathbb{C}^* .

2.6. THE DELIGNE–HITCHIN MODULI SPACE. The compact Riemann surface X provides an underlying real C^∞ manifold $X_{\mathbb{R}}$, and an almost complex structure $J_X : TX_{\mathbb{R}} \rightarrow TX_{\mathbb{R}}$. Since any almost complex structure in real dimension two is integrable,

$$\overline{X} := (X_{\mathbb{R}}, -J_X)$$

is a compact Riemann surface as well. It has the opposite orientation, so

$$(5) \quad \mathcal{M}_{\text{Rep}}^d(X, G) = \mathcal{M}_{\text{Rep}}^{-d}(\overline{X}, G).$$

The *Deligne–Hitchin moduli space* $\mathcal{M}_{\text{DH}}^d(X, G)$ is the complex analytic space obtained by gluing $\mathcal{M}_{\text{Hod}}^d(X, G)$ and $\mathcal{M}_{\text{Hod}}^{-d}(\overline{X}, G)$ along their common open subspace

$$\mathcal{M}_{\text{Hod}}^d(X, G) \xleftarrow{j_X} \mathbb{C}^* \times \mathcal{M}_{\text{Rep}}^d(X, G) \cong \mathbb{C}^* \times \mathcal{M}_{\text{Rep}}^{-d}(\overline{X}, G) \xrightarrow{j_{\overline{X}}} \mathcal{M}_{\text{Hod}}^{-d}(\overline{X}, G)$$

where the isomorphism in the middle sends (λ, ρ) to $(1/\lambda, \rho)$; see [Si3, De]. The projections pr_X on $\mathcal{M}_{\text{Hod}}^d(X, G)$ and $1/\text{pr}_{\overline{X}}$ on $\mathcal{M}_{\text{Hod}}^{-d}(\overline{X}, G)$ patch together to a holomorphic map

$$\mathcal{M}_{\text{DH}}^d(X, G) \longrightarrow \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}.$$

Its fiber over any $\lambda \in \mathbb{C}^*$ is biholomorphic to the representation space (5), whereas its fibers over $\lambda = 0$ and $\lambda = \infty$ are $\mathcal{M}_{\text{Higgs}}^d(X, G)$ and $\mathcal{M}_{\text{Higgs}}^{-d}(\overline{X}, G)$, respectively.

3. FIXED POINTS OF THE NATURAL \mathbb{C}^* -ACTION

The group \mathbb{C}^* acts algebraically on the moduli space $\mathcal{M}_{\text{Higgs}}^d(X, G)$, via the formula

$$(6) \quad t \cdot (E, \theta) := (E, t\theta).$$

The fixed point locus $\mathcal{M}_{\text{Higgs}}^d(X, G)^{\mathbb{C}^*}$ contains the closed subvariety $\mathcal{M}^d(X, G)$.

PROPOSITION 3.1. *Let Z be an irreducible component of $\mathcal{M}_{\text{Higgs}}^d(X, G)^{\mathbb{C}^*}$. Then one has*

$$\dim Z \leq (g - 1) \cdot \dim_{\mathbb{C}} G,$$

with equality holding only for $Z = \mathcal{M}^d(X, G)$.

Proof. Let (E, θ) be a stable Higgs G -bundle over X . Its infinitesimal deformations are, according to [BR, Theorem 2.3], governed by the complex of vector bundles

$$(7) \quad C^0 := \text{ad}(E) \xrightarrow{\text{ad}(\theta)} \text{ad}(E) \otimes K_X =: C^1$$

over X . Since (E, θ) is stable, it has no infinitesimal automorphisms, so

$$\mathbb{H}^0(X, C^\bullet) = 0.$$

The Killing form on \mathfrak{g} induces isomorphisms $\mathfrak{g}^* \cong \mathfrak{g}$ and $\text{ad}(E)^* \cong \text{ad}(E)$. Hence the vector bundle $\text{ad}(E)$ has degree 0. Serre duality allows us to conclude

$$\mathbb{H}^2(X, C^\bullet) = 0.$$

Using all this, the Riemann–Roch formula yields

$$(8) \quad \dim \mathbb{H}^1(X, C^\bullet) = 2(g - 1) \cdot \dim_{\mathbb{C}} G.$$

From now on, we assume that the point (E, θ) is fixed by \mathbb{C}^* , and we also assume $\theta \neq 0$. Then $(E, \theta) \cong (E, t\theta)$ for all $t \in \mathbb{C}^*$, so the sequence of complex algebraic groups

$$1 \longrightarrow \text{Aut}(E, \theta) \longrightarrow \text{Aut}(E, \mathbb{C}\theta) \longrightarrow \text{Aut}(\mathbb{C}\theta) = \mathbb{C}^* \longrightarrow 1$$

is exact. Because (E, θ) is stable, $\text{Aut}(E, \theta)$ is finite. Consequently, the identity component of $\text{Aut}(E, \mathbb{C}\theta)$ is isomorphic to \mathbb{C}^* . This provides an embedding

$$\mathbb{C}^* \hookrightarrow \text{Aut}(E), \quad t \longmapsto \varphi_t,$$

and an integer $w \neq 0$ with $\varphi_t(\theta) = t^w \cdot \theta$ for all $t \in \mathbb{C}^*$. We may assume that $w \geq 1$.

Choose a point $e_0 \in E$. Then there is a unique group homomorphism

$$\iota : \mathbb{C}^* \longrightarrow G$$

such that $\varphi_t(e_0) = e_0 \cdot \iota(t)$ for all $t \in \mathbb{C}^*$. The conjugacy class of ι doesn't depend on e_0 , since the space of conjugacy classes $\text{Hom}(\mathbb{C}^*, G)/G$ is discrete. The subset

$$E_H := \{e \in E : \varphi_t(e) = e \cdot \iota(t) \text{ for all } t \in \mathbb{C}^*\}$$

of E is a holomorphic reduction of structure group to the centralizer H of $\iota(\mathbb{C}^*)$ in G . Let

$$(9) \quad \mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_n$$

denote the eigenspace decomposition given by the adjoint action of \mathbb{C}^* on \mathfrak{g} via ι .

Let $N \in \mathbb{Z}$ be maximal with $\mathfrak{g}_N \neq 0$. Let $P \subset G$ be the parabolic subgroup with

$$\text{Lie}(P) = \bigoplus_{n \geq 0} \mathfrak{g}_n \subset \mathfrak{g}.$$

Since $H \subset G$ has Lie algebra \mathfrak{g}_0 , it is a Levi subgroup in P . Choose subgroups

$$\iota(\mathbb{C}^*) \subseteq T \subseteq B \subseteq P \subset G$$

such that T is a maximal torus in $H \subset G$ and B is a Borel subgroup in G . Let

$$\alpha_j : T \longrightarrow \mathbb{C}^* \quad \text{and} \quad \alpha_j^\vee : \mathbb{C}^* \longrightarrow T$$

be the resulting simple roots and coroots of G . We denote by $\langle \cdot, \cdot \rangle$ the natural pairing between characters and cocharacters of T . Let α_j be a simple root of G with $\langle \alpha_j, \iota \rangle > 0$, and let β be a root of G with $\langle \beta, \iota \rangle = N$. Then the elementary reflection

$$s_j(\beta) = \beta - \langle \beta, \alpha_j^\vee \rangle \alpha_j$$

is a root of G , so $\langle s_j(\beta), \iota \rangle \leq N$; this implies that $\langle \beta, \alpha_j^\vee \rangle \geq 0$. The sum of all such roots β with $\langle \beta, \iota \rangle = N$ is the restriction $\chi|_T$ of the determinant

$$(10) \quad \chi : P \longrightarrow \text{Aut}(\mathfrak{g}_N) \xrightarrow{\det} \mathbb{C}^*$$

of the adjoint action of P on \mathfrak{g}_N . Hence we conclude $\langle \chi|_T, \alpha_j^\vee \rangle \geq 0$ for all simple roots α_j with $\langle \alpha_j, \iota \rangle > 0$. This means that the character χ of P is dominant. The decomposition (9) of \mathfrak{g} induces a vector bundle decomposition

$$\text{ad}(E) = \bigoplus_{n \in \mathbb{Z}} E_H \times^H \mathfrak{g}_n.$$

Since \mathbb{C}^* acts with weight w on the Higgs field θ by construction, we have

$$(11) \quad \theta \in H^0(X, (E_H \times^H \mathfrak{g}_w) \otimes K_X).$$

In particular, $\theta \in H^0(X, \text{ad}(E_P) \otimes K_X)$ for the reduction $E_P := E_H \times^H P \subseteq E$ of the structure group to P . The Higgs version of the stability criterion [Ra, Lemma 2.1] yields

$$\text{degree}(E_H \times^H \mathfrak{g}_N) \leq 0$$

since P acts on $\det(\mathfrak{g}_N)$ via the dominant character χ in (10). Now Riemann–Roch implies the following:

$$(12) \quad \dim H^1(X, E_H \times^H \mathfrak{g}_N) \geq (g - 1) \cdot \dim_{\mathbb{C}} \mathfrak{g}_N > 0.$$

The complex C^\bullet in (7) is, due to (11), the direct sum of its subcomplexes C_n^\bullet given by

$$C_n^0 := E_H \times^H \mathfrak{g}_n \xrightarrow{\text{ad}(\theta)} (E_H \times^H \mathfrak{g}_{n+w}) \otimes K_X =: C_n^1.$$

Thus the hypercohomology of C^\bullet decomposes as well; in particular, we have

$$H^1(X, C^\bullet) = \bigoplus_{n \in \mathbb{Z}} H^1(X, C_n^\bullet).$$

In the last nonzero summand C_N^\bullet , we have $C_N^1 = 0$ and hence

$$\dim H^1(X, C_N^\bullet) = \dim H^1(X, E_H \times^H \mathfrak{g}_N) > 0$$

due to (12). Since $\mathfrak{g}_n^* \cong \mathfrak{g}_{-n}$ via the Killing form on \mathfrak{g} , Serre duality yields in particular

$$\dim \mathbb{H}^1(X, C_0^\bullet) = \dim \mathbb{H}^1(X, C_{-w}^\bullet).$$

Taken together, the last three formulas and the equation (8) imply that

$$\dim \mathbb{H}^1(X, C_0^\bullet) < \frac{1}{2} \dim \mathbb{H}^1(X, C^\bullet) = (g - 1) \cdot \dim_{\mathbb{C}} G.$$

But $\mathbb{H}^1(X, C_0^\bullet)$ parameterizes infinitesimal deformations of pairs (E_H, θ) consisting of a principal H -bundle E_H and a section θ as in (11); see [BR, Theorem 2.3]. This proves that

$$\dim Z < (g - 1) \cdot \dim_{\mathbb{C}} G$$

for every irreducible component Z of the fixed point locus $\mathcal{M}_{\text{Higgs}}^d(X, G)^{\mathbb{C}^*}$ such that Z contains stable Higgs G -bundles (E, θ) with $\theta \neq 0$.

The non-stable points in $\mathcal{M}_{\text{Higgs}}^d(X, G)$ correspond to polystable Higgs G -bundles (E, θ) . Polystability means that E admits a reduction of structure group E_L to a Levi subgroup $L \subsetneq G$ of a parabolic subgroup in G such that θ is a section of the subbundle

$$\text{ad}(E_L) \otimes K_X \subset \text{ad}(E) \otimes K_X$$

and the pair (E_L, θ) is stable. Let $C \subseteq L$ be the identity component of the center, and let $\mathfrak{c} \subseteq \mathfrak{l}$ be their Lie algebras. Then $E_{L/C} := E_L/C$ is a principal (L/C) -bundle over X , and

$$\text{ad}(E_L) \cong (\mathfrak{c} \otimes \mathcal{O}_X) \oplus \text{ad}(E_{L/C})$$

since $\mathfrak{l} = \mathfrak{c} \oplus [\mathfrak{l}, \mathfrak{l}]$, where the subalgebra $[\mathfrak{l}, \mathfrak{l}] \subseteq \mathfrak{l}$ is also the Lie algebra of L/C . We have

$$\dim_{\mathbb{C}} G - \dim_{\mathbb{C}} L \geq 2 \dim_{\mathbb{C}} C$$

because maximal Levi subgroups in G have 1-dimensional center and at least one pair of opposite roots less than G ; the other Levi subgroups can be reached by iterating this.

Now suppose that \mathbb{C}^* fixes the point (E, θ) . Then $(E_L, \theta) \cong (E_L, t\theta)$ for all $t \in \mathbb{C}^*$. But the action of $\text{Aut}(E_L)$ on the direct summand $\mathfrak{c} \otimes \mathcal{O}_X$ of $\text{ad}(E_L)$ is trivial, since the adjoint action of L on \mathfrak{c} is trivial. So θ lives in the other summand of $\text{ad}(E_L)$, meaning

$$\theta \in H^0(X, \text{ad}(E_{L/C}) \otimes K_X).$$

The Higgs (L/C) -bundle $(E_{L/C}, \theta)$ is still stable and fixed by \mathbb{C}^* ; we have already proved that the locus of such has dimension $\leq (g - 1) \cdot \dim_{\mathbb{C}}(L/C)$. The abelian variety $\mathcal{M}^0(X, C)$ acts simply transitively on lifts of $E_{L/C}$ to a principal L -bundle E_L , so these lifts form a family of dimension $g \cdot \dim_{\mathbb{C}} C$. Hence the pairs (E_L, θ) in question have at most

$$(g - 1) \cdot \dim_{\mathbb{C}}(L/C) + g \cdot \dim_{\mathbb{C}} C < (g - 1) \cdot \dim_{\mathbb{C}} G$$

moduli. This implies that $\dim Z < (g - 1) \cdot \dim_{\mathbb{C}} G$ for each non-stable component Z of the fixed point locus, since there are only finitely many possibilities for L up to conjugation. \square

The algebraic \mathbb{C}^* -action (6) on $\mathcal{M}_{\text{Higgs}}^d(X, G)$ extends naturally to an algebraic \mathbb{C}^* -action on $\mathcal{M}_{\text{Hod}}^d(X, G)$, which is given by the formula

$$(13) \quad t \cdot (\lambda, E, D) := (t\lambda, E, tD).$$

A point (λ, E, D) can only be fixed by this action if $\lambda = 0$, so Proposition 3.1 yields the following corollary:

COROLLARY 3.2. *Let Z be an irreducible component of $\mathcal{M}_{\text{Hod}}^d(X, G)^{\mathbb{C}^*}$. Then one has*

$$\dim Z \leq (g - 1) \cdot \dim_{\mathbb{C}} G,$$

with equality only for $Z = \mathcal{M}^d(X, G)$.

The algebraic \mathbb{C}^* -action (13) on $\mathcal{M}_{\text{Hod}}^d(X, G)$ extends naturally to a holomorphic \mathbb{C}^* -action on $\mathcal{M}_{\text{DH}}^d(X, G)$, which is on the other open patch $\mathcal{M}_{\text{Hod}}^{-d}(\overline{X}, G)$ given by the formula

$$t \cdot (\lambda, E, D) := (t^{-1}\lambda, E, t^{-1}D).$$

Applying Corollary 3.2 to both $\mathcal{M}_{\text{Hod}}^d(X, G)$ and $\mathcal{M}_{\text{Hod}}^{-d}(\overline{X}, G)$, one immediately gets

COROLLARY 3.3. *Let Z be an irreducible component of $\mathcal{M}_{\text{DH}}^d(X, G)^{\mathbb{C}^*}$. Then one has*

$$\dim Z \leq (g - 1) \cdot \dim_{\mathbb{C}} G,$$

with equality only for $Z = \mathcal{M}^d(X, G)$ and for $Z = \mathcal{M}^{-d}(\overline{X}, G)$.

4. VECTOR FIELDS ON THE MODULI SPACES

A stable principal G -bundle E over X is called *regularly stable* if the automorphism group $\text{Aut}(E)$ is just the center of G . The regularly stable locus

$$\mathcal{M}^{d, \text{rs}}(X, G) \subseteq \mathcal{M}^d(X, G)$$

is open, and coincides with the smooth locus of $\mathcal{M}^d(X, G)$; see [BH, Corollary 3.4].

PROPOSITION 4.1. *There are no nonzero holomorphic vector fields on $\mathcal{M}^{d, \text{rs}}(X, G)$.*

Proof. This statement is contained in [Fa, p. 549, Corollary III.3]. □

PROPOSITION 4.2. *There are no nonzero holomorphic 1-forms on $\mathcal{M}^{d, \text{rs}}(X, G)$.*

Proof. The moduli space of Higgs G -bundles is equipped with the *Hitchin map*

$$\mathcal{M}_{\text{Higgs}}^d(X, G) \longrightarrow \bigoplus_{i=1}^{\text{rank}(G)} H^0(X, K_X^{\otimes n_i})$$

where the n_i are the degrees of generators for the algebra $\text{Sym}(\mathfrak{g}^*)^G$; see [Hi1, § 4], [La]. Any sufficiently general fiber of this Hitchin map is a complex abelian variety A (see [Do], [Fa], [DP] for the details), and

$$\varphi : A \dashrightarrow \mathcal{M}^{d, \text{rs}}(X, G), \quad (E, \theta) \longmapsto E,$$

is a dominant rational map. This rational map φ is defined outside a closed subscheme of codimension at least two; see [Fa, p. 534, Theorem II.6].

Let ω be a holomorphic 1-form on $\mathcal{M}^{d,rs}(X, G)$. Then $\varphi^*\omega$ extends to a holomorphic 1-form on A by Hartog’s theorem. As any holomorphic 1-form on A is closed, it follows that ω is closed. Since $H^1(\mathcal{M}^{d,rs}(X, G), \mathbb{C}) = 0$ by [AB, Ch. 10], we conclude $\omega = df$ for a holomorphic function f on $\mathcal{M}^{d,rs}(X, G)$. But any such function f is constant, so $\omega = 0$. \square

We denote by $\mathcal{M}_{\text{Higgs}}^{d,rs}(X, G) \subseteq \mathcal{M}_{\text{Higgs}}^d(X, G)$ the open locus of Higgs G -bundles (E, θ) for which E is regularly stable. The forgetful map

$$(14) \quad \mathcal{M}_{\text{Higgs}}^{d,rs}(X, G) \longrightarrow \mathcal{M}^{d,rs}(X, G), \quad (E, \theta) \longmapsto E,$$

is an algebraic vector bundle with fibers $H^0(X, \text{ad}(E) \otimes K_X) \cong H^1(X, \text{ad}(E))^*$, so it is the cotangent bundle of $\mathcal{M}^{d,rs}(X, G)$.

COROLLARY 4.3. *The restriction of the algebraic tangent bundle*

$$T\mathcal{M}_{\text{Higgs}}^{d,rs}(X, G) \longrightarrow \mathcal{M}_{\text{Higgs}}^{d,rs}(X, G)$$

to the subvariety $\mathcal{M}^{d,rs}(X, G) \subseteq \mathcal{M}_{\text{Higgs}}^{d,rs}(X, G)$ has no nonzero holomorphic sections.

Proof. The subvariety in question is the zero section of the vector bundle (14). Given a vector bundle $V \longrightarrow M$ with zero section $M \subseteq V$, there is a natural isomorphism

$$(15) \quad (TV)|_M \cong TM \oplus V$$

of vector bundles over M . In our situation, both summands have no nonzero holomorphic sections, according to Proposition 4.1 and Proposition 4.2. \square

Let $\mathcal{M}_{\text{conn}}^{d,rs}(X, G) \subseteq \mathcal{M}_{\text{conn}}^d(X, G)$ denote the open locus of holomorphic G -connections (E, D) for which E is regularly stable.

PROPOSITION 4.4. *There are no holomorphic sections for the forgetful map*

$$(16) \quad \mathcal{M}_{\text{conn}}^{d,rs}(X, G) \longrightarrow \mathcal{M}^{d,rs}(X, G), \quad (E, D) \longmapsto E.$$

Proof. The map (16) is a holomorphic torsor under the cotangent bundle of $\mathcal{M}^{d,rs}(X, G)$. As such, it is isomorphic to the torsor of holomorphic connections on the line bundle

$$\mathcal{L} \longrightarrow \mathcal{M}^{d,rs}(X, G)$$

with fibers $\det H^1(X, \text{ad}(E))$; see [Fa, p. 554, Lemma IV.4]. Since \mathcal{L} is ample [KNR], its first Chern class is nonzero, so \mathcal{L} admits no global holomorphic connections. \square

Let $\mathcal{M}_{\text{Hod}}^{d,rs}(X, G) \subseteq \mathcal{M}_{\text{Hod}}^d(X, G)$ denote the open locus of triples (λ, E, D) for which E is regularly stable. The forgetful maps in (14) and (16) extend to the forgetful map

$$(17) \quad \mathcal{M}_{\text{Hod}}^{d,rs}(X, G) \longrightarrow \mathcal{M}^{d,rs}(X, G), \quad (\lambda, E, D) \longmapsto E,$$

which is an algebraic vector bundle. It contains the cotangent bundle (14) as a subbundle; the quotient is a line bundle, which is trivialized by the projection pr in (4).

COROLLARY 4.5. *The vector bundle (17) has no nonzero holomorphic sections.*

Proof. Let s be a holomorphic section of the vector bundle (17). Then $\text{pr} \circ s$ is a holomorphic function on $\mathcal{M}^{d,\text{rs}}(X, G)$, and hence constant. This constant vanishes because of Proposition 4.4. So $\text{pr} \circ s = 0$, which implies that $s = 0$ using Proposition 4.2. \square

COROLLARY 4.6. *The restriction of the algebraic tangent bundle*

$$T\mathcal{M}_{\text{Hod}}^{d,\text{rs}}(X, G) \longrightarrow \mathcal{M}_{\text{Hod}}^{d,\text{rs}}(X, G)$$

to the subvariety $\mathcal{M}^{d,\text{rs}}(X, G) \subseteq \mathcal{M}_{\text{Hod}}^{d,\text{rs}}(X, G)$ has no nonzero holomorphic sections.

Proof. Use the decomposition (15), Proposition 4.1, and Corollary 4.5. \square

5. TORELLI THEOREMS

Let X, X' be compact connected Riemann surfaces of genus ≥ 3 . Let G, G' be nontrivial connected semisimple linear algebraic groups over \mathbb{C} . Fix $d \in \pi_1(G)$ and $d' \in \pi_1(G')$.

THEOREM 5.1. *If $\mathcal{M}_{\text{Higgs}}^{d'}(X', G')$ is biholomorphic to $\mathcal{M}_{\text{Higgs}}^d(X, G)$, then $X' \cong X$.*

Proof. Corollary 4.3 implies that the subvariety $\mathcal{M}^d(X, G)$ is fixed pointwise by every holomorphic \mathbb{C}^* -action on $\mathcal{M}_{\text{Higgs}}^d(X, G)$. All other complex analytic subvarieties with that property have smaller dimension, due to Proposition 3.1. Thus we get a biholomorphic map from $\mathcal{M}^{d'}(X', G')$ to $\mathcal{M}^d(X, G)$ by restriction. Using [BH], this implies that $X' \cong X$. \square

THEOREM 5.2. *If $\mathcal{M}_{\text{Hod}}^{d'}(X', G')$ is biholomorphic to $\mathcal{M}_{\text{Hod}}^d(X, G)$, then $X' \cong X$.*

Proof. The argument is exactly the same as in the previous proof. It suffices to replace Corollary 4.3 by Corollary 4.6, and Proposition 3.1 by Corollary 3.2. \square

THEOREM 5.3. *If $\mathcal{M}_{\text{DH}}^{d'}(X', G')$ is biholomorphic to $\mathcal{M}_{\text{DH}}^d(X, G)$, then $X' \cong X$ or $X' \cong \overline{X}$.*

Proof. The argument is similar. Corollary 4.6 implies that the two subvarieties $\mathcal{M}^d(X, G)$ and $\mathcal{M}^{-d}(\overline{X}, G)$ are fixed pointwise by every holomorphic \mathbb{C}^* -action on $\mathcal{M}_{\text{DH}}^d(X, G)$. All other complex analytic subvarieties with that property have smaller dimension, due to Corollary 3.3. Thus we get a biholomorphic map from $\mathcal{M}^{d'}(X', G')$ to either $\mathcal{M}^d(X, G)$ or $\mathcal{M}^{-d}(\overline{X}, G)$ by restriction. Using [BH], this implies that either $X' \cong X$ or $X' \cong \overline{X}$. \square

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