Essential Whittaker Functions for $GL(n)$

Nadir Matringe

Received: July 6, 2012
Revised: July 19, 2013

Communicated by Patrick Delorme

Abstract. We give a constructive proof of the existence of the essential Whittaker function of a generic representation of $GL(n, F)$, for F a non-archimedean local field, using mirabolic restriction techniques.

2010 Mathematics Subject Classification: 22E50

Introduction

Let F be nonarchimedean local field, we denote by $\mathfrak{O}$ its ring of integers, and by $\mathfrak{P} = \varpi\mathfrak{O}$ the maximal ideal of this ring, where ϖ is a uniformiser of F. We denote by q the cardinality of $\mathfrak{O}/\mathfrak{P}$ and by $|.|$ the absolute value on F normalised such that $|\varpi|$ is equal to q^{-1}.
For $n \geq 1$, we denote the group $GL(n, F)$ by G_n, the group $GL(n, \mathfrak{O})$ by $G_n(\mathfrak{O})$, and we set $G_0 = \{1\}$. We denote by A_n the torus of diagonal matrices in G_n, and by N_n the unipotent radical of the Borel subgroup of G_n given by upper triangular matrices. For $m \geq 1$, we denote by $K_n(m)$ the subgroup of G_n, given by matrices $\begin{pmatrix} g & v \\ l & t \end{pmatrix}$, for g in $G_{n-1}(\mathfrak{O})$, v in $\mathfrak{O}^{n-1}$, l with every coefficient in $\mathfrak{P}^m$, and t in $1 + \mathfrak{P}^m$. We set $K_n(0) = G_n(\mathfrak{O})$.

If π is a generic representation of G_2, the essential vector of π was first considered in [C], for G_n with $n \geq 2$, it was studied in [J-P-S]. Here is one of its main properties: if one calls d the conductor (the power of q^{-s} in the ϵ factor with respect to an unramified additive character of F) of the representation π, the complex vector space $\pi^{K_n(d)}$ of vectors in π fixed under $K_n(d)$, is generated by the essential vector of π, and $\pi^{K_n(k)}$ becomes the null space for $k < d$.
However, to prove its existence, one has to study properties of the Rankin-Selberg integrals associated to the pairs (π, π'), where π' varies through the

set of unramified generic representations of G_{n-1}.

We set a few more notations before explaining this.
We denote by ν the positive character $|.| \circ det$ of G_n. We use the product notation for normalised parabolic induction (see Section 1.2). For any sequence of complex numbers $s_1, \dots, s_n$, the representation $|.|^{s_1} \times \cdots \times |.|^{s_n}$ of G_n is unramified, and its subspace of $G_n(\mathfrak{O})$-invariant vectors is of dimension 1 (see Section 1.5).
We choose a character θ of $(F, +)$ trivial on $\mathfrak{O}$ but not on $\mathfrak{P}^{-1}$, and use it to define a non degenerate character, still denoted θ, of the standard unipotent subgroup N_n of G_n, by $\theta(n) = \theta(\sum_{i=1}^{n-1} n_{i,i+1})$.

For $n \geq 2$, let π and π' be representations of Whittaker type (see Section 1.4) of G_n and G_{n-1} respectively, and denote by $W(\pi, \theta)$ and $W(\pi', \theta^{-1})$ their respective Whittaker models (which are quotients of π and π') with respect to θ and θ^{-1}.
If W and W' belong respectively to $W(\pi, \theta)$ and $W(\pi', \theta^{-1})$, we denote $I(W, W', s)$ the associated Rankin-Selberg integral (see Section 1.4).
For example, for a sequence of complex numbers $a_1, \dots, a_m$, the induced representation $|.|^{a_1} \times \cdots \times |.|^{a_m}$ of G_m, is of Whittaker type. If moreover $Re(a_1) \geq \cdots \geq Re(a_m)$, the representation $|.|^{a_1} \times \cdots \times |.|^{a_m}$ is of Langlands' type, and its Whittaker model contains a unique normalised spherical Whittaker function $W(q^{-a_1}, \dots, q^{-a_m})$. It is the unique Whittaker function on G_m, fixed by $G_m(\mathfrak{O})$, which equals 1 on $G_m(\mathfrak{O})$, and associated to the Satake parameter $\{q^{-a_1}, \dots, q^{-a_m}\}$ (see [S]). For fixed g in G_m, the function $W(q^{-s_1}, \dots, q^{-s_m})(g)$ is an element of the ring $\mathbb{C}[q^{\pm s_1}, \dots, q^{\pm s_m}]^{\mathcal{S}_m}$ of invariant Laurent polynomials. To define the essential vector of π, one needs to show as in [J-P-S], the following theorem (see [G-J] for the definition of the L function of an irreducible representation of G_n):

Theorem. *Let π be a generic representation of G_n with Whittaker model $W(\pi, \theta)$, then there exists in $W(\pi, \theta)$ a unique $G_{n-1}(\mathfrak{O})$-invariant function W_π^{ess}, such that for every sequence of complex numbers $s_1, \dots, s_{n-1}$, one has the equality $I(W_\pi^{ess}, W(q^{-s_1}, \dots, q^{-s_{n-1}}), s) = \prod_{i=1}^{n-1} L(\pi, s + s_i)$.*

Hence, the statement of the theorem is equivalent to say that for any unramified representation π' of Langlands' type of G_{n-1} with normalised spherical Whittaker function $W_{\pi'}^0$ in $W(\pi', \theta)$, one has the equality $I(W_\pi^{ess}, W_{\pi'}^0, s) = L(\pi, \pi', s)$ (see Section 1.4 for the definition of $L(\pi, \pi', s)$ and the equality $L(\pi, \pi', s) = \prod_{i=1}^{n-1} L(\pi, s + s_i)$ when $\pi' = |.|^{s_1} \times \cdots \times |.|^{s_m}$).
Using this theorem, it is then shown in [J-P-S], using the functional equation of $L(\pi, \pi', s)$, that the space $W(\pi, \theta)^{K_n(d)}$ is a complex line spanned by W_π^{ess}, and that $W(\pi, \theta)^{K_n(k)}$ is zero for $k < d$.

In this paper, we will show the following result, using the interpretation in terms of restriction of Whittaker functions of the Bernstein-Zelevinsky derivatives.

Let π be a ramified generic representation of G_n, and π_u be the unramified component of the first nonzero spherical Bernstein-Zelevinsky derivative $\pi^{(n-r)}$ of π (see Definition 1.3 for the precise definition). The representation π_u is an unramified representation of Langlands' type of G_r when $r \geq 1$. In this situation, we show in Corollary 3.2, that there is a unique Whittaker function W_π^{ess} in $W(\pi, \theta)$, which is right $G_{n-1}(\mathfrak{O})$-invariant, and which satisfies, for $a = diag(a_1, \dots, a_{n-1}) \in A_{n-1}$ and $a' = diag(a_1, \dots, a_r) \in A_r$, the equality:

$$W(diag(a,1)) = W^0_{\pi_u}(a')\nu(a')^{(n-r)/2}\mathbf{1}_{\mathfrak{O}}(a_r)\prod_{r<i<n}\mathbf{1}_{\mathfrak{O}^*}(a_i), \quad (1)$$

when $r \geq 1$, and

$$W(diag(a,1)) = \prod_{0<i<n}\mathbf{1}_{\mathfrak{O}^*}(a_i) \quad (2)$$

when $r = 0$.
Computing the integral $I(W_\pi^{ess}, W^0_{\pi'}, s)$ for an unramified representation π' of Langlands' type of G_{n-1}, we will obtain in Corollary 3.3 the statement (more precisely a slightly more general statement) of the theorem stated above.

For $GL(2,F)$, a detailed account about newforms can be found in [Sc], the author obtains Formula (1) (see Section 2.4 of [loc. cit.]) up to normalisation by an ϵ-factor. For $GL(n,F)$, Miyauchi ([Mi]) recently obtained Formula (1), assuming the existence of the essential vector, by using Hecke algebras, i.e. generalising Shintani's method for spherical representations.

Remark. The reason why we got interested in reproving the existence of such a vector is the following. In [J-P-S], the uniqueness of such a vector is proved. The proof of the existence is valid only for generic representations π appearing as subquotients of representations parabolically induced by ramified characters of $GL(1,F)$ and cuspidal representations of $GL(r,F)$ for $r \geq 2$, i.e. generic representations whith L-function equal to one.
Before we explain this, let us mention that Jacquet (see [J]) found a simple fix for the proof of [J-P-S], so that the motivation of writing our note is really to give a constructive proof of the existence of this vector, which provides a nice application of the techniques developed in [C-P].

In [J-P-S], the following is shown: for fixed W in $W(\pi, \theta)$, the function

$$P(W, q^{-s_1}, \dots, q^{-s_{n-1}}) = I(W, W(q^{-s_1}, \dots, q^{-s_{n-1}}), 0) / \prod_{i=1}^{n-1} L(\pi, s_i)$$

belongs to the ring $\mathbb{C}[q^{\pm s_1}, \dots, q^{\pm s_{n-1}}]^{\mathcal{S}_{n-1}}$ of symmetric Laurent polynomials in the variables q^{-s_i}. It is also shown that the existence of the essential vector is equivalent to the fact that the vector space

$$I(\pi) = \{P(W, q^{-s_1}, \dots, q^{-s_{n-1}}), W \in W(\pi, \theta)\},$$

which is actually an ideal, is equal to the the full ring

$$\mathbb{C}[q^{\pm s_1},\dots,q^{\pm s_{n-1}}]^{\mathcal{S}_{n-1}}.$$

The argument used to prove it goes like this:
For W well chosen, $P(W,q^{s_1},\dots,q^{-s_{n-1}})$ is equal to

$$\prod_{i=1}^{n-1} 1/L(\pi,s_i).$$

We denote by Q the element $1/L(\pi,s)$ of $\mathbb{C}[q^{-s}]$, so that

$$P(W,q^{-s_1},\dots,q^{-s_{n-1}})=\prod_{i=1}^{n-1} Q(q^{-s_i}).$$

Because of the functional equation of the L-function $L(\pi,|.|^{s_1}\times\cdots\times|.|^{s_{n-1}},s)$, denoting $\pi^\vee$ the smooth contragredient of π, one shows that $I(\pi)$ also contains the product $\prod_{i=1}^{n-1} Q'(q^{-1}q^{s_i})$, where

$$Q'(q^{-s})=1/L(\pi^\vee,s).$$

Proposition 2.1. of the paper then shows that $Q'(q^{-1}q^s)$ and $Q(q^{-s})$ are prime to one another in $\mathbb{C}[q^{\pm s}]$, and they deduce from this that no maximal ideal

$$I_{q^{-a_1},\dots,q^{-a_{n-1}}}=\{R\in\mathbb{C}[q^{\pm a_1},\dots,q^{\pm a_{n-1}}],R(q^{\pm a_1},\dots,q^{\pm a_{n-1}})=0\}$$

for $(a_1,\dots,a_{n-1})$ in $\mathbb{C}^{n-1}$, contains $\prod_{i=1}^{n-1} Q'(q^{-1}q^{s_i})$ and $\prod_{i=1}^{n-1} Q(q^{-s_i})$ together, which implies the result.
This last step is false as soon as $n\geq 3$, and there are a and b in $\mathbb{C}^*$ such that $Q(a)=Q'(q^{-1}b^{-1})=0$, because then both products belong to any ideal $I_{a,b,\dots,x_{n-1}}$. This is the case as soon as the degree $d^\circ(Q)$ of Q satisfies $d^\circ(Q)\geq 1$. However, using the functional equation of $L(\pi,|.|^{z_1}\times\cdots\times|.|^{z_{n-1}})$, and the cyclicity of $W(q^{-z_1},\dots,q^{-z_{n-1}})$ in $W(|.|^{z_1}\times\cdots\times|.|^{z_{n-1}},\theta)$ when $Re(z_i)\geq Re(z_{i+1})$, Jacquet noticed (see [J]) that one can find for every $(a_1,\dots,a_{n-1})$ in $\mathbb{C}^{n-1}$, a polynomial in $I(\pi)$, taking the value 1 when evaluated at $(q^{\pm a_1},\dots,q^{\pm a_{n-1}})$, so $I(\pi)$ is indeed equal to $\mathbb{C}[q^{\pm s_1},\dots,q^{\pm s_{n-1}}]^{\mathcal{S}_{n-1}}$.

1 Preliminaries

In this section, we first recall basic facts about smooth representations of locally profinite groups. We then focus on G_n, recall results from [B-Z] about derivatives, then introduce the L-function of a pair of representations of Whittaker type, we discuss espacially the unramified case.

1.1 Smooth representations, restriction and induction

When G is an l-group (locally compact totally disconnected group), we denote by $Alg(G)$ the category of smooth complex G-modules. We denote by $\widehat{G}$ the group of smooth characters (smooth representations of dimension 1) of G. If (π, V) belongs to $Alg(G)$, H is a closed subgroup of G, and χ is a character of H, we denote by $V(H,\chi)$ the subspace of V generated by vectors of the form $\pi(h)v - \chi(h)v$ for h in H and v in V. This space is stable under the action of the subgroup $N_G(\chi)$ of the normalizer $N_G(H)$ of H in G, which fixes χ.
We denote by δ_G the positive character of G such that if μ is a right Haar measure on G, and int is the action of G on smooth functions f with compact support in G, given by $(int(g)f)(x) = f(g^{-1}xg)$, then $\mu \circ int(g) = \delta_G(g)\mu$ for g in G.
The space $V(H,\chi)$ is $N_G(\chi)$-stable. Thus, if L is a closed-subgroup of $N_G(\chi)$, and δ' is a (smooth) character of L (which will be a normalising character dual to that of normalised induction later), the quotient $V_{H,\chi} = V/V(H,\chi)$ (that we simply denote by V_H when χ is trivial) becomes a smooth L-module for the (normalised) action $l.(v + V(H,\chi)) = \delta'(l)\pi(l)v + V(H,\chi)$ of L on $V_{H,\chi}$.
We denote by V^H the subspace of vectors of V fixed by H; for H compact and open, the functor $V \mapsto V^H$ from $Alg(G)$ to $Alg(G_0)$ is exact ([B-H], 2.3., Corollary 1).
We say that (π, V) in $Alg(G)$ is *admissible* if for any compact open subgroup H of G, the vector space V^H is finite dimensional.
If H is a closed subgroup of an l-group G, and (ρ, W) belongs to $Alg(H)$, we define the objects $(ind_H^G(\rho), V_c = ind_H^G(W))$ and $(Ind_H^G(\rho), V = Ind_H^G(W))$ of $Alg(G)$ as follows. The space V is the space of smooth functions from G to W, fixed under right translation by the elements of a compact open subgroup U_f of G, and satisfying $f(hg) = \rho(h)f(g)$ for all h in H and g in G. The space V_c is the subspace of V, consisting of functions with support compact mod H, in both cases, the action of G is by right translation on the functions.
We recall that by Frobenius reciprocity law ([B-H], 2.4.), the spaces $Hom_G(\pi, Ind_H^G(\rho))$ and $Hom_H(\pi_{|H}, \rho)$ are isomorphic when π (resp. ρ) belongs to $Alg(G)$ (resp. $Alg(H)$).
If the group G is exhausted by compact subsets (which is the case of closed subgroups of G_n), and (π, V) is irreducible, it is known ([B-H], 2.6., Corollary 1) that the center Z of G acts on V by the so-called central character of π which we will denote c_π. When $G = G_n$, then Z identifies with F^*. By definition, the real part $Re(\chi)$ of a character χ of F^* is the real number r such that $|\chi(t)|_{\mathbb{C}} = |t|^r$, where $|z|_{\mathbb{C}} = \sqrt{z\bar{z}}$ for z in $\mathbb{C}$.

1.2 Parabolic induction and segments for GL(n)

Now we focus on the case $G = G_n$, we will only consider smooth representations of its closed subgroups. It is known that irreducible representations of G_n are admissible (see [C2]).

If $n \geq 1$, let $\bar{n} = (n_1, \dots, n_t)$ be a partition of n of length t (i.e. an ordrered set of t positive integers whose sum is n), we denote by $M_{\bar{n}}$ to be the Levi subgroup of G_n, of matrices $diag(g_1, \dots, g_t)$, with each g_i in G_{n_i}, by $N_{\bar{n}}$ the unipotent subgroup of matrices $\begin{pmatrix} I_{n_1} & \star & \star \\ & \ddots & \star \\ & & I_{n_t} \end{pmatrix}$, and by $P_{\bar{n}}$ the standard parabolic subgroup $M_{\bar{n}}N_{\bar{n}}$ (where $M_{\bar{n}}$ normalises $N_{\bar{n}}$). Note that $M_{(1,\dots,1)}$ is equal to A_n, and $N_{(1,\dots,1)} = N_n$. For each i, let π_i be a smooth representation of G_{n_i}, then the tensor product $\pi_1 \otimes \cdots \otimes \pi_t$ is a representation of $M_{\bar{n}}$, which can be considered as a representation of $P_{\bar{n}}$ trivial on $N_{\bar{n}}$. We will use the product notation

$$\pi_1 \times \cdots \times \pi_t = Ind_{P_{\bar{n}}}^{G_n}(\delta_{P_{\bar{n}}}^{1/2}\pi_1 \otimes \cdots \otimes \pi_t)$$

for the normalised parabolic induction. Parabolic induction preserves finite length and admissibility (see [B-Z] or [C2]).
We say that an irreducible representation (ρ, V) of G_n is cuspidal, if the Jacquet module $V_{N_{\bar{n}}}$ is zero whenever $\bar{n}$ is a proper partition of n (i.e. we exclude $\bar{n} = (n)$).
Suppose that $\bar{n} = (m, \dots, m)$ is a partition of n of length l, and that ρ is a cuspidal representation of G_m. Then Theorem 9.3. of [Z] implies that the G_n-module $\nu^{-(l-1)}\rho \times \nu^{-(l-2)}\rho \times \cdots \times \nu^{-1}\rho \times \rho$ has a unique irreducible quotient which we denote $[\nu^{-(l-1)}\rho, \nu^{-(l-2)}\rho, \dots, \nu^{-1}\rho, \rho]$. We will call such a representation a segment, it is known that segments are the quasi square integrable representations of G_n, but we won't need this result.
We end this paragraph with a word about induced representations of Langlands' type:

Definition 1.1. *Let $\Delta_1, \dots, \Delta_t$ be segments of respectively $G_{n_1}, \dots, G_{n_t}$, and suppose that $Re(c_{\Delta_i}) \geq Re(c_{\Delta_{i+1}})$. Let $n = n_1 + \cdots + n_t$, then the representation $\Delta_1 \times \cdots \times \Delta_t$ of G_n is said to be induced of Langlands' type.*

These representations enjoy many remarkable properties, some of which we will recall later, here is a first one (which is the main result of [Sil]).

Proposition 1.1. *Let π be induced of Langlands' type, then π has a unique irreducible quotient $Q(\pi)$. Moreover, considering that isomorphic representations are equal, the map $\pi \mapsto Q(\pi)$ gives a bijection between the set of induced representations of Langlands' type of G_n, and the set of irreducible representations of G_n.*

1.3 Berstein-Zelevinsky derivatives

For $n \geq 2$ we denote by U_n the group of matrices of the form $\begin{pmatrix} I_{n-1} & v \\ & 1 \end{pmatrix}$.

For $n > k \geq 1$, the group G_k embeds naturally in G_n, and is given by matrices of the form $diag(g, I_{n-k})$. We denote by P_n the mirabolic subgroup $G_{n-1}U_n$ of G_n for $n \geq 2$, and $P_1 = \{1_{G_1}\}$. If one sees P_{n-1} as a subgroup of G_{n-1} itself embedded in G_n, then P_{n-1} is the normaliser of $\theta_{|U_n}$ in G_{n-1} (i.e. if $g \in G_{n-1}$, then $\theta(g^{-1}ug) = \theta(u)$ for all $u \in U_n$ if and only if $g \in P_{n-1}$). We define the following functors:

- The functor Φ^- from $Alg(P_k)$ to $Alg(P_{k-1})$ such that, if (π, V) is a smooth P_k-module, $\Phi^- V = V_{U_k,\theta}$, and P_{k-1} acts on $\Phi^-(V)$ by $\Phi^-\pi(p)(v + V(U_k, \theta)) = \delta_{P_k}(p)^{-1/2}\pi(p)(v + V(U_k, \theta))$.
- The functor Φ^+ from $Alg(P_{k-1})$ to $Alg(P_k)$ such that, for π in $Alg(P_{k-1})$, one has $\Phi^+\pi = ind_{P_{k-1}U_k}^{P_k}(\delta_{P_k}^{1/2}\pi \otimes \theta)$.
- The functor $\hat{\Phi}^+$ from $Alg(P_{k-1})$ to $Alg(P_k)$ such that, for π in $Alg(P_{k-1})$, one has $\hat{\Phi}^+\pi = Ind_{P_{k-1}U_k}^{P_k}(\delta_{P_k}^{1/2}\pi \otimes \theta)$.
- The functor Ψ^- from $Alg(P_k)$ to $Alg(G_{k-1})$, such that if (π, V) is a smooth P_k-module, $\Psi^- V = V_{U_k,1}$, and G_{k-1} acts on $\Psi^-(V)$ by $\Psi^-\pi(g)(v + V(U_k, 1)) = \delta_{P_k}(g)^{-1/2}\pi(g)(v + V(U_k, 1))$.
- The functor Ψ^+ from $Alg(G_{k-1})$ to $Alg(P_k)$, such that for π in $Alg(G_{k-1})$, one has $\Psi^+\pi = ind_{G_{k-1}U_k}^{P_k}(\delta_{P_k}^{1/2}\pi \otimes 1) = \delta_{P_k}^{1/2}\pi \otimes 1$.

These functors have the following properties which can be found in [B-Z]:

PROPOSITION 1.2. *a) The functors Φ^-, Φ^+, Ψ^-, and Ψ^+ are exact.*
b) Ψ^- is left adjoint to Ψ^+.
b') Φ^- is left adjoint to $\hat{\Phi}^+$.
c) $\Phi^-\Psi^+ = 0$ and $\Psi^-\Phi^+ = 0$.
d) $\Psi^-\Psi^+ \simeq Id$ and $\Phi^-\Phi^+ \simeq Id$.
e) One has the exact sequence $0 \to \Phi^+\Phi^- \to Id \to \Psi^+\Psi^- \to 0$.

Following [C-P], if τ belongs to $Alg(P_n)$, we will denote $(\Phi^-)^k\tau$ by $\tau_{(k)}$, and as usual, $\tau^{(k)}$ will be defined as $\Psi^-\tau_{(k-1)}$.

Because of e), τ has a natural filtration of P_n-modules $0 \subset \tau_n \subset \cdots \subset \tau_1 = \tau$, where $\tau_k = {\Phi^+}^{k-1}{\Phi^-}^{k-1}\tau$. We will use the notation $\tau_{(k),i}$ for $(\tau_{(k)})_i$. The following observation is just a restatement of the definitions:

LEMMA 1.1. *If τ belongs to $Alg(P_n)$, then $\tau_k = \Phi^+(\tau_{(1),k-1})$ for $k \geq 1$.*

1.4 Representations of Whittaker type and their L-functions

We recall that we fixed a character θ of conductor $\mathfrak{O}$ in the introduction.

Definition 1.2. *Let π be an admissible representation of G_n, we say that π is of Whittaker type π if $Hom(\pi, Ind_{N_n}^{G_n}(\theta))$ is of dimension 1, or equivalently, according to Frobenius reciprocity law, if the space $Hom_{N_n}(\pi,\theta)$ is of dimension 1. We denote by $W(\pi,\theta)$ the image of π in $Ind_{N_n}^{G_n}(\theta)$, it is called the Whittaker model of π (with respect to θ), it is a quotient of π.*

Being of Whittaker type does not depend on the character θ of $(F,+)$, as another non trivial character θ' of $(F,+)$ will give birth to a character θ' of N_n, conjugate to θ by A_n.

In terms of derivatives, as the representation $Ind_{N_n}^{P_n}(\theta)$ is isomorphic to $(\hat{\Phi}^+)^{n-1}\Psi^+(\mathbf{1})$, where $\mathbf{1}$ is the trivial representation of G_0, applying $b)$ and $b')$ of Proposition 1.2, we obtain that $Hom_{N_n}(\pi,\theta)\simeq\mathbb{C}$ if and only if $\pi^{(n)}=\mathbf{1}$. Applying this to product of segments, and using the rules of "derivation" given in Lemma 3.5 of [B-Z] and Proposition 9.6. of [Z], we obtain that if $\Delta_1,\dots,\Delta_t$ are segments of $G_{n_1},\dots,G_{n_t}$ respectively, the representation $\pi=\Delta_1\times\dots\times\Delta_t$ of G_n (for $n=n_1+\dots+n_t$) is of Whittaker type. If the segments Δ_i are ordered so that π is of Langlands' type, we can say more according to the main result of [J-S 3].

Proposition 1.3. *For $n\geq 1$, let π be a representation of G_n, which is induced of Langlands' type, then it has an injective Whittaker model, i.e. $\pi\simeq W(\pi,\theta)$ (equivalently π embeds in $Ind_{N_n}^{G_n}(\theta)$).*

If π is irreducible and embeds in $Ind_{N_n}^{G_n}(\theta)$, it is a well-known theorem of Gelfand and Kazhdan ([G-K]) that the multiplicity of π in $Ind_{N_n}^{G_n}(\theta)$ is 1, we then say that π is *generic*. We recall (Theorem 9.7 of [Z]), that every generic representation π of $GL(n,F)$ can be written uniquely, up to permutation of the terms in the product, as a commutative product of unlinked (see 4.1. of [Z]) segments

$$[\nu^{-(k_1(\pi)-1)}\rho_1(\pi),\dots,\rho_1(\pi)]\times\dots\times[\nu^{-(k_t(\pi)-1)}\rho_t(\pi),\dots,\rho_t(\pi)].$$

In particular, generic representations are the representations of Langlands' type which are irreducible.

We now recall from [J-P-S 2], some results about the L-function of a pair of representations of Whittaker type. Let π be a representation of G_n of Whittaker type, and π' be a representation of Whittaker type of G_m, with respective Whittaker models $W(\pi,\theta)$ and $W(\pi',\theta^{-1})$, for $n\geq m\geq 1$.
When $n>m$, and W and W' are respectively in $W(\pi,\theta)$ and $W(\pi',\theta^{-1})$, we write

$$I(W,W',s)=\int_{N_m\backslash G_m} W\begin{pmatrix} g & \\ & I_{n-m}\end{pmatrix} W'(g)\nu(g)^{s-(n-m)/2}dg.$$

When $n = m$, and W and W' are respectively in $W(\pi, \theta)$ and $W(\pi', \theta^{-1})$, ϕ is in $\mathcal{C}_c^\infty(F^n)$, and η is the row vector $(0, \dots, 0, 1)$ in the space $\mathcal{M}(1, n, F)$ of row matrices 1 by n with entries in F, we write

$$I(W, W', \phi, s) = \int_{N_n \backslash G_n} W(g)W'(g)\phi(\eta g)\nu(g)^s dg.$$

It is shown in [J-P-S 2] that these integrals converge absolutely for $Re(s)$ large, and define elements of $\mathbb{C}(q^{-s})$.
If $n > m$, the integrals $I(W, W', s)$ (which we shall also write $I(W', W, s)$ when convenient) span, when (W, W') varies in $W(\pi, \theta) \times W(\pi', \theta^{-1})$, a fractional ideal of $\mathbb{C}[q^s, q^{-s}]$, which is generated by a unique Euler factor $L(\pi, \pi', s)$. If $n = m$, the integrals $I(W, W', \phi, s)$ span, when (W, W', ϕ) varies in $W(\pi, \theta) \times W(\pi', \theta^{-1}) \times \mathcal{C}_c^\infty(F^n)$, a fractional ideal of $\mathbb{C}[q^s, q^{-s}]$, which is generated by a unique Euler factor $L(\pi, \pi', s)$. If $n < m$, we define $L(\pi, \pi', s)$ to be $L(\pi', \pi, s)$. We recall Proposition 9.4 of [J-P-S 2].

Proposition 1.4. *For $n \geq m \geq 1$, if $\pi = \Delta_1 \times \cdots \times \Delta_t$ is a representation of G_n induced of Langlands' type, and $\pi' = \Delta'_1 \times \cdots \times \Delta'_u$ is a representation of G_m induced of Langlands' type, then $L(\pi, \pi', s) = \prod_{i,j} L(\Delta_i, \Delta'_j, s)$.*

Finally, we recall that it is proved in Section 5 of [J-P-S 2], that if π is a generic representation of G_n, and χ is a character of G_1, one has the equality $L(\pi, \chi, s) = L(\chi \otimes \pi, s)$ between the Rankin-Selberg L-function on the left, and the Godement-Jacquet L-function on the right.

1.5 Unramified representations

We say that a representation (π, V) of G_n is *unramified* (or *spherical*), if it admits a nonzero $G_n(\mathfrak{O})$-fixed vector in its space. If it is the case, we recall that the Hecke convolution algebra $\mathcal{H}_n$ (whose elements are the functions with compact support on G_n, which are left and right invariant under $G_n(\mathfrak{O})$), acts on $V^{G_n(\mathfrak{O})}$ (when $V^{G_n(\mathfrak{O})}$ is of dimension 1, the action is necessarily by a character). The Hecke algebra $\mathcal{H}_n$ is commutative and isomorphic by the Satake isomorphism to the algebra $\mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]^{\mathcal{S}_n}$ according to [Sat]. Hence, a character of $\mathcal{H}_n$ is associated to a unique set of nonzero complex numbers $\{z_1, \dots, z_n\}$, corresponding to the evaluation $P \mapsto P(z_1^{\pm 1}, \dots, z_n^{\pm 1})$ from $\mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]^{\mathcal{S}_n}$ to $\mathbb{C}$. It is known that when π is unramified and irreducible (see Section 4.6 in [Bu] for example), then $V^{G_n(\mathfrak{O})}$ is one dimensional, and that the corresponding character of $\mathcal{H}_n$ determines π, in which case the associated set of nonzero complex numbers is called the *Satake parameter* of π.
We recall with proofs, some classical facts about parabolically induced spherical representations.

Proposition 1.5. *Let π_i be a representation of G_{n_i} for i between 1 and t, $n = n_1 + \cdots + n_t$, $M_{\bar{n}}$ be the standard Levi subgroup of n corresponding to the partition $(n_1, \dots, n_t)$, and $\pi = \pi_1 \times \cdots \times \pi_t$. The vector space $Hom_{G_n(\mathfrak{O})}(\mathbf{1}, \pi)$ is isomorphic to $Hom_{M_{\bar{n}} \cap G_n(\mathfrak{O})}(\mathbf{1}, \pi_1 \otimes \cdots \otimes \pi_t)$.*

Proof. From Mackey's theory ([B-Z], Theorem 5.2.), as G_n is equal to the double-class $P_{\bar{n}}G_n(\mathfrak{O})$, the restriction of π to $G_n(\mathfrak{O})$ is equal to $Ind_{M_{\bar{n}}\cap G_n(\mathfrak{O})}^{G_n(\mathfrak{O})}(\pi_1\otimes\cdots\otimes\pi_t)$. The result then follows from Frobenius reciprocity law. □

We have the following corollaries to this.

COROLLARY 1.1. *Let $\chi_1,\dots,\chi_n$ be unramified characters of F^*, then the representation $\pi=\chi_1\times\cdots\times\chi_n$ of G_n is unramified, and $\pi^{G_n(\mathfrak{O})}$ is one dimensional. In particular, by the exactness of the functor $V\mapsto V^{G_n(\mathfrak{O})}$ from $Alg(G_n)$ to $Alg(G_0)$, the representation π has only one irreducible spherical subquotient.*

When $\pi=|.|^{s_1}\times\cdots\times|.|^{s_n}$ is an unramified representation of G_n induced from the Borel subgroup, we just saw that $\pi^{G_n(\mathfrak{O})}$ is of dimension 1. The character of $\mathcal{H}_n$, given by its action on $\pi^{G_n(\mathfrak{O})}$ corresponds to the set $\{q^{-s_1},\dots,q^{-s_n}\}$. If π is moreover of Langlands' type, it is the Satake parameter of $Q(\pi)$ thanks to the next corollary.

COROLLARY 1.2. *If $\pi=\chi_1\times\cdots\times\chi_n$ is as above, but moreover of Langlands' type, then it is the irreducible quotient of π which is spherical. In particular, π is spanned by $\pi^{G_n(\mathfrak{O})}$.*

Proof. One checks by induction on n the following assertion: the representation π can be written $\pi_1\times\cdots\times\pi_t$, where for each i, the representation π_i is equal to $\nu^{(l_i-1)/2}\mu_i\times\nu^{(l_i-3)/2}\mu_i\cdots\times\nu^{(1-l_i)/2}\mu_i$ for a positive integer l_i and an unramified character μ_i of F^*, and such that the segments $[\pi_i']=[\nu^{(1-l_i)/2}\mu_i,\dots,\nu^{(l_i-1)/2}\mu_i]$ are unlinked, the character $\nu^{(1-l_t)/2}\mu_t$ is equal to χ_n, and the segment $[\pi_t']$ contains any other $[\pi_i']$ in which χ_n occurs (i.e. if $\chi_n=\nu^r\mu_i$ for $r\in\frac{1}{2}\mathbb{Z}$ between $(1-l_i)/2$ and $(l_i-1)/2$, then $r=(1-l_i)/2$ and $l_i\leq l_t$). From [Z], Section 2, the irreducible quotient of π_i is the irreducible submodule of $\nu^{(1-l_i)/2}\mu_i\times\cdots\times\nu^{(l_i-1)/2}\mu_i$, i.e. the character $\tilde{\mu_i}=\mu_i\circ det$ of G_{l_i}. The representation $\tilde{\mu_1}\times\cdots\times\tilde{\mu_l}$ is thus a quotient of π, which is irreducible by Theorem 4.2 of [Z], hence it is its irreducible quotient. It is spherical from Proposition 1.5, as the representations $\tilde{\mu_i}$ are spherical. □

COROLLARY 1.3. *A segment Δ of G_n, for $n\geq 2$, is always ramified.*

Proof. If Δ was unramified, as it is irreducible, its Satake parameter would be equal to a set $\{q^{-s_1},\dots,q^{-s_n}\}$, hence the same as that of $Q(\pi)$, for $\pi=|.|^{s_1}\times\cdots\times|.|^{s_n}$ (if we order the s_i's in a correct way). In particular, Δ would be equal to $Q(\pi)$, which is absurd according to Proposition 1.1. □

For unramified representations of Langlands' type, normalised spherical Whittaker functions are test functions for L factors. If π is an unramified representation of Langlands' type, we denote by W_π^0 the spherical function in $W(\pi,\theta)$ which is equal to 1 on $G_n(\mathfrak{O})$, and call it the normalised spherical Whittaker function of π. In [S], Shintani gave an explicit formula for W_π^0 in terms of the Satake parameter of π.

Using this formula, Jacquet and Shalika found (Proposition 2.3 of [J-S], and equality (3) in Section 1 of [J-S 2]), for π and π' two unramified representations of Langlands' type (see the discussion after Equations (3) and (4) below) of G_n and G_m respectively, and for correct normalisations of Haar measures, the equalities:

$$L(\pi,\pi',s) = I(W_\pi^0, W_{\pi'}^0, \mathbf{1}_{\mathfrak{O}^n}, s)$$

$$= \int_{A_n} W_\pi^0(a) W_{\pi'}^0(a) \mathbf{1}_{\mathfrak{O}}(a_n) \delta_{B_n}(a)^{-1} \nu(a)^s d^*a, \tag{3}$$

when $n = m$, and

$$L(\pi,\pi',s) = I(W_\pi^0, W_{\pi'}^0, s)$$

$$= \int_{A_m} W_\pi^0 \begin{pmatrix} a & \\ & I_{n-m} \end{pmatrix} W_{\pi'}^0(a) \delta_{B_m}(a)^{-1} \nu(a)^{s-(n-m)/2} d^*a \tag{4}$$

when $n > m$.

In the aforementioned papers, Jacquet and Shalika work with generic representations, however, their proofs extend verbatim to unramified representations of Langlands' type. Indeed, the formula for W_π^0 in terms of Satake parameters is still valid, and thanks to Proposition 1.4, the factor $L(\pi,\pi',s)$ is still equal to (using notations of [J-S] and [J-S 2]) the Artin factor $1/det(1-q^{-s}A \otimes A')$ for A and A' diagonal matrices corresponding to the Satake parameters of π and π' respectively. In particular, when π is generic, W_π^0 is the essential vector of π.

Now let $\pi = \Delta_1 \times \cdots \times \Delta_t$ be a generic representation of G_n, written as a unique product of the unlinked segments $\Delta_i = [\nu^{-(k_i(\pi)-1)}\rho_i(\pi), \dots, \rho_i(\pi)]$, and $\pi' = \mu_1 \times \cdots \times \mu_m$ be an unramified representation of G_m of Langlands' type, for $1 \le m \le n$. One has, according to Proposition 1.4 and Theorem 8.2. of [J-P-S 2] (whose proof is independant of [J-P-S]), the equality of Rankin-Selberg L-functions $L(\pi,\pi',s) = \prod_{i,j} L(\rho_i(\pi),\mu_j,s)$.

We notice that $L(\rho_i(\pi),\mu_j,s)$ is equal to 1 unless $\rho_i(\pi)$ is an unramified character of G_1. Hence, one has the equality

$$L(\pi,\pi',s) = \prod_{\{i,\rho_i(\pi)\in \widehat{F^*/\mathfrak{O}^*},j\}} L(\rho_i(\pi),\mu_j,s).$$

This incites us to introduce the following representation.

Definition 1.3. *Let $\pi = \Delta_1 \times \cdots \times \Delta_t$ be a generic representation of G_n, with $\Delta_i = [\nu^{-(k_i(\pi)-1)}\rho_i(\pi), \dots, \rho_i(\pi)]$. Let r be the cardinality of the set $\{\rho_j(\pi), \rho_j(\pi) \in \widehat{F^*/\mathfrak{O}^*}\}$. When this set is non empty, denote by $\chi_1, \dots, \chi_r$ its elements ordered such that $Re(\chi_i) \ge Re(\chi_{i+1})$ for $1 \le i \le r-1$. We define π_u as the trivial representation of G_0 when $r = 0$, and as the unramified representation of Langlands type $\chi_1 \times \cdots \times \chi_r$ of G_r when $r \ge 1$.*

Let π a generic representation of G_n, and π' be an unramified representation of G_m of Langlands' type, with $1 \leq m \leq n$. If we set $L(\pi_u, \pi', s) = 1$ when π_u is the trivial representation of G_0, we have, according to Proposition 1.4:

$$L(\pi, \pi', s) = L(\pi_u, \pi', s). \tag{5}$$

From now on, we will order the segments Δ_i in the generic representation π, such that $\rho_i(\pi)$ is an unramified character χ_i of G_1 for $1 \leq i \leq r$, is not such a character for $i \geq r+1$, and $Re(\chi_i) \geq Re(\chi_{i+1})$ for $1 \leq i \leq r-1$.

2 Mirabolic restriction, sphericity, and restriction of Whittaker functions

In this section, we first give results on the derivative functors and how they act on subspaces fixed by compact subgroups, then we recall some results from [C-P] about their interpretation in terms of restriction of Whittaker functions. We introduce a few more notations, in order to get a handy parametrisation of the diagonal torus of G_n, in terms of simple roots. For $k \leq n$, let Z_k be the center of G_k naturally embedded in G_n; we parametrise it by F^* using the morphism $\beta_k : z_k \mapsto diag(z_k I_k, I_{n-k})$. Hence the maximal torus A_n of G_n is the direct product $Z_1.Z_2 \dots Z_{n-1}.Z_n$. We will sometimes (but not always) omit the β_k's in this parametrisation and write $(z_1, \dots, z_n)$ for the element $\beta_1(z_1) \dots \beta_n(z_n)$ of A_n. Notice that the i-th simple root α_i has the property that $\alpha_i(z_1, \dots, z_n) = z_i$.

2.1 Mirabolic restriction and sphericity

We first give a corollary of Proposition 1.2, about a concrete interpretation of Φ^-, when restricted to $\Phi^+(Alg(P_n))$. Property d) of the aforementioned proposition says that Φ^- sends $\Phi^+\tau$ surjectively onto a P_n-module isomorphic to τ. Writing $\Phi^+\tau$ as $Ind_{P_nU_{n+1}}^{P_{n+1}}(\delta_{P_{n+1}}^{1/2}\tau \otimes \theta)$, we want to make the map Φ^- explicit between $Ind_{P_nU_{n+1}}^{P_{n+1}}(\delta_{P_{n+1}}^{1/2}\tau \otimes \theta)$ and τ.

Proposition 2.1. *For $n \geq 1$, if τ belongs to $Alg(P_n)$, then Φ^- identifies with the map $f \mapsto f(I_{n+1})$ from $\Phi^+\tau$ to τ.*

Proof. Call E_I the map $f \mapsto f(I_{n+1})$ from $\Phi^+\tau$ to τ. Let's first show that E_I is surjective. If v belongs to the space V of τ, let $U = I_n + \mathcal{M}(n, \mathfrak{P}^l)$ be a congruence subgroup of G_n, with l large enough for v to be fixed by $U' = U \cap P_n$. Call f the function from P_n to V, defined by

$$f\begin{pmatrix} pu & x \\ & 1 \end{pmatrix} = \delta_{P_{n+1}}^{1/2}(p)\theta(x_n)\tau(p)v,$$

for $p \in P_n$, $u \in U$, $x \in F^n$ with bottom coordinate x_n, and $f(p') = 0$ when p' is not in $P_nU_{n+1}U$. The map f is well defined, because v is fixed by U'. It

is smooth as it is right invariant under $U_{n+1}(\mathfrak{O})U$ (as θ is trivial on $\mathfrak{O}$). One checks (see the proof of next Propositon for a detailed similar computation), that f satsifies the requested left invariance under P_nU_{n+1}, so that $f \in \Phi^+(V)$. Finally $f(I_{n+1}) = v$, thus f is a preimage of v via E_I.
An easy adaptation of the Proposition 1.1. of [C-P] then shows that the P_n-submodule $\Phi^+\tau(U_{n+1},\theta)$ of $\Phi^+(\tau)$ is equal to $Ker(E_I)$. As a consequence, the map E_I induces an isomorphism $\overline{E_I}$ between $\Phi^-\Phi^+(\tau)$ and τ, which is P_n-equivariant. Hence, the following diagram, with the right isomorphism equal to $\overline{E_I}$, commutes:

$$\begin{array}{ccc} \Phi^+(\tau) & \overset{\Phi^-}{\twoheadrightarrow} & \Phi^-\Phi^+(\tau) \\ \| & & \wr\| \\ \Phi^+(\tau) & \overset{E_I}{\twoheadrightarrow} & \tau \end{array}$$

□

We recall that for $n \geq 1$, as a consequence of the Iwasawa decomposition, any element g of G_n can be written in the form zpk with z in F^*, p in P_n, and k in $G_n(\mathfrak{O})$. We now notice that the restriction of Φ^- to $(\Phi^+\tau)^{P_{n+1}(\mathfrak{O})}$ is surjective onto $\tau^{P_n(\mathfrak{O})}$.

Proposition 2.2. *For $n \geq 1$, the map $f \mapsto f(I_{n+1})$ from $(\Phi^+\tau)^{P_{n+1}(\mathfrak{O})}$ to $\tau^{P_n(\mathfrak{O})}$ is surjective.*

Proof. Let v_0 be a vector in the space of τ which is $P_n(\mathfrak{O})$-invariant, we claim that the function f defined by

$$f\begin{pmatrix} zpk & x \\ & 1 \end{pmatrix} = \delta_{P_{n+1}}^{1/2}(p)\theta(x_n)\mathbf{1}_{\mathfrak{O}^*}(z)\tau(p)v_0,$$

for z in F^*, p in P_n, k in $G_n(\mathfrak{O})$, and x in F^n (with x the transpose of $(x_1,\dots,x_n)$), is a preimage of v_0 in $(\Phi^+\tau)^{P_{n+1}(\mathfrak{O})}$.
First we check that f is well-defined: if $zpk = z'p'k'$, this implies that z' is equal to z mod $\mathfrak{O}^*$, and p' is equal to p mod $P_n(\mathfrak{O})$. Hence $\delta_{P_{n+1}}^{1/2}(p) = \delta_{P_{n+1}}^{1/2}(p')$, $\mathbf{1}_{\mathfrak{O}^*}(z') = \mathbf{1}_{\mathfrak{O}^*}(z)$, and $\tau(p')v_0 = \tau(p)v_0$ as v_0 is $P_n(\mathfrak{O})$-invariant.
Then we check that f indeed belongs to $\Phi^+(\tau)$. Let p_0 belong to P_n embedded in P_{n+1} by $p \mapsto diag(p,1)$, let u_0 belong to U_{n+1}, and p_1 belong to P_{n+1}, we need to check the relations

$$f(p_0p_1) = \delta_{P_{n+1}}^{1/2}(p_0)\tau(p_0)f(p_1)$$

and

$$f(u_0p_1) = \theta(u_0)f(p_1).$$

Write $p_1 = \begin{pmatrix} zpk & x \\ & 1 \end{pmatrix}$, we have $f(p_0p_1) = f\begin{pmatrix} zp_0pk & p_0x \\ & 1 \end{pmatrix}$. As $\theta((p_0x)_n) = \theta(x_n)$, we have

$$f(p_0p_1) = \delta_{P_{n+1}}^{1/2}(p_0p)\theta(x_n)\mathbf{1}_{\mathfrak{O}^*}(z)\tau(p_0p)v_0 = \delta_{P_{n+1}}^{1/2}(p_0)\tau(p_0)f(p_1)$$

as wanted.
Write $u_0 = \begin{pmatrix} I_n & x_0 \\ & 1 \end{pmatrix}$, we also have

$$f(u_0p_1) = f\begin{pmatrix} zpk & x_0 + x \\ & 1 \end{pmatrix}$$

$$= \delta_{P_{n+1}}^{1/2}(p)\theta((x_0 + x)_n)\mathbf{1}_{\mathfrak{O}^*}(z)\tau(p)v_0 = \theta(u_0)f(p_1)$$

as wanted.
If $\begin{pmatrix} k_0 & x_0 \\ & 1 \end{pmatrix}$ belongs to $P_{n+1}(\mathfrak{O})$ (i.e. $k_0 \in G_n(\mathfrak{O})$ and $x_0 \in \mathfrak{O}^n$), as $\theta((zpkx_0)_n) = 1$ when z belongs to $\mathfrak{O}^*$, we have

$$f(\begin{pmatrix} zpk & x \\ & 1 \end{pmatrix}\begin{pmatrix} k_0 & x_0 \\ & 1 \end{pmatrix}) = f\begin{pmatrix} zpkk_0 & zpkx_0 + x \\ & 1 \end{pmatrix} = f\begin{pmatrix} zpk & x \\ & 1 \end{pmatrix},$$

and f is right $P_{n+1}(\mathfrak{O})$-invariant. Finally, it is obvious that $f(I_{n+1}) = v_0$. □

Now we are able to prove the following property of Φ^-, that we will be of great use later.

Proposition 2.3. *For $n \geq 1$, if τ belongs to $Alg(P_n)$, then Φ^- maps $\tau^{P_n(\mathfrak{O})}$ surjectively onto $\tau_{(1)}^{P_{n-1}(\mathfrak{O})}$, and Ψ^- maps $\tau^{G_{n-1}(\mathfrak{O})}$ surjectively onto ${\tau^{(1)}}^{G_{n-1}(\mathfrak{O})}$.*

Proof. For the first part, we use the filtrations $0 \subset \tau_n \subset \cdots \subset \tau_1 = \tau$ of τ, and $0 \subset \tau_{(1),n-1} \subset \cdots \subset \tau_{(1),1} = \tau_{(1)}$ of $\tau_{(1)}$. But τ_i equals $\Phi^+(\tau_{(1),i-1})$ because of Lemma 1.1, so that Φ^- maps $\tau_i^{P_n(\mathfrak{O})}$ onto $\tau_{(1),i-1}^{P_{n-1}(\mathfrak{O})}$ surjectively according to Proposition 2.2. In particular, Φ^- maps $\tau_2^{P_n(\mathfrak{O})}$, hence $\tau^{P_n(\mathfrak{O})}$ (as $\tau_2^{P_n(\mathfrak{O})} \subset \tau^{P_n(\mathfrak{O})}$), onto $\tau_{(1),1}^{P_{n-1}(\mathfrak{O})} = \tau_{(1)}^{P_{n-1}(\mathfrak{O})}$ surjectively.
Ψ^- maps $\tau^{G_{n-1}(\mathfrak{O})}$ surjectively onto ${\tau^{(1)}}^{G_{n-1}(\mathfrak{O})}$, because Ψ^- is surjective from τ to $\tau^{(1)}$, and the functor $V \mapsto V^{G_{n-1}(\mathfrak{O})}$ is exact from $Alg(G_{n-1})$ to $Alg(G_0)$ as $G_{n-1}(\mathfrak{O})$ is compact open in G_{n-1} (τ is a G_{n-1}-module by restriction). □

2.2 Mirabolic restriction for Whittaker functions

We start by recalling Proposition 1.1 of [C-P], which gives an interpretation of Φ^- in terms of restriction of Whittaker functions.

Proposition 2.4. *For $k \geq 2$, and any submodule τ of $(\rho, C^\infty(N_k\backslash P_k, \theta))$ (where ρ denotes the action of P_k by right translation), the map $R : W \mapsto \delta_{P_k}^{-1/2}W_{|P_{k-1}}$ is P_{k-1}-equivariant from $(\rho, C^\infty(N_k\backslash P_k, \theta))$ to $(\rho, C^\infty(N_{k-1}\backslash P_{k-1}, \theta))$, with kernel $\tau(U_k, \theta)$. Hence it inducues a P_{k-1}-modules isomorphism between $\Phi^-\tau$ and $Im(R) \subset C^\infty(N_{k-1}\backslash P_{k-1}, \theta)$, so that $(\rho, Im(R))$ is a model for $\Phi^-\tau$.*

Notice that for $k \geq 2$, if $g \in G_{k-1}$ equals zpk with $z \in F^*, p \in P_{k-1}$, and $k \in G_{n-1}(\mathfrak{O})$, then the absolute value of z depends only on g, so we can write it $|z(g)|$. We now state a proposition that follows from the proofs of Proposition 1.6. of [C-P], about the interpretation of Ψ^- in terms of Whittaker functions.

Proposition 2.5. *Let τ be a P_k-submodule of $C^\infty(N_k\backslash P_k, \theta)$, and suppose that $\tau^{(1)}$ is a G_{k-1}-module with central character c. Then, for any W in τ, for any g in G_{k-1}, the quantity $c^{-1}(z)|z|^{-(k-1)/2}W(diag(zg,1))$ is constant whenever z is in a punctured neighbourhood of zero (maybe depending on g) in F^*.*

Remark 2.1. Notice that in the proof of the Proposition 1.6. of [C-P], τ is of a particular form, and $\tau^{(1)}$ is supposed to be irreducible. The only fact that is actually needed is that $\tau^{(1)}$ has a central character.

This has the following consequence.

Corollary 2.1. *For $k \geq 2$, let τ be a P_k-submodule of $(\rho, \mathcal{C}^\infty(N_k\backslash P_k, \theta))$, and suppose that $\tau^{(1)}$ is a G_{k-1}-module with central character c, then Ψ^- identifies with the map*

$$S : W \mapsto [g \mapsto \lim_{z\to 0} c^{-1}(z)|z|^{(1-k)/2}W(diag(zg,1))\delta_{P_k}^{-1/2}(g)]$$

from τ to to $\mathcal{C}^\infty(N_{k-1}\backslash G_{k-1}, \theta)$. To be more precise, S has kernel $\tau(U_k, 1)$, and it induces a G_{k-1}-modules isomorhism $\overline{S}$ between $\tau^{(1)}$ and $S(\tau) \subset \mathcal{C}^\infty(N_{k-1}\backslash G_{k-1}, \theta)$.

Proof. For W in τ, call $\overline{W}$ its image in $\tau^{(1)}$, and call $S(W)$ the function

$$[g \mapsto \lim_{z\to 0} c^{-1}(z)|z|^{(1-k)/2}W(diag(zg,1))\delta_{P_k}^{-1/2}(g)]$$

in $C^\infty(N_{k-1}\backslash G_{k-1}, \theta)$, which is well defined according to Proposition 2.5. If $u(x) = \begin{pmatrix} I_{k-1} & x \\ & 1 \end{pmatrix}$ belongs to U_k, then $\rho(u(x))W(diag(zg,1)) = \theta(z(gx)_{k-1})W(diag(zg,1))$. As $\theta(z(gx)_{k-1}) = 1$ for z small enough, we deduce that $S(\rho(u(x))W) = S(W)$, hence the kernel of S contains $\tau(U_k, 1)$. Conversely, if $S(W) = 0$, the smoothness of W and the Iwasawa decomposition imply that $W(g)$ is null for $|z(g)|$ in a punctured neighbourhood of zero depending only on W. According to Proposition 2.3. of [M] (which is a restatement of Proposition 1.3. of [C-P]), this means that W belongs to $\tau(U_k, 1)$.
The $\mathbb{C}$-linear map $S : W \mapsto S(W)$ induces a $\mathbb{C}$-linear isomorphism $\overline{S} : \overline{W} \mapsto S(W)$ between $\tau^{(1)}$ and its image in $(\rho, \mathcal{C}^\infty(N_{k-1}\backslash G_{k-1}, \theta))$. Moreover, it is a G_{k-1}-equivariant because for $g_0 \in G_{k-1}$, one has that $\overline{S}(\tau^{(1)}(g_0)\overline{W})$ equals

$$\overline{S}(\delta_{P_k}^{-1/2}(g_0)\overline{\rho(g_0)W}) = \delta_{P_k}^{-1/2}(g_0)S(\rho(g_0)W) = \rho(g_0)S(W) = \rho(g_0)\overline{S}(\overline{W}).$$

$\square$

We end this section by stating two technical lemmas about Whittaker functions fixed under a maximal compact subgroup, the first is inspired from Lemma 9.2 of [J-P-S 2].

Lemma 2.1. *For $n \geq 2$, let τ be a P_n-submodule of $\mathcal{C}^\infty(N_n\backslash P_n, \theta)$, and let W belong to $\tau^{P_n(\mathfrak{O})}$, then there exists W' in $\tau^{P_n(\mathfrak{O})}$, such that $W'(p\beta_{n-1}(z)) = W(p)\mathbf{1}_{\mathfrak{O}^*}(z)$ for p in P_{n-1} and z in F^*.*

Proof. For l in $\mathbb{Z}$, we denote by ϕ_l the characteristic function $\mathbf{1}_{\mathfrak{P}^l}$ of $\mathfrak{P}^l$. We fix a Haar measure dt on F. The Fourier transform $\widehat{\phi_l}^\theta$ with respect to θ and dt is equal to $\lambda_l\phi_{-l}$ for $\lambda_l = dt(\mathfrak{P}^l) > 0$. We denote by Φ_l the function $\otimes_{i=1}^{n-1}\phi_l$, which is the characteristic of the lattice $\varpi^l\mathfrak{O}^{n-1}$ in F^{n-1}. We denote by u the natural isomorphism between F^{n-1} and U_n. We also recall that any element of τ is determined by its restriction to G_{n-1}.
We set $W^l(p) = \int_{x\in F^{n-1}} W(pu(x))\Phi_l(x)dx$ for p in P_n and $dx = dt_1 \otimes \dots \otimes dt_{n-1}$, hence W^l belongs to τ. Moreover if k belongs to $G_{n-1}(\mathfrak{O})$, and g belongs to G_{n-1}, then $W^l(gk)$ is equal to $\int_{x\in F^{n-1}} W(gku(x))\Phi_l(x)dx = \int_{x\in F^{n-1}} W(gu(kx))\Phi_l(x)dx$ because W is $P_n(\mathfrak{O})$-invariant, and this last integral is equal to $\int_{x\in F^{n-1}} W(gu(x))\Phi_l(k^{-1}x)dx = \int_{x\in F^{n-1}} W(gu(x))\Phi_l(x)dx = W^l(g)$ because of the invariance of dx and Φ_l under $G_{n-1}(\mathfrak{O})$. It is also clear that W^l is invariant $U_n(\mathfrak{O})$ because W is, hence W^l belongs to $\tau^{P_n(\mathfrak{O})}$.
Now, for p in $P_{n-1} \subset P_n$, and z in F^*, we obtain:

$$\begin{aligned}
W^l(p\beta_{n-1}(z)) &= \int_{x\in F^{n-1}} W(p\beta_{n-1}(z)u(x))\Phi_l(x)dx \\
&= \int_{x\in F^{n-1}} W(u(zpx)p\beta_{n-1}(z))\Phi_l(x)dx \\
&= \int_{x\in F^{n-1}} \theta((zpx)_{n-1})W(p\beta_{n-1}(z))\Phi_l(x)dx \\
&= W(p\beta_{n-1}(z)) \int_{x\in F^{n-1}} \Phi_l(x)\theta(zx_{n-1})dx \\
&= W(p\beta_{n-1}(z)) \int_{y\in\varpi^l\mathfrak{O}^{n-2}} dy \int_{t\in F} \phi_l(t)\theta(zt)dt \\
&= \lambda_l^{n-2} W(p\beta_{n-1}(z))\widehat{\phi_l}^\theta(z) = \lambda_l^{n-1}W(p\beta_{n-1}(z))\phi_{-l}(z)
\end{aligned}$$

The function $W' = W^0/\lambda_0^{n-1} - W^{-1}/\lambda_{-1}^{n-1}$ thus satisfies

$$W'(p\beta_{n-1}(z)) = W(p\beta_{n-1}(z))(\phi_0 - \phi_1)(z) = W(p\beta_{n-1}(z))\mathbf{1}_{\mathfrak{O}^*}(z) = W(p)\mathbf{1}_{\mathfrak{O}^*}(z)$$

because W is invariant under $\beta_{n-1}(\mathfrak{O}^*) \subset P_n(\mathfrak{O})$. □

Lemma 2.2. *For $n \geq 2$, let τ be a P_n-submodule of $\mathcal{C}^\infty(N_n\backslash P_n, \theta)$, and let W belong to $\tau^{G_{n-1}(\mathfrak{O})}$, then there exists W' in $\tau^{P_n(\mathfrak{O})}$, such that $W'(z_1, \dots, z_{n-1}, 1) = W(z_1, \dots, z_{n-1}, 1)1_{\mathfrak{O}}(z_{n-1})$ for z_i in F^*.*

Proof. Let du be the Haar measure on U_n, corresponding to the Haar measure $dx = dt_1 \otimes \cdots \otimes dt_{n-1}$ on F^{n-1}, normalised by $dt_i(\mathfrak{O}) = 1$ for i between 1 and $n-1$. Now set $W'(g) = \int_{u \in U_n(\mathfrak{O})} W(gu)du$. The vector W' is a linear combination of right translates of W by elements of $U_n(\mathfrak{O})$, so it belongs to τ. It is clearly invariant under $U_n(\mathfrak{O})$, and still invariant under $G_{n-1}(\mathfrak{O})$, as $G_{n-1}(\mathfrak{O})$ normalises $U_n(\mathfrak{O})$. The following computation then gives the result:

$$W'(z_1, \ldots, z_{n-1}, 1) = \int_{x \in \mathfrak{O}^{n-1}} W(\beta_1(z_1) \ldots \beta_{n-1}(z_{n-1})u(x))dx$$

$$= \int_{x \in \mathfrak{O}^{n-1}} W(u(\beta_1(z_1) \ldots \beta_{n-1}(z_{n-1})x)\beta_1(z_1) \ldots \beta_{n-1}(z_{n-1}))dx$$

$$= \int_{x \in \mathfrak{O}^{n-1}} \theta(z_{n-1}x_{n-1})W(\beta_1(z_1) \ldots \beta_{n-1}(z_{n-1}))dx$$

$$= \widehat{\mathbf{1}_{\mathfrak{O}}}^{\theta}(z_{n-1})W(z_1, \ldots, z_{n-1}, 1) = \mathbf{1}_{\mathfrak{O}}(z_{n-1})W(z_1, \ldots, z_{n-1}, 1),$$

the last equality because of the normalisation of the Haar measure on F. □

3 Construction of the essential Whittaker function

We are now going to produce the essential vector of a generic representation π of G_n, which will now be fixed untill the end. We recall recall that we associated to π, an integer $0 \leq r \leq n$, and an unramified representation of Langlands' type π_u of G_r in Section 1.5.

We first notice that the subspace of $\pi^{(n-r)}$ fixed by $G_r(\mathfrak{O})$ is a complex line.

Proposition 3.1. *Let π be generic representation of G_n. Then $(\pi^{(n-r)})^{G_r(\mathfrak{O})}$ is of dimension 1. If v^0 is a generator of $(\pi^{(n-r)})^{G_r(\mathfrak{O})}$, then the submodule $< G_r.v^0 >$ of $\pi^{(n-r)}$ spanned by v^0 surjects onto π_u.*

Proof. Write $\pi = \Delta_1 \times \cdots \times \Delta_t$ for the ordering of the Δ_i's fixed after Definition 1.3. According to Lemma 3.5. of [B-Z] the representation $\pi^{(n-r)}$ has a filtration with subquotients $\Delta_1^{(a_1)} \times \cdots \times \Delta_t^{(a_t)}$, with $\sum_i a_i = n - r$. According to Proposition 9.6 of [Z], π_u appears as one of these subquotients, and by the choice of r, the other nonzero subquotients amongst them all contain either a segment as a factor (in the product notation) of some G_k for $k \geq 2$, or a ramified character of G_1. According to Proposition 1.3, and Proposition 1.5 1.5, these other subquotients contain no nonzero $G_n(\mathfrak{O})$-invariant vector. The result then follows from the exactness of the functor $V \mapsto V^{G_n(\mathfrak{O})}$ from $Alg(G_n)$ to $Alg(G_0)$. □

We also notice the following facts. First, from the theory of Kirillov models (see [B-Z], Theorem 4.9), for $n \geq 2$, the map $W \in W(\pi, \theta) \mapsto W_{|P_n}$ is injective, we denote by $W(\pi_{(0)}, \theta)$ its image. We choose this notation because P_n-module $\pi_{(0)} = \pi_{|P_n}$ is isomorphic to the submodule $W(\pi_{(0)}, \theta)$ of

$(\rho, \mathcal{C}^\infty(N_n\backslash P_n, \theta))$. Now if one applies Proposition 2.4 repeatedly to $\pi_{(0)}$, then for $r \leq n-1$, the P_{r+1}-module $\pi_{(n-r-1)}$ is isomorphic to the submodule of $(\rho, \mathcal{C}^\infty(N_{r+1}\backslash P_{r+1}, \theta))$, whose vectors are the functions $(\prod_{k=r+2}^n \delta_{P_k}^{-1/2}) W_{|P_{r+1}}$ for $W \in W(\pi, \theta)$ (where P_{r+1} is embedded in P_n via $p \mapsto diag(p, I_{n-r-1})$), we denote by $W(\pi_{(n-r-1)}, \theta)$ this P_{r+1}-module.

Proposition 3.1 has the following corollary.

Corollary 3.1. *Under the condition* $1 \leq r \leq n-1$, *there exists* $\tilde{W}_0$ *in* $W(\pi_{(n-r-1)}, \theta)^{G_r(\mathfrak{O})}$ *such that*

$$W_{\pi_u}^0(g) = \lim_{z \to 0} c^{-1}(z)|z|^{-r/2}\tilde{W}_0(diag(zg, 1))\delta_{P_{r+1}}^{-1/2}(g)$$

for all g *in* G_r. *This implies that the representation* π_u *occurs as a submodule of* $\pi^{(n-r)}$.

Proof. We take $\Psi^-(W(\pi_{(n-r-1)}, \theta))$ as a model for $\pi^{(n-r)}$, i.e. $\pi^{(n-r)} = \Psi^-(W(\pi_{(n-r-1)}, \theta))$. Let $\tilde{W}_0$ be a preimage (which we shall normalise later), of v^0 under Ψ^-, which we take in $W(\pi_{(n-r-1)}, \theta)^{G_r(\mathfrak{O})}$ thanks to Proposition 2.3. We denote by $< P_{r+1}.\tilde{W}_0 >$ the P_{r+1}-submodule of $W(\pi_{(n-r-1)}, \theta)$ spanned by $\tilde{W}_0$. By definition of Ψ^-, we have

$$\Psi^-(< P_{r+1}.\tilde{W}_0 >) = \Psi^-(< G_r.\tilde{W}_0 >) =< G_r.v^0 > .$$

Now, Z_n acts by a character c on $< G_r.v^0 >$ (which is the central character of π_u as well according to Proposition 3.1). Let S be the map defined in Corollary 2.1 from $< P_{r+1}.\tilde{W}_0 >$ to $\mathcal{C}^\infty(N_r\backslash G_r, \theta)$. We know from this corollary, that S factors to give an isomorphism $\overline{S}$ between $< G_r.v^0 >$ and $S(< P_{r+1}.\tilde{W}_0 >)$. Define W^0 as $W^0 = S(\tilde{W}_0)$ in $\mathcal{C}^\infty(N_r\backslash G_r, \theta)$, so that $S(< P_{r+1}.\tilde{W}_0 >)$ is equal to $G_r.W^0$. As the G_r-module $< G_r.W^0 >$ is isomorphic to $< G_r.v^0 >$, there is a surjective G_r-module morphism from $< G_r.W^0 >$ onto $W(\pi_u, \theta)$ according to Proposition 3.1. It sends W^0 to a nonzero multiple of $W_{\pi_u}^0$. We normalise $\tilde{W}_0$, such that the Whittaker function $W^0 = S(\tilde{W}_0)$ is equal to 1 on $G_n(\mathfrak{O})$. The Hecke algebra $\mathcal{H}_r$ thus multiplies W^0 and $W_{\pi_u}^0$ by the same character, as $W_{\pi_u}^0$ is the image of W^0 via a G_r-intertwining operator. Both are normalised spherical Whittaker functions, they are thus equal according to [S]. In particular, we have $S(\tilde{W}_0) = W_{\pi_u}^0$, which is the first statement of the corollary. Next, this implies the equalities $< G_r.W^0 >=< G_r.W_{\pi_u}^0 >= W(\pi_u, \theta)$, so the surjection from $< G_r.v^0 >$ onto $W(\pi_u, \theta)$ is actually equal to the isomorphism $\overline{S}$, hence π_u occurs as a submodule of $\pi^{(n-r)}$. □

The following proposition then holds.

Proposition 3.2. *Under the condition* $1 \leq r \leq n-1$, *there exists in* $W(\pi_{(n-r-1)}, \theta)^{P_{r+1}(\mathfrak{O})}$ *an element* W_0, *such that* $W_0(z_1, \dots, z_r, 1) = \delta_{P_{r+1}}^{1/2}(z_1, \dots, z_r) W_{\pi_u}^0(z_1, \dots, z_r) 1_{\mathfrak{O}}(z_r)$ *for* z_r *in* F^*.

Proof. Let $\tilde{W}_0$ be as in Corollary 3.1. By the claim in the proof of Theorem 2.1 of [M] (the arguments of the claim in Proposition 1.6. of [C-P] are actually sufficient here), there is N in $\mathbb{N}$, such that $\tilde{W}_0(z_1,\dots,z_r a,1) = c_{\pi_u}(a)|a|^{r/2}\tilde{W}_0(z_1,\dots,z_r,1)$ (parametrizing A_{r+1} with the β_i's) for $|z_r| \le q^{-N}$ and $|a| \le 1$. For b in F^*, call $\tilde{W}_{0,b}$ the function $p \mapsto \tilde{W}_0(p\beta_r(b))/(c_{\pi_u}(b)|b|^{r/2})$ defined on P_{r+1}, then $\tilde{W}_{0,b}$ still belongs to $W(\pi_{(n-r-1)},\theta)^{G_r(\mathfrak{O})}$, and $\tilde{W}_{0,b}(z_1,\dots,z_r,1)/(c_{\pi_u}(z_r)|z_r|^{r/2})$ is constant with respect to z_r whenever $|z_r| \le q^{-N}/|b|$. We choose b in F^* satisfying $|b| = q^{-N}$, so that the function $\tilde{W}_{0,b}(z_1,\dots,z_r,1)/(c_{\pi_u}(z_r)|z_r|^{r/2})$ is constant with respect to z_r for $|z_r| \le 1$. Hence, according to Corollary 3.1, for $|z_r| \le 1$, we have the equalities

$$\tilde{W}_{0,b}(z_1,\dots,z_r,1)/(c_{\pi_u}(z_r)|z_r|^{r/2}) = \tilde{W}_0(z_1,\dots,z_r b,1)/(c_{\pi_u}(z_r b)|z_r b|^{r/2})$$

$$= \delta_{P_{r+1}}^{1/2}(z_1,\dots,z_{r-1},b)W_{\pi_u}^0(z_1,\dots,z_{r-1},b)/(c_{\pi_u}(b)|b|^{r/2})$$

$$= \delta_{P_{r+1}}^{1/2}(z_1,\dots,z_{r-1},1)W_{\pi_u}^0(z_1,\dots,z_{r-1},1).$$

They imply the equality

$$\tilde{W}_{0,b}(z_1,\dots,z_r,1) = \delta_{P_{r+1}}^{1/2}(z_1,\dots,z_{r-1},z_r)W_{\pi_u}^0(z_1,\dots,z_{r-1},z_r)$$

for $|z_r| \le 1$. On the other hand, applying Lemma 2.2, there is W_0 is in $W(\pi_{(n-r-1)},\theta)^{P_{r+1}(\mathfrak{O})}$, such that $W_0(z_1,\dots,z_r,1)$ is equal to $\tilde{W}_{0,b}(z_1,\dots,z_r,1)1_{\mathfrak{O}}(z_r)$, it is then clear that W_0 has the desired property. ☐

We now prove the main result of this paper.

THEOREM 3.1. *For $n \ge 2$, let π be a ramified generic representation of G_n (i.e. $r \le n-1$). Then one can produce a $G_{n-1}(\mathfrak{O})$-invariant function W_π^{ess} in $W(\pi,\theta)$, whose restriction to A_{n-1} (when A_{n-1} is parametrised by its simple roots), is given by formula*

$$W_\pi^{ess}(z_1,\dots,z_{n-1},1)$$

$$= W_{\pi_u}^0(z_1,\dots,z_r)\nu(z_1,\dots,z_r)^{(n-r)/2}\mathbf{1}_{\mathfrak{O}}(z_r)\prod_{j=r+1}^{n-1}\mathbf{1}_{\mathfrak{O}^*}(z_j) \qquad (6)$$

when $r \ge 1$, and by

$$W_\pi^{ess}(z_1,\dots,z_{n-1},1) = \prod_{j=1}^{n-1}\mathbf{1}_{\mathfrak{O}^*}(z_j) \qquad (7)$$

when $r = 0$. A function W_π^{ess} with such properties is unique, and has image $W_{\pi_u}^0$ in $\pi^{(n-r)}$.

Proof. Suppose first that we have $r \geq 1$. We already constructed in the previous proposition a vector W_0 in $W(\pi_{(n-r-1)}, \theta)^{P_{r+1}(\mathfrak{O})}$ such that

$$W_0(z_1, \ldots, z_r, 1) = \delta_{P_{r+1}}^{1/2}(z_1, \ldots, z_r) W_{\pi_u}^0(z_1, \ldots, z_r) 1_{\mathfrak{O}}(z_r).$$

Then, applying Proposition 2.3 and then Lemma 2.1, we obtain W_1 in $W(\pi_{(n-r-2)}, \theta)^{P_{r+2}(\mathfrak{O})}$, that satisfies

$$W_1(z_1, \ldots, z_{r+1}, 1) = \delta_{P_{r+2}}^{1/2}(z_1, \ldots, z_{r+1}) W_0(z_1, \ldots, z_r, 1) \mathbf{1}_{\mathfrak{O}^*}(z_{r+1})$$
$$= \delta_{P_{r+2}}^{1/2}(z_1, \ldots, z_r, 1) \delta_{P_{r+1}}^{1/2}(z_1, \ldots, z_r) W_{\pi_u}^0(z_1, \ldots, z_r) 1_{\mathfrak{O}}(z_r) \mathbf{1}_{\mathfrak{O}^*}(z_{r+1}).$$

Repeating this last step (Proposition 2.3 and then Lemma 2.1), we obtain by induction for all k between 1 and $n-r-1$, an element W_k in $W(\pi_{(n-r-1-k)}, \theta)^{P_{r+k+1}(\mathfrak{O})}$, that satisfies

$$W_k(z_1, \ldots, z_{r+k}, 1) = \delta_{P_{r+k+1}}^{1/2}(z_1, \ldots, z_{r+k}) W_{k-1}(z_1, \ldots, z_{r+k-1}, 1) \mathbf{1}_{\mathfrak{O}^*}(z_{r+k})$$

$$= W_{\pi_u}^0(z_1, \ldots, z_r) \mathbf{1}_{\mathfrak{O}}(z_r) \prod_{j=r+1}^{r+k} \mathbf{1}_{\mathfrak{O}^*}(z_j) \prod_{i=r+1}^{r+k+1} \delta_{P_i}^{1/2}(z_1, \ldots, z_r, \underbrace{1, \ldots, 1}_{i-(r+1)\times}).$$

We define W_π^{ess} to be the element of $W(\pi, \theta)$ which restricts to P_n as W_{n-r-1}, it is thus $G_{n-1}(\mathfrak{O})$-invariant and satisfies Equation (6) of the statement of the theorem, because

$$\prod_{i=r+1}^{n} \delta_{P_i}^{1/2}(z_1, \ldots, z_r, \underbrace{1, \ldots, 1}_{i-(r+1)\times}) = |det(z_1, \ldots, z_r)|^{\frac{n-r}{2}}$$

as $\delta_{P_i}(z_1, \ldots, z_r, 1, \ldots, 1) = |det(z_1, \ldots, z_r)|$ for $i > r$.
If $r = 0$, we take for W_0 the constant function on the trivial group P_1 equal to 1 in $W(\pi_{(n-1)}, \theta) = W(\pi_{(n-1)}, \theta)^{P_1(\mathfrak{O})}$. Again, thanks to Proposition 2.3 and Lemma 2.1, there is W_1 in $W(\pi_{(n-2)}, \theta)^{P_2(\mathfrak{O})}$ such that $W_1(z_1, 1) = \mathbf{1}_{\mathfrak{O}^*}(z_1)$ for z_1 in F^*, and we end as in the case $r \geq 1$.
The function W_π^{ess} is unique by the theory of Kirillov models, and its image in $\pi^{(n-r)}$ is $W_{\pi_u}^0$ by construction. □

The expression of the restriction of W_π^{ess} to A_{n-1} in the usual coordinates is the same.

Corollary 3.2. *Let π be a ramified generic representation of G_n, then if $a = diag(a_1, \ldots, a_{n-1})$ belongs to A_{n-1}, and $a' = diag(a_1, \ldots, a_r) \in A_r$, we obtain Formulas (1) and (2) of the introduction:*

$$W_\pi^{ess}(diag(a, 1)) = W_{\pi_u}^0(a') \nu(a')^{(n-r)/2} \mathbf{1}_{\mathfrak{O}}(a_r) \prod_{j=r+1}^{n-1} \mathbf{1}_{\mathfrak{O}^*}(a_j)$$

when $r \geq 1$, and

$$W_\pi^{ess}(diag(a,1)) = \prod_{j=1}^{n-1} \mathbf{1}_{\mathfrak{O}^*}(a_j)$$

when $r = 0$.

Proof. We do the case $r \geq 1$, the case $r = 0$ being simpler. If $diag(a_1,\dots,a_{n-1}) = (z_1,\dots,z_{n-1})$ belongs to A_{n-1}, we have $a_i = z_i \dots z_{n-1}$, hence if $a' = diag(a_1,\dots,a_r)$, we have $(z_1,\dots,z_r) = (\prod_{i=r+1}^{n-1} z_i)^{-1}a'$ in A_r. Equation (6) can thus be read:

$$W_\pi^{ess}(diag(a,1))$$

$$= W_{\pi_u}^0((\prod_{i=r+1}^{n-1} z_i)^{-1}a')\nu((\prod_{i=r+1}^{n-1} z_i)^{-1}a')^{(n-1)/2}\mathbf{1}_{\mathfrak{O}}(z_r)\prod_{i=r+1}^{n-1}\mathbf{1}_{\mathfrak{O}^*}(z_i)$$

$$= W_{\pi_u}^0(a')\nu(a')^{(n-r)/2}\mathbf{1}_{\mathfrak{O}}(a_r)\prod_{i=r+1}^{n-1}\mathbf{1}_{\mathfrak{O}^*}(a_i),$$

the last equality because if it is not $0=0$, this means that $z_{r+1},\dots,z_{n-1}$ all belong to $\mathfrak{O}^*$, hence the inverse of their product as well, and $W_{\pi_u}^0$, ν, $\mathbf{1}_{\mathfrak{O}^*}$ and $\mathbf{1}_{\mathfrak{O}}$ are all invariant under $\mathfrak{O}^*$. □

We then have the following corollary.

Corollary 3.3. *Let π be a generic representation of G_n with Whittaker model $W(\pi,\theta)$. There exists in $W(\pi,\theta)$ a unique $G_{n-1}(\mathfrak{O})$-invariant function W_π^{ess} equal to 1 on $G_{n-1}(\mathfrak{O})$, such that for every $1 \leq m \leq n-1$, and every unramified representation π' of Langlands' type of G_m, with normalised spherical function $W_{\pi'}^0$ in $W(\pi',\theta^{-1})$, the equality $I(W_\pi^{ess},W_{\pi'}^0,s) = L(\pi,\pi',s)$ holds for an appropriate normalisation of the invariant measure on $N_m\backslash G_m$.*

Proof. The unicity of a function W_π^{ess} with such properties follows from [J-P-S]. If π is unramified (i.e. $r=n$), we set $W_\pi^{ess} = W_\pi^0$ and Equations (3) and (4) show that it is the correct choice.
When $r \leq n-1$, we again only treat the case $r \geq 1$, the case $r=0$ being similar, but simpler (using Equation (2) instead of Equation (1)). We show that the function W_π^{ess} from the previous corollary satisfies the wanted equalities.
Thanks to Iwasawa decomposition, we have

$$I(W_\pi^{ess},W_{\pi'}^0,s) = \int_{A_m} W_\pi^{ess}(diag(a,I_{n-m}))W_{\pi'}^0(a)\delta_{B_m}^{-1}(a)\nu(a)^{s-\frac{(n-m)}{2}}d^*a'.$$

If $m > r$, using Equations (1) and $\delta_{B_m}\begin{pmatrix} a & \\ & I_{m-r}\end{pmatrix} = \delta_{B_m}(a)\nu(a)^{m-r}$, we find

$$I(W_\pi^{ess},W_{\pi'}^0,s) = \int_{A_r} W_{\pi_u}^0(a')W_{\pi'}^0\begin{pmatrix} a' & \\ & I_{m-r}\end{pmatrix}\delta_{B_r}^{-1}(a')\mathbf{1}_{\mathfrak{O}}(a_r)\nu(a')^{s-\frac{(m-r)}{2}}d^*a'$$

$$= \int_{A_r} W^0_{\pi_u}(a')W^0_{\pi'}\begin{pmatrix} a' & \\ & I_{m-r}\end{pmatrix}\delta_{B_r}^{-1}(a')\nu(a')^{s-\frac{(m-r)}{2}}d^*a' = I(W^0_{\pi'}, W^0_{\pi_u}, s),$$

as $W^0_{\pi'}\begin{pmatrix} a' & \\ & I_{m-r}\end{pmatrix}$ vanishes for $|a_r| > 1$. Hence, by Equations (4) and (5), we obtain

$$I(W^{ess}_\pi, W^0_{\pi'}, s) = L(\pi_u, \pi', s) = L(\pi, \pi', s).$$

If $m = r$, using Equation (1), we find

$$I(W^{ess}_\pi, W^0_{\pi'}, s) = \int_{A_r} W^0_{\pi_u}(a')W^0_{\pi'}\left(a'\right)\delta_{B_r}^{-1}(a')\mathbf{1}_{\mathfrak{O}}(a_r)\nu(a')^s d^*a',$$

but this integral is equal to

$$I(W^0_{\pi_u}, W^0_{\pi'}, \mathbf{1}_{\mathfrak{O}^m}, s) = L(\pi_u, \pi', s) = L(\pi, \pi', s)$$

by Equations (3) and (5).
If $m < r$, Equation (1) gives

$$I(W^{ess}_\pi, W^0_{\pi'}, s) = \int_{A_m} W^0_{\pi_u}(diag(a, I_{r-m}))W^0_{\pi'}(a)\delta_{B_m}^{-1}(a)\nu(a)^{s-\frac{(r-m)}{2}}d^*a,$$

and this integral is equal to

$$I(W^0_{\pi_u}, W^0_{\pi'}) = L(\pi_u, \pi', s) = L(\pi, \pi', s)$$

by Equations (4) and (5).
In all cases, we have

$$I(W^{ess}_\pi, W^0_{\pi'}, s) = L(\pi, \pi', s).$$

□

Acknowledgements. *I thank the p-adic workgroup in Poitiers for allowing me to give talks about [J-P-S]. I thank Guy Henniart, Hervé Jacquet, and Michitaka Miyauchi, for useful comments and corrections about this note. Finally and most importantly, I thank the referee for his very careful reading of the paper, which allowed to correct some mistakes, and improve many arguments.*

References

[B-Z] J. N. Bernstein and A.V. Zelevinsky, *Induced representations of reductive p-adic groups,*. Ann. Sc. E.N.S. 10, 1977, 441-472.

[B] A. Borel, *Admissible representations of a semi-simple group over a local field with vectors fixed under an Iwahori subgroup*, Invent. Math. 35 (1976), 233-259.

[Bu] D. Bump, *Automorphic forms and representations*, Cambridge Studies in Advanced Mathematics, 55. Cambridge University Press, Cambridge, 1997.

[B-H] C.J. Bushnell, G. Henniart, *The local Langlands conjecture for* GL(2), Grundlehren der Mathematischen Wissenschaften, 335. Springer-Verlag, Berlin, 2006.

[C] W. Cassellman, *On Some Results of Atkin and Lehner*, Math. Ann. 201 (1973), 301-314.

[C2] W. Cassellman, *Introduction to the theory of admissible representations of reductive p-adic groups*, (1995), http://www.math.ubc.ca/ cass/research/pdf/p-adic-book.pdf.

[C-P] J. W. Cogdell, I.I. Piatetski-Shapiro, *Derivatives and L-functions for GL(n)*, to appear in The Heritage of B. Moishezon, IMCP. http://www.math.osu.edu/cogdell.1/moish-www.pdf.

[G-K] I. Gelfand, D. Kazhdan, *Representations of the group GL(n,K) where K is a local field*, in Lie groups and their representations (Proc. Summer School, Bolyai János Math. Soc., Budapest, 1971), pp. 95-118. Halsted, New York, 1975.

[G-J] R. Godement, H. Jacquet, *Zeta functions of simple algebras. Lecture Notes in Mathematics*, Vol. 260. Springer-Verlag, Berlin-New York, 1972.

[J] H. Jacquet, *A correction to "Conducteur des représentations du groupe linéaire"*, Pacific J. Math. 260 (2012), no. 2, 515-525.

[J-P-S] H. Jacquet, I.I. Piatetski-Shapiro, J. Shalika, *Conducteur des représentations du groupe linéaire*, Math. Ann. 256 (1981), no. 2, 199-214.

[J-P-S 2] H.Jacquet, I.I.Piatetskii-Shapiro and J. Shalika, *Rankin-Selberg Convolutions*, Amer. J. Math., 105, (1983), 367-464.

[J-S] H. Jacquet and J. Shalika, *On Euler products and the classification of automorphic representations. I.*, Amer. J. Math. 103 (1981), no. 3, 499-558.

[J-S 2] H. Jacquet and J. Shalika, *On Euler products and the classification of automorphic forms. II.*, Amer. J. Math. 103 (1981), no. 4, 777-815.

[J-S 3] H. Jacquet and J. Shalika, *The Whittaker models of induced representations*, Pacific J. Math. 109 (1983), no. 1, 107-120.

[M] N. Matringe, *Derivatives and Asymptotics of Whittaker functions*, Represent. Theory, 2011, vol. 15, 646-669.

[Mi] M. Miyauchi, *Whittaker functions associated to newforms for GL(n) over p-adic fields*, to appear in J. Math. Soc. Japan, arXiv:1201.3507

[Sat] I. Satake, *Theory of spherical functions on reductive algebraic groups over p-adic fields*, Publ. Math. I.H.E.S. 18, (1963) p. 5-69.

[Sc] R. Schmidt, *Some remarks on local newforms for GL(2)*, Ramanujan Math. Soc. 17 (2002), no. 2, 115-147.

[S] T. Shintani, *On an explicit formula for class-*1 *"Whittaker functions" on* GL_n *over p-adic fields*, Proc. Japan Acad. 52 (1976), no. 4, 180-182.

[Sil] A.J. Silberger, *The Langlands quotient theorem for p-adic groups*, Math. Ann. 236 (1978), no. 2, 95-104.

[Z] A.V. Zelevinsky, *Induced representations of reductive p-adic groups II*, Ann. Sci. Éc. Norm. Sup., 13 (1980), no. 2, 165-210.

Nadir Matringe
Université de Poitiers
Laboratoire de Mathématiques
et Applications
Téléport 2 - BP 30179
Boulevard Marie et Pierre Curie
86962, Futuroscope Chasseneuil
France
matringe@math.univ-poitiers.fr