Holomorphic Connections on Filtered Bundles over Curves

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Abstract. Let X be a compact connected Riemann surface and E_P a holomorphic principal P–bundle over X, where P is a parabolic subgroup of a complex reductive affine algebraic group G. If the Levi bundle associated to E_P admits a holomorphic connection, and the reduction $E_P \subset E_P \times^P G$ is rigid, we prove that E_P admits a holomorphic connection. As an immediate consequence, we obtain a sufficient condition for a filtered holomorphic vector bundle over X to admit a filtration preserving holomorphic connection. Moreover, we state a weaker sufficient condition in the special case of a filtration of length two.

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1. Introduction

Let X be a compact connected Riemann surface. A holomorphic vector bundle E over X admits a holomorphic connection if and only if every indecomposable component of E is of degree zero [We], [At]. This criterion generalizes to the context of principal bundles over X with a complex reductive affine algebraic group as the structure group [AB1]. Note that since there are no nonzero $(2,0)$–forms on X, holomorphic connections on a holomorphic bundle on X are the same as flat connections compatible with the holomorphic structure of the bundle.

Our aim here is to consider flat connections on vector bundles compatible with a given filtration of the bundle. Let

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{\ell-1} \subset E_\ell = E \tag{1.1}$$

be a filtration of a holomorphic vector bundle E on X. If E admits a flat connection

$$D : E \longrightarrow E \otimes \Omega^1_X$$

preserving the filtration, meaning $D(E_i) \subset E_i \otimes \Omega^1_X$ for every i, then this connection induces a flat connection D_i on each successive quotient E_i/E_{i-1} with $i \in [1\,,\ell]$. The question is the following: which supplementary condition is needed in order to ensure the existence of a filtration preserving holomorphic connection D? Suppose for example that E is semi-stable of degree zero such that each successive quotient in (1.1) admits a flat connection. Then it follows immediately that each subbundle E_i, $i \in [1\,,\ell]$, is also semi-stable of degree zero. According to Corollary 3.10 in [Si, p. 40], the filtered vector bundle E then admits a filtration preserving holomorphic connection D. In this paper, we show that the rigidity of the filtration (1.1) is another sufficient supplementary condition for the existence of a filtration preserving holomorphic connection on E. We note that a related example is quoted in [Bi] (see [Bi, p. 119, Example 3.6]).

More generally, we consider holomorphic connections on principal bundles with a parabolic group as the structure group. Let P be a parabolic subgroup of a complex reductive affine algebraic group G, and let E_P be a holomorphic principal P–bundle over X. Let $L(P) := P/R_u(P)$ be the Levi quotient of P, where $R_u(P)$ is the unipotent radical of P. Assume that the associated holomorphic principal $L(P)$–bundle $E_P/R_u(P)$ admits a holomorphic connection. We are interested in the question of finding sufficient conditions for the existence of a holomorphic connection on E_P.

Let $E_P \times^P G$ be the holomorphic principal G–bundle obtained by extending the structure group E_P using the inclusion of P in G. We shall prove that the rigidity of the reduction of structure group $E_P \subset E_P \times^P G$ ensures the existence of a holomorphic connection on E_P (see Theorem 2.1).

2. Connections on principal bundles with parabolic structure group

Let G be a connected reductive affine algebraic group defined over $\mathbb{C}$. Let $P \subset G$ be a parabolic subgroup, *i.e.*, P is a Zariski closed connected algebraic subgroup of G such that the quotient variety G/P is complete. The unipotent radical of P will be denoted by $R_u(P)$. The quotient $L(P) := P/R_u(P)$, which is a connected reductive complex affine algebraic group, is called the *Levi quotient* of P. The Lie algebra of G (respectively, P) will be denoted by $\mathfrak{g}$ (respectively, $\mathfrak{p}$).

Let X be a compact connected Riemann surface. Let

$$f : E_P \longrightarrow X \tag{2.1}$$

be a holomorphic principal P–bundle. The quotient

$$E_{L(P)} := E_P/R_u(P) \tag{2.2}$$

is a holomorphic principal $L(P)$–bundle on X. We note that $E_{L(P)}$ is identified with the principal $L(P)$-bundle obtained by extending the structure group of E_P using the quotient map $P \longrightarrow L(P)$.

Let

$$E_G := E_P \times^P G \longrightarrow X$$

be the holomorphic principal G–bundle obtained by extending the structure group of E_P using the inclusion of P in G. Let

$$\mathrm{ad}(E_G) := E_G \times^G \mathfrak{g} \quad \text{and} \quad \mathrm{ad}(E_P) := E_P \times^P \mathfrak{p}$$

be the adjoint vector bundles for E_G and E_P respectively. The reduction of structure group $E_P \subset E_G$ is called *rigid* if

$$\mathrm{H}^0(X,\, \mathrm{ad}(E_G)/\mathrm{ad}(E_P)) = 0\,.$$

Let us give a brief geometric interpretation of this property. Recall that the space of infinitesimal deformations of the principal bundle E_G (respectively, E_P) can be identified with $\mathrm{H}^1(X,\, \mathrm{ad}(E_G))$ (respectively, $\mathrm{H}^1(X,\, \mathrm{ad}(E_P))$) [SU]. We have a short exact sequence of vector bundles

$$0 \longrightarrow \mathrm{ad}(E_P) \longrightarrow \mathrm{ad}(E_G) \longrightarrow \mathrm{ad}(E_G)/\mathrm{ad}(E_P) \longrightarrow 0\,.$$

The rigidity of the reduction of structure group $E_P \subset E_G$ thus translates as

$$\mathrm{H}^1(X,\, \mathrm{ad}(E_P)) \hookrightarrow \mathrm{H}^1(X,\, \mathrm{ad}(E_G)),$$

i.e. the infinitesimal deformations of E_P are uniquely determined by the infinitesimal deformations of E_G that they induce. In other words, if we fix the principal bundle E_G, then the parabolic subbundle E_P cannot be deformed.

Theorem 2.1. *Assume that the holomorphic principal $L(P)$–bundle $E_{L(P)}$ in* (2.2) *admits a holomorphic connection, and the reduction of structure group $E_P \subset E_G$ is rigid. Then the holomorphic principal P–bundle E_P admits a holomorphic connection.*

Proof. Let $\mathrm{At}(E_P) := (f_* TE_P)^P \subset f_* TE_P$ be the Atiyah bundle for E_P, where f is the projection in (2.1) [At]. It fits in a short exact sequence of holomorphic vector bundles on X

$$0 \longrightarrow \mathrm{ad}(E_P) \longrightarrow \mathrm{At}(E_P) \xrightarrow{p_0} TX \longrightarrow 0\,, \tag{2.3}$$

where p_0 is given by the differential $df : TE_P \longrightarrow f^*TX$ of f. We recall that a holomorphic connection on E_P is a holomorphic splitting of (2.3) [At].
Let $R_n(\mathfrak{p})$ be the Lie algebra of the unipotent radical $R_u(P)$. We note that $R_n(\mathfrak{p})$ is the nilpotent radical of the Lie algebra $\mathfrak{p}$. Let

$$\mathcal{V}_0 := E_P \times^P R_n(\mathfrak{p}) \longrightarrow X \tag{2.4}$$

be the holomorphic vector bundle associated to the principal P–bundle E_P for the P–module $R_n(\mathfrak{p})$.
Let $\widehat{f} : E_{L(P)} \longrightarrow X$ be the projection induced by f. Let

$$\mathrm{At}(E_{L(P)}) := (\widehat{f}_* TE_{L(P)})^{L(P)} \subset \widehat{f}_* TE_{L(P)}$$

be the Atiyah bundle for $E_{L(P)}$. We have a commutative diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{V}_0 & = & \mathcal{V}_0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathrm{ad}(E_P) & \longrightarrow & \mathrm{At}(E_P) & \xrightarrow{p_0} & TX & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle q} & & \| & & \\
0 & \longrightarrow & \mathrm{ad}(E_{L(P)}) & \longrightarrow & \mathrm{At}(E_{L(P)}) & \xrightarrow{p_1} & TX & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
\tag{2.5}
$$

where $\mathcal{V}_0$ is defined in (2.4).

By assumption, $E_{L(P)}$ admits a holomorphic connection. Hence there is a holomorphic homomorphism

$$\beta : TX \longrightarrow \mathrm{At}(E_{L(P)}) \tag{2.6}$$

such that $p_1 \circ \beta = \mathrm{Id}_{TX}$, where p_1 is the projection in (2.5). Therefore, we have a short exact sequence of holomorphic vector bundles

$$0 \longrightarrow \mathcal{V}_0 \longrightarrow \mathcal{V} := q^{-1}(\beta(TX)) \xrightarrow{p_0} TX \longrightarrow 0\,, \tag{2.7}$$

where q is the projection in (2.5).

The short exact sequence in (2.3) splits holomorphically if the the short exact sequence in (2.7) splits holomorphically. The obstruction for splitting of (2.7) is a cohomology class

$$\psi \in \mathrm{H}^1(X, \mathcal{V}_0 \otimes (TX)^*) = \mathrm{H}^0(X, \mathcal{V}_0^*)^* \tag{2.8}$$

(by Serre duality).

Since the group G is reductive, its Lie algebra $\mathfrak{g}$ has a G–invariant symmetric non-degenerate bilinear form. For example, let B be the direct sum of the Killing form on $[\mathfrak{g}\,,\mathfrak{g}]$ and a symmetric non-degenerate bilinear form on the center of $\mathfrak{g}$. Note that $\mathfrak{p} \subset R_n(\mathfrak{p})^\perp$ (the annihilator of $R_n(\mathfrak{p})^\perp$) and actually

$$\mathfrak{p} = R_n(\mathfrak{p})^\perp$$

since they have the same dimension. We thus have

$$R_n(\mathfrak{p})^* = \mathfrak{g}/R_n(\mathfrak{p})^\perp = \mathfrak{g}/\mathfrak{p}\,.$$

As the above isomorphism between $R_n(\mathfrak{p})^*$ and $\mathfrak{g}/\mathfrak{p}$ is P–equivariant, it follows that

$$\mathcal{V}_0^* = E_P \times^P R_n(\mathfrak{p})^* = \mathrm{ad}(E_G)/\mathrm{ad}(E_P)\,.$$

Now the given condition that $E_P \subset E_G$ is rigid implies that that

$$\mathrm{H}^0(X, \mathcal{V}_0^*) = 0\,.$$

Therefore, ψ in (2.8) vanishes. Consequently, the short exact sequence in (2.7) splits, implying that the short exact sequence in (2.3) splits. □

Some criteria for the existence of a holomorphic connection on $E_{L(P)}$ can be found in [AB1] and [AB2]. Theorem 2.1 has the following immediate corollary:

Corollary 2.2. *Let*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{\ell-1} \subset E_\ell = E$$

be a filtration of holomorphic vector bundles on X, *and let* $\mathrm{End}(E_\bullet) \subset \mathrm{End}(E)$ *be the subbundle defined by the sheaf of filtration preserving endomorphisms. Assume that each successive quotient* E_i/E_{i-1}, *with* $i \in [1\,,\ell]$, *admits a holomorphic connection, and*

$$\mathrm{H}^0(X,\, \mathrm{End}(E)/\mathrm{End}(E_\bullet)) = 0\,. \tag{2.9}$$

Then E *admits a holomorphic connection* D *such that* D *preserves each subbundle* E_i *with* $i \in [1\,,\ell]$.

Note that (2.9) is not a necessary condition for the existence of a filtration preserving connection D, as one can see by the example of trivial bundles filtered by trivial subbundles. In the next section, we state a weaker sufficient condition when the of length ℓ of the filtration is two.

3. Holomorphic connections on extensions

Let E and F be holomorphic vector bundles on X admitting holomorphic connections. A holomorphic connection on E and a holomorphic connection on F together define a holomorphic connection on the vector bundle $\mathrm{Hom}(E\,,F) = E^* \otimes F$.

Proposition 3.1. *Assume that* E *and* F *admit holomorphic connections* D_E *and* D_F *respectively, such that every holomorphic section of* $\mathrm{Hom}(E\,,F)$ *is flat with respect to the connection on* $\mathrm{Hom}(E\,,F)$ *given by* D_E *and* D_F. *Then for any holomorphic extension*

$$0 \longrightarrow E \longrightarrow W \longrightarrow F \longrightarrow 0\,,$$

the holomorphic vector bundle W *admits a holomorphic connection that preserves the subbundle* E.

Proof. Let r_1 and r_2 be the ranks of E and F respectively. Take the group

$$G = \mathrm{GL}(r_1 + r_2, \mathbb{C})\,;$$

let $P \subset G$ be the parabolic subgroup that preserves the subspace $\mathbb{C}^{r_1} \subset \mathbb{C}^{r_1+r_2}$ given by the first r_1 vectors of the standard basis. We note that $L(P) = \mathrm{GL}(r_1) \times \mathrm{GL}(r_2)$. Take an extension W as in the proposition. Then the pair $(W\,, E)$ defines a holomorphic principal P–bundle E_P over X and $E \oplus F$ defines the associated $L(P)$−bundle $E_{L(P)}$. The holomorphic connection $D_E \oplus D_F$ on $E \oplus F$ gives a section β as in (2.6).

After we fix the above set-up, the vector bundle $\mathcal{V}_0$ in (2.4) is $E\otimes F^*$. Consider

$$\psi \in \mathrm{H}^1(X,\, E\otimes F^*\otimes K_X) = \mathrm{H}^0(X,\, E^*\otimes F)^* = \mathrm{H}^0(X,\, \mathrm{Hom}(E\,,F))^*$$

in (2.8). Given any $T \in \mathrm{H}^0(X,\, \mathrm{Hom}(E\,,F))$, we will explicitly describe the evaluation $\psi(T) \in \mathbb{C}$.
Fix a C^∞ splitting

$$\eta : F \longrightarrow W$$

of the short exact sequence in the proposition. We will identity F with $\eta(F) \subset W$. Let $\overline{\partial}_E$ (respectively, $\overline{\partial}_F$) be the Dolbeault operator defining the holomorphic structure of E (respectively, F). Using the C^∞ isomorphism

$$W \longrightarrow E\oplus F \tag{3.1}$$

given by η, the Dolbeault operator of W is

$$\begin{pmatrix} \overline{\partial}_E & A \\ 0 & \overline{\partial}_F \end{pmatrix},$$

where A is a smooth section of $\mathrm{Hom}(F\,,E)\otimes\Omega^{0,1}_X$.
Let $D_{F,E}$ be the holomorphic connection on $\mathrm{Hom}(F\,,E)$ given by D_E and D_F. We have

$$D_{F,E}(A) \in C^\infty(X;\, \mathrm{Hom}(F\,,E)\otimes\Omega^{1,1}_X)\,.$$

Take any $T \in \mathrm{H}^0(X,\, \mathrm{Hom}(E\,,F))$. We will show that

$$\psi(T) = \int_X \mathrm{trace}(D_{F,E}(A)\circ T) \in \mathbb{C}\,. \tag{3.2}$$

To prove this, consider the holomorphic connection $D_E\oplus D_F$ on $E\oplus F$. Using the C^∞ isomorphism in (3.1), this connection produces a C^∞ connection ∇^W on W. We should clarify that ∇^W is holomorphic if and only if the isomorphism in (3.1) is holomorphic. Let

$$\mathcal{K}(\nabla^W) \in C^\infty(X;\, \mathrm{End}(W)\otimes\Omega^{1,1}_X)$$

be the curvature of the connection ∇^W. Since $D_E\oplus D_F$ is a flat connection on $E\oplus F$, and the inclusion of E in W is holomorphic, it follows that $\mathcal{K}(\nabla^W)$ lies in the subspace

$$C^\infty(X;\, E\otimes F^*\otimes\Omega^{1,1}_X) \subset C^\infty(X;\, \mathrm{End}(W)\otimes\Omega^{1,1}_X)$$

constructed using the inclusion of the vector bundle $\mathrm{Hom}(F\,,E)$ in $\mathrm{End}(W)$. From the definition of the cohomology class $\psi \in \mathrm{H}^1(X,\, E\otimes F^*\otimes K_X)$ it follows that the Dolbeault cohomology class in $\mathrm{H}^1(X,\, E\otimes F^*\otimes K_X)$ represented by the form $\mathcal{K}(\nabla^W) \in C^\infty(X;\, E\otimes F^*\otimes\Omega^{1,1}_X)$ coincides with ψ. On the other hand, the form

$$D_{F,E}(A) \in C^\infty(X;\, E\otimes F^*\otimes\Omega^{1,1}_X)$$

coincides with the curvature $\mathcal{K}(\nabla^W)$. Therefore, the equality in (3.2) follows.
We note that $\int_X \mathrm{trace}(D_{F,E}(A)\circ T)$ is independent of the choice of the homomorphism η. Indeed, for a different choice of η, the section A is replaced by

$A + \overline{\partial}_{E\otimes F^*}(A')$, where A' is a smooth section of $\text{Hom}(F\,,E)$, and $\overline{\partial}_{F,E}$ is the Dolbeault operator defining the holomorphic structure of $\text{Hom}(F\,,E)$. Now

$$\int_X \text{trace}(D_{F,E}(\overline{\partial}_{F,E}(A')) \circ T) \;=\; \int_X \text{trace}(\overline{\partial}_{F,E}(D_{F,E}(A')) \circ T)$$

since the connection $D_{F,E}$ is flat and compatible with the holomorphic structure, and we also have

$$\int_X \text{trace}(\overline{\partial}_{F,E}(D_{F,E}(A')) \circ T) = \int_X \overline{\partial}(\text{trace}(D_{F,E}(A') \circ T)) \;=\; 0$$

because the section T is holomorphic. Therefore, $\int_X \text{trace}(D_{F,E}(A) \circ T)$ is independent of the choice of η.

We also note that $\text{trace}(D_{F,E}(A) \circ T) \;=\; \text{trace}(T \circ D_{E,F}(A))$.

Let $D_{E,E}$ be the holomorphic connection on $\text{End}(E)$ induced by D_E. Let $D_{E,F}$ be the holomorphic connection on $\text{Hom}(E\,,F)$ induced by D_E and D_F. Note that

$$D_{E,F}(T) \;=\; 0$$

by the condition given in the proposition. Therefore, we have

$$D_{F,E}(A) \circ T \;=\; D_{F,E}(A) \circ T + A \circ D_{E,F}(T) \;=\; D_{E,E}(A \circ T)\,.$$

On the other hand,

$$\int_X \text{trace}(D_{E,E}(A \circ T)) \;=\; \int_X \partial(\text{trace}(A \circ T)) \;=\; 0\,.$$

Combining these, from (3.2) it follows that $\psi \;=\; 0$. The principal P–bundle E_P thus admits a holomorphic connection. In other words, the holomorphic vector bundle W admits a holomorphic connection that preserves the subbundle E. □

Corollary 3.2. *Let E be a holomorphic vector bundle on X of degree zero such that*

$$\text{H}^0(X,\, \text{End}(E)) \;=\; \mathbb{C} \cdot \text{Id}_E.$$

Then given any short exact sequence of holomorphic vector bundles

$$0 \longrightarrow E \longrightarrow W \longrightarrow E \longrightarrow 0\,,$$

the holomorphic vector bundle W admits a holomorphic connection that preserves the subbundle E.

Proof. The holomorphic vector bundle E is indecomposable because

$$\text{H}^0(X,\, \text{End}(E)) \;=\; \mathbb{C} \cdot \text{Id}_E\,.$$

Therefore, the given condition that $\text{degree}(E) \;=\; 0$ implies that E admits a holomorphic connection [We], [At, p. 203, Theorem 10]. For any holomorphic connection on E, the corresponding connection on $\text{End}(E)$ has the property that the section Id_E is flat with respect to it. Hence Proposition 3.1 completes the proof. □

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