An Application of Hermitian K-Theory: Sums-of-Squares Formulas

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Received: April 4, 2013
Revised: January 29, 2014

Communicated by Max Karoubi

Abstract. By using Hermitian K-theory, we improve D. Dugger and D. Isaksen's condition (some powers of 2 dividing some binomial coefficients) for the existence of sums-of-squares formulas.

2010 Mathematics Subject Classification: 19G38; 11E25; 15A63
Keywords and Phrases: Hermitian K-theory, sums-of-squares formulas, comparison maps and deleted quadrics.

1 Introduction

A sums-of-squares formula of type $[r, s, n]$ over a field F of characteristic $\neq 2$ (with strictly positive integers r, s and n) is a formula

$$\Big(\sum_{i=1}^{r} x_i^2\Big) \cdot \Big(\sum_{i=1}^{s} y_i^2\Big) = \Big(\sum_{i=1}^{n} z_i^2\Big) \in F[x_1, \ldots, x_r, y_1, \ldots, y_s] \qquad (1)$$

where $z_i = z_i(X, Y)$ for each $i \in \{1, \ldots, n\}$ is a bilinear form in X and Y (with coefficients in F), i.e. $z_i \in F[x_1, \ldots, x_r, y_1, \ldots, y_s]$ is homogeneous of degree 2 and F-linear in X and Y. Here, $X = (x_1, \ldots, x_r)$ and $Y = (y_1, \ldots, y_s)$ are coordinate systems. To be specific, $z_i = \sum_{k,j} c_{kj}^{(i)} x_k y_j$ for $c_{kj}^{(i)} \in F$. An old problem of Adolf Hurwitz concerns the existence of sums-of-squares formulas. Historical remarks can be found in [18] and [20]. For any $m \in \mathbb{Z}_{>0}$, we let $\varphi(m)$ denote the cardinality of the set $\{l \in \mathbb{Z} : 0 < l \leq m \text{ and } l \equiv 0, 1, 2 \text{ or } 4 \pmod 8\}$. The aim of this paper is to introduce the following result.

Theorem 1.1. *If a sums-of-squares formula of type $[r, s, n]$ exists over a field F of characteristic $\neq 2$, then $2^{\varphi(s-1)-i+1}$ divides $\binom{n}{i}$ for $n - r < i \leq \varphi(s-1)$.*

The proof of Theorem 1.1 over $\mathbb{R}$ was provided by [2] and [21]. It involves computations of topological KO-theory of real projective spaces and γ^i-operations. The statement of Theorem 1.1 over $\mathbb{R}$ can be extended to any field of characteristic 0 by an algebraic remark of T.Y. Lam and K.Y. Lam, cf. Theorem 3.3 [18]. By using algebraic K-theory, D. Dugger and D. Isaksen prove a similar result over an arbitrary field of characteristic $\neq 2$, where $\varphi(s-1)$ in the above theorem is replaced by $\lfloor \frac{s-1}{2} \rfloor$, cf. Theorem 1.1 [7]. They actually conjectured the above statement. Since $\varphi(s-1) \geq \lfloor \frac{s-1}{2} \rfloor$, our main theorem generalizes theirs. One may wish to look at the following table.

n	1	2	3	4	5	6	7	8	9	$\cdots$
$\varphi(n)$	1	2	2	3	3	3	3	4	5	$\cdots$
$\lfloor \frac{n}{2} \rfloor$	0	1	1	2	2	3	3	4	4	$\cdots$

EXAMPLE 1.1. Consider the triplet $[15, 10, 16]$ which does not exist over F by the above theorem. Neither Hopf's condition [8] nor the weaker condition in [7] can give the non-existence of $[15, 10, 16]$.

REMARK 1.1. The necessary condition of our main theorem does not imply the existence of $[r, s, n]$. To illustrate, $[3, 5, 5]$ does not exist over the field F by the Hurwitz-Radon theorem. However, it satisfies the necessary condition.

REMARK 1.2. The algebraic K-theory analog (cf. Theorem 1.1 [7]) of our main theorem works even if the assumption 'if a sums-of-squares formula of type $[r, s, n]$ exists over F' is replaced by 'if a nonsingular bilinear map of size $[r, s, n]$ exists over F'. The statement with the latter assumption is 'stronger'. However, this is not the case under our proof, since we will use the sums-of-squares identity (1).

REMARK 1.3. The triplet $[r, s, n]$ is independent of the base fields whenever $r \leq 4$ and whenever $s \geq n-2$ (cf. Corollary 14.21 [20]), so that the main theorem is true. There is a bold conjecture which states that the existence of $[r, s, n]$ is independent of the base field F (of characteristic $\neq 2$), cf. Conjecture 3.8 [18] or Conjecture 14.22 [20]. Our main theorem and Dugger-Isaksen's Hopf condition (cf. [8]) suggest this conjecture to some extent. However, as Shapiro points out in Chapter 14 [20], there is indeed very little evidence to support this conjecture.

In [22], it is shown that the Grothendieck-Witt group of a complex cellular variety is isomorphic to the KO-theory of its set of $\mathbb{C}$-rational points with analytic topology. The set of $\mathbb{C}$-rational points of a deleted quadric is homotopy equivalent to the real projective space of the same dimension, cf. Lemma 6.3 [15]. Moreover, the computation of topological KO-theory of a real projective space is well-known, cf. Theorem 7.4 [1]. We therefore have motivations to work on the Grothendieck-Witt group of a deleted quadric and on the γ^i-operations. The proof of our main theorem requires the computation of Grothendieck-Witt group of a deleted quadric which will be explored in Section 3.

2 Terminology, notation and remark

Let $(\mathcal{E}, *, \eta)$ be a $\mathbb{Z}[\frac{1}{2}]$-linear exact category with duality. For $i \in \mathbb{Z}$, Walter's Grothendieck-Witt groups $GW^i(\mathcal{E}, *, \eta)$ are defined in Section 4.3 [16]. The triplet $(\mathrm{Vect}(X), Hom(\ \ , \mathcal{L}), \mathrm{can})$ (notation in Example 2.3 [16]) is an exact category with duality. If X is any $\mathbb{Z}[\frac{1}{2}]$-scheme, then we define

$$GW^i(X, \mathcal{L}) := GW^i(\mathrm{Vect}(X), Hom(\ \ , \mathcal{L}), \mathrm{can}).$$

By the symbols $GW^i(X)$, we mean the groups $GW^i(X, \mathcal{O})$. Note that $GW^0(X)$ is just Knebusch's $L(X)$ which is defined in [14]. The notation in [3] is used for the Witt theory. For KO-theory and comparison maps, we refer to [22].

Definition 2.1. Let T be a scheme. For us, a smooth T-variety X is called *T-cellular* if it has a filtration by closed subvarieties

$$X = Z_0 \supset Z_1 \supset \cdots \supset Z_N = \emptyset$$

such that $Z_{k-1} - Z_k \cong \mathbb{A}_T^{n_k}$ for each k.

In this paper, the following notations are introduced for convenience:

F	—	a field of characteristic $\neq 2$;
K	—	an algebraically closed field of characteristic $\neq 2$;
V	—	the ring of Witt vectors over K;
L	—	the field of fractions of V;
X_F	—	the base-change scheme $X \times_{\mathbb{Z}[\frac{1}{2}]} F$ for any $\mathbb{Z}[\frac{1}{2}]$-scheme X;
S	—	the polynomial ring $F[y_1, \dots, y_s]$;
$\mathbb{P}^{s-1}$	—	the scheme Proj $\mathbb{Z}[\frac{1}{2}][y_1, \dots, y_s]$;
q_s	—	the quadratic polynomial $q_s(y) = y_1^2 + \ldots + y_s^2$;
$V_+(q_s)$	—	the closed subscheme of $\mathbb{P}^{s-1}$ defined by q_s;
$D_+(q_s)$	—	the open subscheme $\mathbb{P}^{s-1} - V_+(q_s)$ of $\mathbb{P}^{s-1}$;
ξ	—	the line bundle $\mathcal{O}(-1)$ of $\mathbb{P}_F^{s-1}$ restricted to $D_+(q_s)_F$;
R	—	the ring of elements of total degrees 0 in S_{q_s};
P	—	the R-module of elements of total degrees -1 in S_{q_s};
Q_n	—	the $\mathbb{Z}[\frac{1}{2}]$-scheme defined by $\sum_{i=0}^{n/2} x_i y_i = 0$ in $\mathbb{P}^{n+1}$, if $n > 0$ is even; $\sum_{i=0}^{(n-1)/2} x_i y_i + c^2 = 0$ in $\mathbb{P}^{n+1}$, if $n > 0$ is odd;
DQ_{n+1}	—	the open subscheme $\mathbb{P}^{n+1} - Q_n$ of $\mathbb{P}^{n+1}$.

Remark 2.1. (i) Let E be a field containing $\sqrt{-1}$ and of characteristic $\neq 2$. Note that $(Q_{s-2})_E$ is isomorphic to the projective variety $V_+(q_s)_E$, cf. Lemma 2.2 [8]. This map induces an isomorphism $i_E : (DQ_{s-1})_E \to D_+(q_s)_E$.
(ii) Observe that V is a complete DVR with the quotient field K, cf. Chapter II [17]. Also, note that the fraction field L of V has characteristic 0, cf. *loc. cit.*.
(iii) The scheme $D_+(q_s)_F$ is affine over the base field F, since $D_+(q_s)_F$ and Spec R are isomorphic, cf. the proof of Proposition 2.2 [7].

3 Proof of Theorem 1.1

Lemma 3.1. *If a sums-of-squares formula of type $[r, s, n]$ exists over F, then there exist a non-degenerate bilinear form $\sigma : \xi \times \xi \to \mathcal{O}$ on $D_+(q_s)_F$ and a bilinear space ζ on $D_+(q_s)_F$ of rank $n - r$ such that*

$$r[\xi, \sigma] + [\zeta] = n \in GW^0(D_+(q_s)_F)$$

where n is the trivial bilinear space of the rank n.

Proof. The K-theory analog has been proved, cf. Proposition 2.2 [7]. It is clear that the group $GW^0(D_+(q_s)_F)$ is isomorphic to $GW_0(R)$ by Remark 2.1 (iii). If the equation (1) exits, we are able to construct a graded S-module homomorphism $(S(-1))^r \to S^n$ by $f = (f_1, \ldots, f_r) \mapsto (z_1(f, Y), \ldots, z_n(f, Y))$ where $Y = (y_1, \ldots, y_s)$ is the coordinate system introduced in Section 1. This map induces a homomorphism $\alpha : P^r \to R^n$ of R-modules by localizing it at q_s.

The isomorphism $P \otimes_R P \to R, f \otimes g \mapsto (fg) \cdot q_s$ gives a non-degenerate bilinear form $\sigma : P \times P \to R$. Let $\langle -, - \rangle_{R^n}$ be the unit bilinear form over R^n. Let $f = (f_1, \ldots, f_r), g = (g_1, \ldots, g_r) \in P^r$. We claim that $\langle \alpha(f), \alpha(g) \rangle_{R^n}$ equals $\sum_{i=1}^r \sigma(f_i, g_i)$. It is enough to show that $\langle \alpha(f), \alpha(f) \rangle_{R^n} = \sum_{i=1}^r \sigma(f_i, f_i)$. Note that $\langle \alpha(f), \alpha(f) \rangle_{R^n} = z_1(f, Y)^2 + \ldots + z_n(f, Y)^2$. By the existence of the triplet $[r, s, n]$, we obtain $z_1(f, Y)^2 + \ldots + z_n(f, Y)^2 = (f_1^2 + \ldots + f_r^2)q_s = \sum_{i=1}^r \sigma(f_i, f_i)$. Note that $(P^r, \sum_{i=1}^r \sigma)$ is non-degenerate. It follows that α is injective and $(P^r, \sum_{i=1}^r \sigma)$ can be viewed as a non-degenerate subspace of $(R^n, \langle -, - \rangle_{R^n})$ via α. Define ζ to be its orthogonal complement $(P^r)^\perp$ with the unit form $\langle -, - \rangle_{R_n}$ restricting to $(P^r)^\perp$. By a basic fact of quadratic form theory, ζ is non-degenerate and $\zeta \perp (P^r, \sum_{i=1}^r \sigma) \cong (R^n, \langle -, - \rangle_{R^n})$. □

Theorem 3.1. *Let ν denote the element $[\xi, \sigma] - 1$ in the ring $GW^0(D_+(q_s)_K)$. Then, the ring $GW^0(D_+(q_s)_K)$ is isomorphic to*

$$\mathbb{Z}[\nu]/(\nu^2 + 2\nu, 2^{\varphi(s-1)}\nu)$$

where $\varphi(s-1)$ is the number defined in Section 1. Therefore, for any rational point $\varsigma : \mathrm{Spec}\ K \to D_+(q_s)_K$, the reduced Grothendieck-Witt ring

$$\widetilde{GW}^0(D_+(q_s)_K) := \ker\left(\varsigma^* : GW^0(D_+(q_s)_K) \to GW^0(\mathrm{Spec}\ K) \cong \mathbb{Z}\right)$$

is isomorphic to $\mathbb{Z}/2^{\varphi(s-1)}$.

Theorem 3.1 will be proved in the next section.

Proof of Theorem 1.1. It is enough to show this theorem over the algebraic closure $\bar{F}$ of F. Indeed, if $[r, s, n]$ exists over F, then it also exists over $\bar{F}$. In order to apply the standard trick (cf. the proof of Theorem 1.3 [7]), we have to take care of γ^i-operations on $GW^0(D_+(q_s)_{\bar{F}})$. To be specific, this standard trick can not be applied without the list of three properties (cf. Properties (i)-(iii) in *loc. cit.*) of γ^i-operations and their generating power series $\gamma_t = 1+\sum_{i>0}\gamma^i t^i$ on $GW^0(D_+(q_s)_{\bar{F}})$. Due to the lack of reference, we will develop γ^i-operations on $K(\mathrm{Bil}(X))$ and prove these three properties (see Appendix A). It is enough for our purpose because $GW^0(X)$ is just $K(\mathrm{Bil}(X))$ if X is affine (see Remark A.1), and the scheme $D_+(q_s)_{\bar{F}}$ is affine by Remark 2.1 (iii). Hence, together with Lemma 3.1, we are allowed to apply the standard trick. One checks that details are the same as in the proof of Theorem 1.3 [7] by replacing K-theory analogs with GW-theory and $\lfloor \frac{s-1}{2} \rfloor$ with $\varphi(s-1)$. Combining with a reformulation of powers of 2 dividing correspondent binomial coefficients (cf. Section 1.2 [7]), we are done. □

4 Proof of Theorem 3.1

4.1 Rigidity and Hermitian K-theory of cellular varieties

By Remark 2.1 (ii), there is always an inclusion map $\overline{\mathbb{Q}} \to \overline{L}$ where $\overline{\mathbb{Q}}$ (resp. $\overline{L}$) is the algebraic closure of $\mathbb{Q}$ (resp. $\overline{L}$). Consider the following diagram (2).

$$\begin{array}{ccccc} K & \longleftarrow & V & \longrightarrow & \overline{L} \\ & & & & \uparrow \\ \mathbb{C} & & \longleftarrow & & \overline{\mathbb{Q}} \end{array} \qquad \begin{array}{ccccc} W^i(K) & \xleftarrow[\cong]{\beta^i} & W^i(V) & \xrightarrow[\cong]{\alpha^i} & W^i(\overline{L}) \\ & & & & {\scriptstyle\cong}\uparrow{\scriptstyle\chi^i} \\ & & W^i(\mathbb{C}) & \xleftarrow[\eta^i]{\cong} & W^i(\overline{\mathbb{Q}}) \end{array} \tag{2}$$

On the right-hand side of the diagram (2), the maps of Witt groups are all induced by the correspondent ring maps of the left-hand side for a fixed $i \in \mathbb{Z}$. All these Witt groups are trivial if $i \not\equiv 0 \pmod 4$, cf. Theorem 5.6 [5]. Note that β^0 is an isomorphism by Satz 3.3 [13]. It is also clear that $W^0(K)$ is isomorphic to $\mathbb{Z}/2$ and that all the maps on the right-hand side of the diagram (2) preserve multiplicative identities for $i = 0$. Since Witt groups are four periodicity in shifting, we obtain

Lemma 4.1. *The map $\eta^i \circ (\chi^i)^{-1} \circ \alpha^i \circ (\beta^i)^{-1}$ yields an isomorphism from $W^i(K)$ to $W^i(\mathbb{C})$. Moreover, by Karoubi induction (cf. Section 3 [6]), the left-hand side of the digram (2) gives an isomorphism $GW^i(K) \to GW^i(\mathbb{C})$ of Grothendieck-Witt groups.* □

Lemma 4.2. *Let X be a smooth $\mathbb{Z}[\frac{1}{2}]$-cellular variety. Let $f : A \to B$ be a map of regular local rings of finite Krull dimensions with $1/2$. Suppose that the map $W^i(A) \to W^i(B)$ induced by f is an isomorphism for each i, then f gives an isomorphism of Witt groups (resp. Grothendieck-Witt groups)*

$$W^i(X_A, \mathcal{L}_A) \to W^i(X_B, \mathcal{L}_B) \text{ (resp. } GW^i(X_A, \mathcal{L}_A) \to GW^i(X_B, \mathcal{L}_B))$$

for each i and any line bundle $\mathcal{L}$ over X.

Proof. We may use $W^i(X,\mathcal{L})_*$ to simplify the notation $W^i(X_*,\mathcal{L}_*)$. We wish to prove the Witt theory case by induction on cells. Firstly, note that the pullback maps $W^i(A) \to W^i(\mathbb{A}^n_A)$ and $W^i(B) \to W^i(\mathbb{A}^n_B)$ are isomorphisms by homotopy invariance, cf. Theorem 3.1 [4]. It follows that

$$W^i(\mathbb{A}^n_A) \cong W^i(\mathbb{A}^n_B).$$

Let $X = Z_0 \supset Z_1 \supset \cdots \supset Z_N = \emptyset$ be the filtration such that

$$Z_{k-1} - Z_k \cong \mathbb{A}^{n_k} =: C_k.$$

In general, the closed subvarieties Z_k may not be smooth. However, let U_k be the open subvariety $X - Z_k$ for each $0 \leq k \leq N$. Every U_k is smooth in X. There is another filtration $X = U_N \supset U_{N-1} \supset \cdots \supset U_0 = \emptyset$ with $U_k - U_{k-1} = Z_{k-1} - Z_k \cong C_k$ closed in U_k of codimension d_k. Consider the following commutative diagram of localization sequences.

$$\begin{array}{ccccccccc} W^{i-1}(U_{k-1})_A & \longrightarrow & W^i_{C_k}(U_k,\mathcal{L})_A & \longrightarrow & W^i(U_k,\mathcal{L})_A & \longrightarrow & W^i(U_{k-1})_A & \longrightarrow & W^{i+1}_{C_k}(U_k,\mathcal{L})_A \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ W^{i-1}(U_{k-1})_B & \longrightarrow & W^i_{C_k}(U_k,\mathcal{L})_B & \longrightarrow & W^i(U_k,\mathcal{L})_B & \longrightarrow & W^i(U_{k-1})_B & \longrightarrow & W^{i+1}_{C_k}(U_k,\mathcal{L})_B \end{array}$$

Here, $W^i_{C_k}(U_k,\mathcal{L})$ means the $\mathcal{L}$-twisted ith-Witt group of U_k with supports on C_k. Note that any line bundle over $(C_k)_A$ is trivial, since

$$\mathrm{Pic}(\mathbb{A}^n_A) \cong \mathrm{Pic}(A) = 0 \text{ (A is regular local and so it is a UFD).}$$

By the dévissage theorem (cf. [10]), we deduce that

$$W^i_{C_k}(U_k,\mathcal{L})_A \cong W^i_{C_k}(U_k,\mathcal{L})_B \text{ for all } i.$$

Moreover, by induction hypothesis,

$$W^i(U_{k-1})_A \cong W^i(U_{k-1})_B \text{ for all } i.$$

Applying the 5-lemma, one sees that the middle vertical map is an isomorphism. Since the K-theory analog of this theorem is also true by induction on cells, the GW-theory cases follow by Karoubi induction, cf. Section 3 [6]. ◻

COROLLARY 4.1. *The Witt group (resp. the Grothendieck-Witt group)*

$$W^i(X,\mathcal{L})_K \textit{ (resp. } GW^i(X,\mathcal{L})_K)$$

is isomorphic to

$$W^i(X,\mathcal{L})_{\mathbb{C}} \textit{ (resp. } GW^i(X,\mathcal{L})_{\mathbb{C}})$$

for each i and any line bundle $\mathcal{L}$ over X.

4.2 Comparison maps and rank one bilinear spaces

If X is a smooth variety over $\mathbb{C}$, we let $X(\mathbb{C})$ be the set of $\mathbb{C}$-rational points of X with analytic topology. One can define comparison maps (cf. Section 2 [22])

$$\begin{array}{rccl} k^0: & K_0(X) & \to & K^0(X(\mathbb{C})) \\ gw^0: & GW^0(X) & \to & KO^0(X(\mathbb{C})) \\ w^0: & W^0(X) & \to & \frac{KO}{K}^0(X(\mathbb{C})) \end{array} \tag{3}$$

where $\frac{KO}{K}^0(X(\mathbb{C}))$ means the cokernel of the realification map from $K^0(X)$ to $KO^0(X(\mathbb{C}))$. Let $GW^0_{\mathrm{top}}(X(\mathbb{C}))$ be the Grothendieck-Witt group of complex bilinear spaces over $X(\mathbb{C})$. The map gw^0 consists of the composition of the following two maps

$$f: GW^0(X) \to GW^0_{\mathrm{top}}(X(\mathbb{C})) \qquad g: GW^0_{\mathrm{top}}(X(\mathbb{C})) \to KO^0(X(\mathbb{C}))$$

where the map f takes a class $[M,\phi]$ on X to the class $[M(\mathbb{C}),\phi(\mathbb{C})]$ on $X(\mathbb{C})$. The map g sends a class $[N,\epsilon]$ on $X(\mathbb{C})$ to the class represented by the underlying real vector bundle $\mathfrak{R}(N,\epsilon)$ such that $\mathfrak{R}(N,\epsilon)\otimes_{\mathbb{R}}\mathbb{C} = N$ and that $\epsilon|_{\mathfrak{R}(N,\epsilon)}$ is real and positive definite, cf. Lemma 1.3 [22]. Let $Q(X)$ (resp. $Q_{\mathrm{top}}(X)$) denote the group of isometry (resp. isomorphism) classes of rank one bilinear spaces (resp. rank one complex bilinear spaces) over X (resp. $X(\mathbb{C})$) with the group law defined by the tensor product. There are maps of sets

$$Q(X) \to GW^0(X), [\mathcal{L},\phi] \mapsto [\mathcal{L},\phi] \qquad Q_{\mathrm{top}}(X(\mathbb{C})) \to GW^0_{\mathrm{top}}(X(\mathbb{C})), [L,\epsilon] \mapsto [L,\epsilon].$$

Let $\mathrm{Pic}_{\mathbb{R}}(X(\mathbb{C}))$ be the group of isomorphism classes of rank one real vector bundles over $X(\mathbb{C})$.

Lemma 4.3. *The following diagram is commutative*

$$\begin{array}{ccccc} GW^0(X) & \xrightarrow{f} & GW^0_{\mathrm{top}}(X(\mathbb{C})) & \xrightarrow{g} & KO^0(X(\mathbb{C})) \\ \uparrow & & {\scriptstyle u}\uparrow & & {\scriptstyle v}\uparrow \\ Q(X) & \xrightarrow{\tilde{f}} & Q_{\mathrm{top}}(X(\mathbb{C})) & \xrightarrow{\tilde{g}} & \mathrm{Pic}_{\mathbb{R}}(X(\mathbb{C})) \end{array}$$

where $\tilde{f}([\mathcal{L},\phi])$ *(resp.* $\tilde{g}([L,\epsilon])$*) is defined as* $[\mathcal{L}(\mathbb{C}),\phi(\mathbb{C})]$ *(resp.* $[\mathfrak{R}(L,\epsilon)]$*).*

Proof. The square on the left-hand side is obviously commutative. It remains to show that the right-hand side square is commutative. Check that the map $\tilde{g}$ is well-defined. Note that, for each couple of complex bilinear spaces (L',ϵ') and (L,ϵ) on $X(\mathbb{C})$, if $\mathfrak{R}(L',\epsilon')$ is isomorphic to $\mathfrak{R}(L,\epsilon)$, then (L',ϵ') is isometric to (L,ϵ). Besides, the map $\tilde{g}$ has image in $\mathrm{Pic}_{\mathbb{R}}(X(\mathbb{C}))$. To see this, suppose $\tilde{g}([L,\epsilon]) = [\mathfrak{R}(L,\epsilon)]$ is not in $\mathrm{Pic}_{\mathbb{R}}(X(\mathbb{C}))$ for some $[L,\epsilon] \in Q_{\mathrm{top}}(X(\mathbb{C}))$. It follows that $X(\mathbb{C})$ has a point with an open neighborhood U such that $\mathfrak{R}(L,\epsilon)|_U$ is isomorphic to $U\times\mathbb{R}^n$ with $n \neq 1$. Then, $L|_U$ is isomorphic to $U\times\mathbb{C}^n$ $(n\neq 1)$, since $\mathfrak{R}(L,\epsilon)\otimes_{\mathbb{R}}\mathbb{C} \cong L$. This contradicts the assumption that the bundle L has rank one. Then, it is clear that $g\circ u = v\circ\tilde{g}$. □

4.3 Comparison maps and cellular varieties

Let $\mathcal{H}(\mathbb{C})$ (resp. $\mathcal{SH}(\mathbb{C})$) be the unstable $\mathbb{A}^1$-homotopy category (resp. the stable $\mathbb{A}^1$-homotopy category) over $\mathbb{C}$. Let $\mathcal{H}_\bullet(\mathbb{C})$ be the pointed version of $\mathcal{H}(\mathbb{C})$. There are objects in $\mathcal{H}_\bullet(\mathbb{C})$:

S^1_s – the constant sheaf represented by $\Delta^1/\partial\Delta^1$ pointed canonically;
S^1_t – the sheaf represented by $\mathbb{A}^1 - \{0\}$ pointed by 1;
T – the sheaf represented by the projective line $\mathbb{P}^1$ pointed by ∞.

Set $S^{p,q} = (S^1_s)^{\wedge(p-q)} \wedge (S^1_t)^{\wedge q}$ with $p \geq q \geq 0$. Then, $S^{2,1}$ and T are $\mathbb{A}^1$-weakly equivalent. See Section 3.2 [9] for details and Section 1.4 [22] for discussion. One may take these objects to $\mathcal{SH}(\mathbb{C})$. The category $\mathcal{SH}(\mathbb{C})$ is triangulated with translation functor $S^{1,0} \wedge -$. Set $\widetilde{KO}^{p,q}(\mathcal{X}) := [\Sigma^\infty \mathcal{X}, S^{p,q} \wedge KO]$ and $KO^{p,q}(X) := [\Sigma^\infty X_+, S^{p,q} \wedge KO]$ where $\mathcal{X} \in \mathcal{H}_\bullet(\mathbb{C})$ and $X \in \mathcal{H}(\mathbb{C})$. The object $KO \in \mathcal{SH}(\mathbb{C})$ is the geometric model of Hermitian K-theory in the $\mathbb{A}^1$-homotopy theory defined by Schlichting and Tripathi (See Section 1.5 [22]). Moreover, there are isomorphisms $GW^q(X) \cong KO^{2q,q}(X)$ and $W^q(X) \cong KO^{2q-1,q-1}(X)$. One defines comparison maps (cf. Section 2 [22])

$$\begin{array}{rccc} \tilde{k}_h^{p,q}(\mathcal{X}): & \widetilde{KO}^{p,q}(\mathcal{X}) & \to & \widetilde{KO}^{p}(\mathcal{X}(\mathbb{C})) \\ k_h^{p,q}(X): & KO^{p,q}(X) & \to & KO^{p}(X(\mathbb{C})). \end{array}$$

In particular, when X is a complex smooth variety, we have

$$\begin{array}{rrccc} gw^q = & k_h^{2q,q}: & GW^q(X) & \to & KO^{2q}(X(\mathbb{C})) \\ w^{q+1} = & k_h^{2q+1,q}: & W^{q+1}(X) & \to & KO^{2q+1}(X(\mathbb{C})). \end{array}$$

Theorem 4.1. *Let X be a complex smooth cellular variety. Assume further that Z is cellular and closed in X, and let $U := X - Z$. Then, the map $k_h^{2q,q}(U)$ is an isomorphism and the map $k_h^{2q+1,q}(U)$ is injective.*

Proof. When $Z = \emptyset$, this theorem is a special case of Theorem 2.6 [22]. We slightly modify the proof of Theorem 2.6 [22] to show this theorem by induction on cells. Let $Z = Z_N \supset Z_{N-1} \supset \cdots \supset Z_0 = \emptyset$ be the filtration such that

$$Z_{k+1} - Z_k \cong \mathbb{A}^{n_k} =: C_k.$$

Set $U_k := X - Z_k$ for each $0 \leq k \leq N$. Note that there is another filtration $X = U_0 \supset U_1 \supset \cdots \supset U_N = U$ with $U_k - U_{k+1} = Z_{k+1} - Z_k \cong C_k$ closed in U_k. Then, the normal bundle $\mathcal{N}_{U_k/C_k}$ of U_k in C_k is trivial. Hence, Thom$(\mathcal{N}_{U_k/C_k})$ and $S^{2d,d}$ are $\mathbb{A}^1$-weakly equivalent, where d is the codimension of C_k in U_k, cf. Proposition 2.17 (page 112) [9]. We can therefore deduce the commutative ladder diagram in Figure 1 (page 486) [22]. Assume by induction, the theorem is true for U_k, and we want to prove it for U_{k+1}. It is known that $\tilde{k}_h^{2q,q}(S^{2d,d})$ and $\tilde{k}_h^{2q+1,q}(S^{2d,d})$ are isomorphisms and that $\tilde{k}_h^{2q+2,q}(S^{2d,d})$ is injective, cf. the proof of Theorem 2.6 [22]. The results follow by the 5-lemma. □

4.4 Grothendieck-Witt group of a deleted quadric

In this subsection, we simply write $X = D_+(q_s), Q = Q_{s-2}$ and $DQ = DQ_{s-1}$. Note that Q is smooth and closed in $\mathbb{P}^{s-1}$ of codimension 1. The normal bundle $\mathcal{N}$ of Q in $\mathbb{P}^{s-1}$ is isomorphic to $\mathcal{O}_Q(2)$.

Theorem 4.2. *The comparison map $gw^q : GW^q(DQ_{\mathbb{C}}) \to KO^{2q}(DQ(\mathbb{C}))$ is an isomorphism for each $q \in \mathbb{Z}$.*

Proof. This theorem is a consequence of Theorem 4.1. □

Lemma 4.4. *The group $GW^0(DQ_{\mathbb{C}})$ is isomorphic to $GW^0(DQ_K)$.*

Proof. Applying Corollary 4.1 and the dévissage theorem, we observe that the vertical maps of W and GW-groups in the following commutative diagram are all isomorphisms

$$\begin{array}{ccccccccc}
GW^0_{Q_K}(\mathbb{P}^{s-1}_K) & \longrightarrow & GW^0(\mathbb{P}^{s-1}_K) & \longrightarrow & GW^0(DQ_K) & \longrightarrow & W^1_{Q_K}(\mathbb{P}^{s-1}_K) & \longrightarrow & W^1(\mathbb{P}^{s-1}_K) \\
\downarrow & & \downarrow & & \Omega\downarrow & & \downarrow & & \downarrow \\
GW^0_{Q_{\mathbb{C}}}(\mathbb{P}^{s-1}_{\mathbb{C}}) & \longrightarrow & GW^0(\mathbb{P}^{s-1}_{\mathbb{C}}) & \longrightarrow & GW^0(DQ_{\mathbb{C}}) & \longrightarrow & W^1_{Q_{\mathbb{C}}}(\mathbb{P}^{s-1}_{\mathbb{C}}) & \longrightarrow & W^1(\mathbb{P}^{s-1}_{\mathbb{C}})
\end{array}$$

where all vertical maps are induced from the left-hand side of the diagram (2) (use the 5-lemma to see the middle map Ω is an isomorphism). □

Recall the isomorphism of varieties $i_K : DQ_K \to X_K$ in Remark 2.1 (i). Note that $i_{\mathbb{C}} : DQ_{\mathbb{C}} \to X_{\mathbb{C}}$ gives a homeomorphism $i_{(\mathbb{C})} : DQ(\mathbb{C}) \to X(\mathbb{C})$ by taking $\mathbb{C}$-rational points. Besides, let $\upsilon : \mathbb{R}P^{s-1} \to X(\mathbb{C})$ be the natural embedding. The space $\mathbb{R}P^{s-1}$ is a deformation retract of the space $X(\mathbb{C})$ in the category of real spaces, cf. Lemma 6.3 [15]. These maps that induce isomorphisms in KO-theory or GW-theory are described in the diagram (4).

$$\begin{array}{ccc|c|ccc}
\multicolumn{3}{c}{\text{Hermitian } K\text{-theory}} & & \multicolumn{3}{c}{\text{Topological } KO\text{-theory}} \\
 & & GW^0(DQ_{\mathbb{C}}) & \xrightarrow{gw^0} & KO^0(DQ(\mathbb{C})) & & \\
 & & \Omega\uparrow & & \downarrow i^*_{(\mathbb{C})} & & \\
GW^0(X_K) & \xrightarrow{i^*_K} & GW^0(DQ_K) & & KO^0(X(\mathbb{C})) & \xrightarrow{\upsilon^*} & KO^0(\mathbb{R}P^{s-1})
\end{array} \qquad (4)$$

Proof of Theorem 3.1. Let ξ_{top} denote the tautological line bundle over $\mathbb{R}P^{s-1}$. Recall that there is an isomorphism of rings

$$KO^0(\mathbb{R}P^{s-1}) \cong \mathbb{Z}[\nu_{\text{top}}]/(\nu_{\text{top}}^2 + 2\nu_{\text{top}}, 2^{\varphi(s-1)}\nu_{\text{top}})$$

where ν_{top} represents the class $[\xi_{\text{top}}]-1$, cf. Section 7 [1] or Chapter IV [12]. Note that $\text{Pic}_{\mathbb{R}}(\mathbb{R}P^{s-1})$ is isomorphic to $\mathbb{Z}/2$. Let $\vartheta : GW^0(X_K) \to KO^0(\mathbb{R}P^{s-1})$ be the composition of maps in the diagram (4). We have known ϑ is an isomorphism. Therefore, to prove Theorem 3.1, we only need to show $\vartheta(\nu) = \nu_{\text{top}}$. To achieve this, we give the following lemma.

LEMMA 4.5. *The group $Q(X_K)$ (cf. Section 4.2) is isomorphic to $\mathbb{Z}/2$.*

Proof. There is an exact sequence (cf. Chapter IV.1 (page 229) [14])

$$1 \longrightarrow \mathcal{O}(X_K)^*/\mathcal{O}(X_K)^{2*} \longrightarrow Q(X_K) \xrightarrow{F} {}_2\mathrm{Pic}(X_K) \longrightarrow 1$$

where ${}_2\mathrm{Pic}(X_K)$ means the subgroup of elements of order ≤ 2 in $\mathrm{Pic}(X_K)$ and where F is the forgetful map. Note that ${}_2\mathrm{Pic}(X_K) \cong \mathbb{Z}/2$, cf. [19]. In addition, observe that $\mathcal{O}(X_K)^* \cong R^* = K^*$ and that the group K^*/K^{2*} is trivial. It follows that the forgetful map F is an isomorphism. In fact, it sends the non-trivial element $[\xi, \sigma]$ (in Lemma 3.1) to the non-trivial element $[\xi]$. □

Proof of Theorem 3.1 (Continued). In light of Lemma 4.3, there is a map

$$\tilde{\vartheta} : Q(X_K) \to \mathrm{Pic}_{\mathbb{R}}(\mathbb{R}P^{s-1})$$

(obtained in an obvious way) such that the following diagram is commutative

$$\begin{array}{ccc} GW^0(X_K) & \xrightarrow{\vartheta} & KO^0(\mathbb{R}P^{s-1}) \\ {\scriptstyle i}\uparrow & & {\scriptstyle j}\uparrow \\ \mathbb{Z}/2 \cong Q(X_K) & \xrightarrow{\tilde{\vartheta}} & \mathrm{Pic}_{\mathbb{R}}(\mathbb{R}P^{s-1}) \cong \mathbb{Z}/2. \end{array}$$

The map i is injective (Note that $[\xi]$ and 1 are distinct elements in $K_0(X_K)$ by its computation in Proposition 2.4 [7]). The map j is injective by the computation of $KO^0(\mathbb{R}P^{s-1})$. Then, we see that $\tilde{\vartheta}$ is bijective and must send $[\xi, \sigma]$ to $[\xi_{\mathrm{top}}]$. Therefore, $\vartheta([\xi, \sigma]) = [\xi_{\mathrm{top}}]$, so that $\vartheta(\nu) = \nu_{\mathrm{top}}$. □

A OPERATIONS ON THE GROTHENDIECK-WITT GROUP

The γ^i-operations on GW^0 of an affine scheme are analogous to those on the topological KO-theory which have been explained in Section 1 and 2 in [2]. For readers' convenience, details have been added.

Let $\mathrm{Bil}(X)$ be the set of isometry classes of bilinear spaces over a scheme X. The orthogonal sum and the tensor product of bilinear spaces over the scheme X make $\mathrm{Bil}(X)$ a semi-ring with a zero and a multiplicative identity. Then, by taking the associated Grothendieck ring $K(\mathrm{Bil}(X))$, we have a homomorphism of the underlying semi-rings

$$\iota : \mathrm{Bil}(X) \to K(\mathrm{Bil}(X))$$

satisfying the universal property (see Chapter I.4 (page 137) [14] for details).

REMARK A.1. For an affine scheme X, the ring $GW^0(X)$ is identified with $K(\mathrm{Bil}(X))$, cf. Chapter I.4 Proposition 1 (page 138) [14].

DEFINITION A.1 (Chapter IV.3 (page 235) [14]). Let $(\mathcal{F}, \phi)$ be a bilinear space over a scheme X. Let i be a strictly positive integer. The *i-th exterior power of* $(\mathcal{F}, \phi)$, denoted by $\Lambda^i(\mathcal{F}, \phi)$, is the symmetric bilinear space $(\Lambda^i\mathcal{F}, \Lambda^i\phi)$ over X, where $\Lambda^i\mathcal{F}$ is the i-th exterior power of the locally free sheaf $\mathcal{F}$ and where

$$\Lambda^i\phi : \Lambda^i\mathcal{F} \times_X \Lambda^i\mathcal{F} \to \mathcal{O}_X$$

is a morphism of sheaves consisting of a symmetric bilinear form

$$\Lambda^i\phi(U) : \Lambda^i\mathcal{F}(U) \times \Lambda^i\mathcal{F}(U) \to \mathcal{O}_X(U)$$

defined by

$$\Lambda^i\phi(U)(x_1 \wedge \cdots \wedge x_i, y_1 \wedge \cdots \wedge y_i) = \det([\phi(U)(x_j, y_k)]_{i\times i})$$

for each open subscheme U of X. The exterior power $\Lambda^0(\mathcal{F}, \phi)$ for every bilinear space $(\mathcal{F}, \phi)$ (over X) is defined as $1 = (\mathcal{O}, \mathrm{id})$.

LEMMA A.1. *Let $(\mathcal{F}, \phi), (\mathcal{G}, \psi)$ be bilinear spaces over X. Then, we have that*

(a) $\Lambda^1(\mathcal{F}, \phi) = (\mathcal{F}, \phi)$;

(b) $\Lambda^k((\mathcal{F}, \phi) \oplus (\mathcal{G}, \psi)) \cong \bigoplus_{r+s=k} \Lambda^r(\mathcal{F}, \phi) \otimes \Lambda^s(\mathcal{G}, \psi)$;

(c) *If $(\mathcal{F}, \phi)$ is of constant rank $\Theta \geq 1$, $\Lambda^i(\mathcal{F}, \phi) = 0$ whenever $i \geq \Theta$.*

Proof. (a) and (c) are clear. For (b), it is enough to show that the canonical isomorphism of locally free sheaves

$$\varrho : \bigoplus_{r+s=k} \Lambda^r\mathcal{F} \otimes \Lambda^s\mathcal{G} \to \Lambda^k(\mathcal{F} \oplus \mathcal{G})$$

respects the symmetric bilinear forms. This may be checked locally. Let U be an affine open subset of the scheme X. One may choose elements

$$x^{(t)} = x_{1,t} \wedge \cdots \wedge x_{r,t} \in \Lambda^r\mathcal{F}(U) \text{ and } y^{(t)} = y_{1,t} \wedge \cdots \wedge y_{s,t} \in \Lambda^s\mathcal{G}(U)$$

for $t \in \{1, 2\}$. Let $a_{i,j} := \phi(U)(x_{i,1}, x_{j,2})$ and $b_{k,l} := \psi(U)(y_{k,1}, y_{l,2})$. We have matrices $A = [a_{i,j}]_{r\times r}$ and $B = [b_{k,l}]_{s\times s}$. On the one hand, we get that

$$\Lambda^r\phi(U) \otimes \Lambda^s\psi(U)(x^{(1)} \otimes y^{(1)}, x^{(2)} \otimes y^{(2)}) = \det(A) \times \det(B). \tag{5}$$

On the other hand, set

$$u^{(t)} := \varrho(U)(x^{(t)} \otimes y^{(t)}) \in \Lambda^{r+s}(\mathcal{F}(U) \oplus \mathcal{G}(U))$$

for $t \in \{1, 2\}$. Consider the elements

$$(x_{j,t}, 0), (0, y_{k,t}) \in \mathcal{F}(U) \oplus \mathcal{G}(U)$$

for $1 \leq j \leq r, 1 \leq k \leq s$ and $t \in \{1, 2\}$. It is clear that

$$u^{(t)} = (x_{1,t}, 0) \wedge \cdots \wedge (x_{r,t}, 0) \wedge (0, y_{1,t}) \wedge \cdots \wedge (0, y_{s,t})$$

for $t \in \{1,2\}$. Then, we deduce that

$$\Lambda^{r+s}(\phi(U) \oplus \psi(U))(u^{(1)}, u^{(2)}) = \det \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \tag{6}$$

Note (5) = (6). The result follows. □

Let $A(X)$ denote the group $1 + tK(\mathrm{Bil}(X))[[t]]$ of formal power series with constant term 1 (under multiplication). Consider a map

$$\Lambda_t : \mathrm{Bil}(X) \to A(X), [\mathcal{F}, \phi] \mapsto 1 + \sum_{i \geq 1} \Lambda^i([\mathcal{F}, \phi])t^i.$$

If $I : (\mathcal{F}, \phi) \to (\mathcal{G}, \psi)$ is an isometry of bilinear spaces, so is the natural map

$$\Lambda^i I : \Lambda^i(\mathcal{F}, \phi) \to \Lambda^i(\mathcal{G}, \psi).$$

Then, the map Λ_t is well-defined. Furthermore, Lemma A.1 (b) implies that Λ_t is a homomorphism of the underlying monoids. By the universal property of K-theory, we can lift Λ_t to a homomorphism of groups

$$\lambda_t : K(\mathrm{Bil}(X)) \to A(X)$$

such that $\lambda_t \circ \iota = \Lambda_t$. Taking coefficients of λ_t, we get operators (not homomorphisms in general)

$$\lambda^i : K(\mathrm{Bil}(X)) \to K(\mathrm{Bil}(X)).$$

Set $\gamma_t = \lambda_{t/(1-t)}$ and write $\gamma_t = 1 + \sum_{i \geq 1} \gamma^i t^i$. Again, we obtain operators

$$\gamma^i : K(\mathrm{Bil}(X)) \to K(\mathrm{Bil}(X)).$$

Explicitly, we deduce

$$\sum_{i \geq 0} \gamma^i t^i = \sum_{i \geq 0} \lambda^i t^i (1-t)^{-i} = 1 + \sum_{i \geq 1} (\sum_{i \geq s \geq 1} \lambda^s \binom{i-1}{s-1}) t^i.$$

Hence, the γ^i are certain $\mathbb{Z}$-linear combinations of the λ^s. By definition, the map γ_t is a homomorphism of groups. Hence, for all $x, y \in K(\mathrm{Bil}(X))$, we have

Corollary A.1. (a) $\gamma_t(x+y) = \gamma_t(x)\gamma_t(y)$;

(b) $\gamma_t([\eta] - 1) = 1 + t([\eta] - 1)$ *where* η *is a bilinear space of rank* 1 *over* X;

(c) *If* $(\mathcal{F}, \phi) \in \mathrm{Bil}(X)$ *is of constant rank* $\Theta \geq 1$, $\gamma^i((\mathcal{F}, \phi) - \Theta) = 0$ *if* $i \geq \Theta$.

Proof. (a) is proved. For (b), we deduce

$$\gamma_t([\eta] - 1) = \frac{\gamma_t([\eta])}{\gamma_t(1)} = \frac{\lambda_{t/(1-t)}([\eta])}{\lambda_{t/(1-t)}(1)} = \frac{1 + [\eta]t/(1-t)}{(1-t)^{-1}} = 1 + t([\eta] - 1).$$

For (c), see the proof of Lemma 2.1 [2]. □

Acknowledgement

This paper has been written during my PhD study. I am grateful to my PhD supervisor Marco Schlichting for introducing to me this interesting problem and guiding me along the way. I appreciate his time and commitment. I want to thank the anonymous referee and the editors for helpful remarks and comments. I also thank Jens Hornbostel, Dmitriy Rumynin and Marcus Zibrowius for useful discussion. Finally, I thank Miles Reid for helping me improve the presentation of this paper.

References

[1] J. F. Adams, *Vector fields on spheres*, Ann. of Math. (3) 75 (1962), 603-632.

[2] M. F. Atiyah, *Immersions and embeddings of manifolds*, Topology 1 (1962), 125-132.

[3] P. Balmer, *Witt groups*, Handbook of K-Theory II, Springer-Verlag Berlin Heidelberg, 2005.

[4] P. Balmer, *Witt cohomology, Mayer-Vietoris, homotopy invariance and the Gersten conjecture*, K-Theory 1 (2001), 15-30.

[5] P. Balmer, *Triangular Witt groups. Part II: From usual to derived*, Math. Z. 263 (2001), 351-382.

[6] A. J. Berrick and M. Karoubi, *Hermitian K-theory of the integers*, American Journal of Mathematics 127 (2005), 785-823.

[7] D. Dugger and D. Isaksen, *Algebraic K-theory and sums-of-squares formulas*, Documenta Math. 10 (2005), 357-366.

[8] D. Dugger and D. Isaksen, *The Hopf condition for bilinear forms over arbitrary fields*, Ann. of Math. (2) 165 (2007), 943-964.

[9] F. Morel and V. Voevodsky, *$\mathbb{A}^1$-homotopy theory of schemes*, Publications Mathématiques de l'I.H.É.S. 90 (1999), 45-143.

[10] S. Gille, *The general dévissage theorem for Witt groups*, Arch. Math. 88 (2007), 333-343.

[11] S. Gille and A. Nenashev, *Pairings in triangular Witt theory*, Journal of Algebra 261 (2003), 292-309.

[12] M. Karoubi, *K-theory: An introduction*, Grundlehren der mathematischen Wissenschaften 226, Springer-Verlag Berlin Heidelberg, 1978.

[13] M. Knebusch, *Isometrien über semilokalen Ringen*, Math. Z., 108 (1969), 255-268.

[14] M. Knebusch, *Symmetric bilinear forms over algebraic varieties*, In: Orzech, G., (ed.) Conference on quadratic forms. Queen's papers in pure and applied mathematics 46 (1977), Queens University, Kingston, Ontario, 103-283.

[15] P. S. Landweber, *Fixed point free conjugations on complex manifolds*, Ann. of Math. (2) 86 (1967), 491-502.

[16] M. Schlichting, *Hermitian K-theory of exact categories*, Journal of K-theory 5 (2010), 105-165.

[17] J. P. Serre, *Local fields*, Springer-Verlag New York Inc., 1979.

[18] D. B. Shapiro, *Products of sums of squares*, Expo. Math. 2 (1984), 235-261.

[19] D. B. Shapiro and M. Szyjewski, *Product formulas for quadratic forms*, Bol. Soc. Mat. Mexicana 37 (1992), 463-474.

[20] D. B. Shapiro, *Composition of quadratic forms*, De Gruyter Expositions in Mathematics 33, Walter de Gruyter, 2000.

[21] S. Yuzvinsky, *Orthogonal pairings of Euclidean spaces*, Michigan Math. J. 28 (1981), 131-145.

[22] M. Zibrowius, *Witt groups of complex cellular varieties*, Documenta Math. 16 (2011), 465-511.

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