

PARTIAL CLASSIFICATION OF THE BAUMSLAG-SOLITAR
GROUP VON NEUMANN ALGEBRAS

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ABSTRACT. We prove that the rational number $|n/m|$ is an invariant of the group von Neumann algebra of the Baumslag-Solitar group $BS(n, m)$. More precisely, if $L(BS(n, m))$ is isomorphic with $L(BS(n', m'))$, then $|n'/m'| = |n/m|^{\pm 1}$. We obtain this result by associating to abelian, but not maximal abelian, subalgebras of a II_1 factor, an equivalence relation that can be of type III. In particular, we associate to $L(BS(n, m))$ a canonical equivalence relation of type $III_{|n/m|}$.

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1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

Some of the deepest open problems in functional analysis center around the classification of group von Neumann algebras $L(G)$ associated with certain natural families of countable groups G . In the case of the free groups, this becomes the famous free group factor problem asking whether $L(\mathbb{F}_n) \cong L(\mathbb{F}_m)$ when $n, m \geq 2$ and $n \neq m$. For property (T) groups with infinite conjugacy classes (icc), this leads to Connes's rigidity conjecture ([Co80]) asserting that an isomorphism $L(G) \cong L(\Lambda)$ between the property (T) factors entails an isomorphism $G \cong \Lambda$ of the groups.

As a consequence of Connes's uniqueness theorem of injective II_1 factors ([Co75]), the group von Neumann algebra $L(G)$ of an amenable icc group G is

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isomorphic with the unique hyperfinite II_1 factor R . In the nonamenable case, many nonisomorphic groups G are known to have nonisomorphic group von Neumann algebras $L(G)$. Nevertheless, concerning the classification of group von Neumann algebras of natural families of groups, e.g. lattices in simple Lie groups, little is known. A notable exception however is [CH88] where it is shown that for $n \neq m$, lattices in $\text{Sp}(n, 1)$, respectively $\text{Sp}(m, 1)$, have nonisomorphic group von Neumann algebras.

Since 2001, Popa has been developing a new arsenal of techniques to study II_1 factors, called deformation/rigidity theory. This theory has provided several classes \mathcal{G} of groups such that an isomorphism $L(G) \cong L(\Lambda)$ with both $G, \Lambda \in \mathcal{G}$ entails the isomorphism $G \cong \Lambda$. By [Po04], this holds in particular when \mathcal{G} is the class of wreath product groups of the form $(\mathbb{Z}/2\mathbb{Z}) \wr \Gamma$ with Γ an icc property (T) group.

In [IPV10], the first W^* -superrigidity theorems for group von Neumann algebras were discovered, yielding icc groups G such that an isomorphism $L(G) \cong L(\Lambda)$ with Λ an *arbitrary* countable group, implies that $G \cong \Lambda$. The groups G discovered in [IPV10] are generalized wreath products of a special form. In [BV12], it was then shown that one can actually take $G = (\mathbb{Z}/2\mathbb{Z})^{(\Gamma)} \rtimes (\Gamma \times \Gamma)$ with Γ ranging over a large family of nonamenable groups including the free groups \mathbb{F}_n , $n \geq 2$.

In this article, we apply Popa's deformation/rigidity theory to partially classify the group von Neumann algebras of the Baumslag-Solitar groups $\text{BS}(n, m)$. Recall that for all $n, m \in \mathbb{Z} - \{0\}$, this group is defined as the group generated by a and b subject to the relation $ba^nb^{-1} = a^m$. So,

$$\text{BS}(n, m) := \langle a, b \mid ba^nb^{-1} = a^m \rangle .$$

The Baumslag-Solitar groups were introduced in [BS62] as the first examples of finitely presented non-Hopfian groups. Ever since, they have been used as examples and counterexamples for numerous group theoretic phenomena. Therefore, it is a natural problem to classify the group von Neumann algebras $L(\text{BS}(n, m))$.

Whenever $|n| = 1$ or $|m| = 1$, the group $\text{BS}(n, m)$ is solvable, hence amenable. So we always assume that $|n| \geq 2$ and $|m| \geq 2$. In that case, $\text{BS}(n, m)$ contains a copy of the free group \mathbb{F}_2 and hence, is nonamenable. In [Mo91], the Baumslag-Solitar groups were classified up to isomorphism: $\text{BS}(n, m) \cong \text{BS}(n', m')$ if and only if $\{n, m\} = \{\varepsilon n', \varepsilon m'\}$ for some $\varepsilon \in \{-1, 1\}$. So, up to isomorphism, we only consider $2 \leq n \leq |m|$. Finally by [St05, Exemple 2.4], the group $\text{BS}(n, m)$ is icc if and only if $|n| \neq |m|$. Therefore, we always assume that $2 \leq n < |m|$.

Using Popa's deformation/rigidity theory and in particular his spectral gap rigidity ([Po06]) and the work on amalgamated free products ([IPP05]), several structural properties of the II_1 factors $M = L(\text{BS}(n, m))$ were proven. In particular, it was shown in [Fi10] that M is not solid, that M is prime and that M has no Cartan subalgebra. More generally, it is proven in [Fi10] that any amenable regular von Neumann subalgebra of M must have a nonamenable relative commutant.

Our main result is the following partial classification theorem for the Baumslag-Solitar group von Neumann algebras $L(\text{BS}(n, m))$. Whenever M is a II_1 factor and $t > 0$, we denote by M^t the *amplification* of M . Up to unitary conjugacy, M^t is defined as $p(M_n(\mathbb{C}) \otimes M)p$ where p is a projection satisfying $(\text{Tr} \otimes \tau)(p) = t$. The II_1 factors M and N are called *stably isomorphic* if there exists a $t > 0$ such that $M \cong N^t$.

THEOREM A. *Let $n, m, n', m' \in \mathbb{Z}$ such that $2 \leq n < |m|$ and $2 \leq n' < |m'|$. If $L(\text{BS}(n, m))$ is stably isomorphic with $L(\text{BS}(n', m'))$, then $\frac{n}{|m|} = \frac{n'}{|m'|}$.*

Note that Theorem A formally resembles, but is independent of, the results in [Ki11] on orbit equivalence relations of essentially free ergodic probability measure preserving actions of Baumslag-Solitar groups, especially [Ki11, Proposition B.2 and Theorem 1.2]. It would be very interesting to find a framework that unifies both types of results.

We prove our Theorem A by associating a canonical equivalence relation to $L(\text{BS}(n, m))$ and proving that it is of type $\text{III}_{n/|m|}$. More precisely, assume that (M, τ) is a von Neumann algebra with separable predual, equipped with a faithful normal tracial state. Whenever $A \subset M$ is an abelian von Neumann subalgebra, the normalizer

$$\mathcal{N}_M(A) := \{u \in \mathcal{U}(M) \mid uAu^* = A\}$$

induces a group of trace preserving automorphisms of A . Writing $A = L^\infty(X, \mu)$ with μ being induced by $\tau|_A$, the corresponding orbit equivalence relation is a countable probability measure preserving (pmp) equivalence relation on (X, μ) .

More generally, we can consider the set of partial isometries

$$\{u \in M \mid u^*u \text{ and } uu^* \text{ are projections in } A' \cap M \text{ and } uAu^* = Auu^*\}. \quad (1)$$

Every such partial isometry induces a partial automorphism of A and hence a partial automorphism of (X, μ) . We denote by $\mathcal{R}(A \subset M)$ the equivalence relation generated by all these partial automorphisms. When $A \subset M$ is maximal abelian, i.e. $A' \cap M = A$, then $\mathcal{R}(A \subset M)$ coincides with the orbit equivalence relation induced by the normalizer $\mathcal{N}_M(A)$. In particular, in that case the equivalence relation $\mathcal{R}(A \subset M)$ preserves the probability measure μ .

If however $A \subset M$ is not maximal abelian, the partial automorphisms of A induced by the partial isometries in the set (1) need not be trace preserving. So in general, $\mathcal{R}(A \subset M)$ can be an equivalence relation of type III.

Our main technical result is Theorem 3.3 below, roughly saying the following. If $A, B \subset M$ are abelian subalgebras such that $\mathcal{Z}(A' \cap M) = A$ and $\mathcal{Z}(B' \cap M) = B$, and if there exist intertwining bimodules $A \prec B$ and $B \prec A$ (in the sense of Popa, see [Po03] and Theorem 2.3 below), then the equivalence relations $\mathcal{R}(A \subset M)$ and $\mathcal{R}(B \subset M)$ must be stably isomorphic. In particular, their types must be the same.

In Section 4, we apply this to $M = L(\text{BS}(n, m))$ and A equal to the abelian von Neumann subalgebra generated by the unitary u_a . We prove that $\mathcal{R}(A \subset M)$ is the unique hyperfinite ergodic equivalence relation of type $\text{III}_{n/|m|}$.

The proof of Theorem A can then be outlined as follows. First we note that the von Neumann algebra $A' \cap M$ is nonamenable. Conversely if $Q \subset M$ is a nonamenable subalgebra, it was proven in [CH08], using spectral gap rigidity ([Po06]) and the structure theory of amalgamated free product factors ([IPP05]), that $Q' \cap M \prec A$. So, up to intertwining-by-bimodules, the position of A inside M is “canonical”. Therefore a stable isomorphism between $L(\text{BS}(n, m))$ and $L(\text{BS}(n', m'))$ will preserve, up to intertwining-by-bimodules, these canonical abelian subalgebras. Hence their associated equivalence relations are stably isomorphic and, in particular, have the same type. This gives us the equality $n/|m| = n'/|m'|$.

2. PRELIMINARIES

We denote by (M, τ) a von Neumann algebra equipped with a faithful normal tracial state τ . We always assume that M has a *separable predual*. If B is a von Neumann subalgebra of (M, τ) , we denote by E_B the unique trace preserving *conditional expectation* of M onto B .

Whenever $x \in M$ is a normal element, we denote by $\text{supp}(x)$ its support, i.e. the smallest projection $p \in M$ that satisfies $xp = x$ (or equivalently, $px = x$).

Let \mathcal{R} be a countable nonsingular (i.e. measure class preserving) equivalence relation on a standard probability space (X, μ) . We denote by $[[\mathcal{R}]]$ the *full pseudogroup* of \mathcal{R} , i.e. the pseudogroup of all partial nonsingular automorphisms φ of X such that the graph of φ is contained in \mathcal{R} . We denote the domain of φ by $\text{dom}(\varphi)$ and its range by $\text{ran}(\varphi)$. We denote by $[x]$ the equivalence class of $x \in X$.

Assume that also \mathcal{R}' is a countable nonsingular equivalence relation on the standard probability space (X', μ') . The equivalence relations \mathcal{R} and \mathcal{R}' are called

- *isomorphic*, if there exists a nonsingular isomorphism $\Delta : X \rightarrow X'$ such that $\Delta([x]) = [\Delta(x)]$ for almost every $x \in X$;
- *stably isomorphic*, if there exist Borel subsets $Z \subset X$ and $Z' \subset X'$ that meet almost every orbit and a nonsingular isomorphism $\Delta : Z \rightarrow Z'$ such that $\Delta([x] \cap Z) = [\Delta(x)] \cap Z'$ for almost every $x \in Z$.

2.1. HNN EXTENSIONS AND BAUMSLAG-SOLITAR GROUPS

Let G be a group, $H < G$ a subgroup and $\theta : H \rightarrow G$ an injective group homomorphism. The *HNN extension* $\text{HNN}(G, H, \theta)$ is defined as the group generated by G and an additional element t subject to the relation $\theta(h) = tht^{-1}$ for all $h \in H$. So,

$$\text{HNN}(G, H, \theta) = \langle G, t \mid \theta(h) = tht^{-1} \text{ for all } h \in H \rangle .$$

Elements of $\text{HNN}(G, H, \theta)$ can be canonically written as “reduced words” using as letters the elements of G and the letters $t^{\pm 1}$. More precisely, we have the following lemma.

LEMMA 2.1 (Britton’s lemma, [Br63]). *Consider the expression $g = g_0 t^{n_1} g_1 t^{n_2} \dots t^{n_k} g_k$ with $k \geq 0$, $g_0, g_k \in G$, $g_1, \dots, g_{k-1} \in G - \{e\}$ and $n_1, \dots, n_k \in \mathbb{Z} - \{0\}$. We call this expression reduced if the following two conditions hold:*

- for every $i \in \{1, \dots, k - 1\}$ with $n_i > 0$ and $n_{i+1} < 0$, we have $g_i \notin H$,
- for every $i \in \{1, \dots, k - 1\}$ with $n_i < 0$ and $n_{i+1} > 0$, we have $g_i \notin \theta(H)$.

If the above expression for g is reduced, then $g \neq e$ in the group $\text{HNN}(G, H, \theta)$, unless $k = 0$ and $g_0 = e$. In particular, the natural homomorphism of G to $\text{HNN}(G, H, \theta)$ is injective.

Recall from the introduction that the Baumslag-Solitar group $\text{BS}(n, m)$ is defined for all $n, m \in \mathbb{Z} - \{0\}$ as

$$\text{BS}(n, m) := \langle a, b \mid ba^n b^{-1} = a^m \rangle .$$

It is one of the easiest examples of an HNN extension. We also recall from the introduction that the $\text{BS}(n, m)$ with $2 \leq n < |m|$ form a complete list of all nonamenable icc Baumslag-Solitar groups up to isomorphism. Since we only want to consider the case where $L(\text{BS}(n, m))$ is a nonamenable II_1 factor, we always assume that $2 \leq n < |m|$.

2.2. HILBERT BIMODULES AND INTERTWINING-BY-BIMODULES

If M and N are tracial von Neumann algebras, then a *left M -module* is a Hilbert space \mathcal{H} endowed with a normal $*$ -homomorphism $\pi : M \rightarrow \text{B}(\mathcal{H})$. A *right N -module* is a left N^{op} -module. An *M - N -bimodule* is a Hilbert space \mathcal{H} endowed with commuting normal $*$ -homomorphisms $\pi : M \rightarrow \text{B}(\mathcal{H})$ and $\varphi : N^{\text{op}} \rightarrow \text{B}(\mathcal{H})$. For $x \in M, y \in N$ and $\xi \in \mathcal{H}$, we write $x\xi y$ instead of $\pi(x)\varphi(y^{\text{op}})(\xi)$. We denote an M - N -bimodule \mathcal{H} by ${}_M\mathcal{H}_N$. We call an M - N -bimodule *bifinite* if it is finitely generated both as a left Hilbert M -module and a right Hilbert N -module.

Let A and B be abelian von Neumann algebras. We denote by $\text{PIso}(A, B)$ the set of all partial isomorphisms from A to B , i.e. isomorphisms $\alpha : Aq \rightarrow Bp$, where $q \in A$ and $p \in B$ are projections. We write $\text{PAut}(A)$ instead of $\text{PIso}(A, A)$. Note that to every $\alpha \in \text{PIso}(A, B)$ we can associate an A - B -bimodule ${}_A\mathcal{H}(\alpha)_B$ given by $\mathcal{H}(\alpha) = L^2(Bp)$ and $a\xi b = \alpha(aq)\xi bp$. The composition of two partial isomorphisms is defined as follows: if $\alpha \in \text{PIso}(B, C)$ and $\beta \in \text{PIso}(A, B)$ are given by $\alpha : Bp \rightarrow Cr$ and $\beta : Aq \rightarrow Bp'$ for projections $q \in A, p, p' \in B$ and $r \in C$, then the composition $\alpha \circ \beta \in \text{PIso}(A, C)$ is defined by $x \mapsto \alpha(\beta(x))$ for all $x \in Aq\beta^{-1}(pp')$.

The following is a well known result.

LEMMA 2.2. *Let A and B be abelian von Neumann algebras. Then every bifinite A - B -bimodule ${}_A\mathcal{H}_B$ is isomorphic to a direct sum of bimodules of the form ${}_A\mathcal{H}(\alpha)_B$ with $\alpha \in \text{PIso}(A, B)$.*

We finally recall Popa's *intertwining-by-bimodules* theorem.

THEOREM 2.3 ([Po03, Theorem 2.1 and Corollary 2.3]). *Let (M, τ) be a tracial von Neumann algebra and let $A, B \subset M$ be possibly nonunital von Neumann subalgebras. Denote their respective units by 1_A and 1_B . The following three conditions are equivalent:*

1. $1_A L^2(M) 1_B$ admits a nonzero A - B -subbimodule that is finitely generated as a right B -module.
2. There exist nonzero projections $p \in A$, $q \in B$, a normal unital $*$ -homomorphism $\psi : pAp \rightarrow qBq$ and a nonzero partial isometry $v \in pMq$ such that $av = v\psi(a)$ for all $a \in pAp$.
3. There is no sequence of unitaries $u_n \in \mathcal{U}(A)$ satisfying $\|E_B(xu_n y^*)\|_2 \rightarrow 0$ for all $x, y \in 1_B M 1_A$.

If one of these equivalent conditions holds, we write $A \prec_M B$.

2.3. QUASI-REGULARITY

Let (M, τ) be a tracial von Neumann algebra and $N \subset M$ a von Neumann subalgebra. We denote by $\text{QN}_M(N)$ the *quasi-normalizer* of N inside M , i.e. the unital $*$ -algebra defined by

$$\left\{ a \in M \mid \exists b_1, \dots, b_k \in M, \exists d_1, \dots, d_r \in M \right. \\ \left. \text{such that } Na \subset \sum_{i=1}^k b_i N \text{ and } aN \subset \sum_{j=1}^r N d_j \right\}.$$

We call $N \subset M$ *quasi-regular* if $\text{QN}_M(N)'' = M$.

If $A, B \subset M$ are abelian von Neumann subalgebras, we define $\text{Q}_M(A, B)$ as

$$\text{Q}_M(A, B) := \{v \in M \mid vv^* \in A' \cap M, v^*v \in B' \cap M \text{ and } Av = vB\}.$$

Whenever $v \in \text{Q}_M(A, B)$, we define $q_v = \text{supp}(E_A(vv^*))$ and $p_v = \text{supp}(E_B(v^*v))$, and we denote by $\alpha_v : Aq_v \rightarrow Bp_v$ the unique $*$ -isomorphism satisfying $av = v\alpha_v(a)$ for all $a \in Aq_v$.

Note that the set $\text{Q}_M(A, B)$ can be $\{0\}$. In Lemma 2.4, we will see that $\text{Q}_M(A, B) \neq \{0\}$ if and only if there exists a bifinite A - B -subbimodule ${}_A\mathcal{H}_B$ of ${}_A L^2(M)_B$.

We denote $\text{Q}_M(A, A)$ by $\text{Q}_M(A)$.

LEMMA 2.4. *Let (M, τ) be a tracial von Neumann algebra and $A, B \subset M$ abelian von Neumann subalgebras. Then the following statements hold.*

1. If $\alpha \in \text{PIso}(A, B)$ and if $\theta : {}_A\mathcal{H}(\alpha)_B \rightarrow {}_A L^2(M)_B$ is an A - B -bimodular isometry, then there exists a partial isometry $v \in \mathcal{Q}_M(A, B)$ such that $\alpha = \alpha_v$ and such that

$$\theta(\mathcal{H}(\alpha)) \subset \overline{v(B' \cap M)}^{\|\cdot\|_2} \subset \overline{\text{span}}^{\|\cdot\|_2} \mathcal{Q}_M(A, B).$$

2. Every bifinite A - B -subbimodule ${}_A\mathcal{H}_B$ of ${}_A L^2(M)_B$ is contained in $\overline{\text{span}}^{\|\cdot\|_2} \mathcal{Q}_M(A, B)$.
3. $\mathcal{Q}_M(A)'' = \text{QN}_M(A)''$.
4. We have $\mathcal{Q}_M(A, B) \neq \{0\}$ if and only if ${}_A L^2(M)_B$ admits a nonzero bifinite A - B -subbimodule.

Proof. 1. Let $\alpha : Aq \rightarrow Bp$ be an element of $\text{PIso}(A, B)$. Define $\xi := \theta(p) \in L^2(M)$ and let $\xi = v|\xi|$ be its polar decomposition. For all $a \in A$, we have $a\xi = \xi\alpha(a)$ and hence, $av = v\alpha(a)$. Furthermore $p = \text{supp}(E_B(v^*v))$ and $q = \alpha^{-1}(p) = \text{supp}(E_A(vv^*))$. So we find that $v \in \mathcal{Q}_M(A, B)$ and $\alpha = \alpha_v$. Because $|\xi| \in L^2(B' \cap M)$, we have that $\xi = v|\xi|$ is an element of $\overline{v(B' \cap M)}^{\|\cdot\|_2}$. Since p generates ${}_A\mathcal{H}(\alpha)_B$ as a right Hilbert B -module, we have proven the first inclusion $\theta(\mathcal{H}(\alpha)) \subset \overline{v(B' \cap M)}^{\|\cdot\|_2}$. Since $v \in \mathcal{Q}_M(A, B)$, also $v(B' \cap M) \subset \mathcal{Q}_M(A, B)$ and the second inclusion in statement 1 is proven as well.

2. Let ${}_A\mathcal{H}_B$ be a bifinite A - B -subbimodule of ${}_A L^2(M)_B$. By Lemma 2.2, ${}_A\mathcal{H}_B$ is isomorphic to a direct sum of bimodules of the form ${}_A\mathcal{H}(\alpha_i)_B$ with $\alpha_i \in \text{PIso}(A, B)$. Using statement 1 of the lemma, we find that \mathcal{H} is generated by subspaces of $\overline{\text{span}}^{\|\cdot\|_2} \mathcal{Q}_M(A, B)$. This proves statement 2.

3. By definition, we have $\mathcal{Q}_M(A)'' \subset \text{QN}_M(A)''$. On the other hand, ${}_A L^2(\text{QN}_M(A)'')_A$ is a direct sum of bifinite A - A -subbimodules of ${}_A L^2(M)_A$. So by statement 2, we have that $L^2(\text{QN}_M(A)'') \subset \overline{\text{span}}^{\|\cdot\|_2}(\mathcal{Q}_M(A))$. Therefore we conclude that $\text{QN}_M(A)'' = \mathcal{Q}_M(A)''$.

Finally, 4 is an immediate consequence of 2. □

We end this subsection with the following lemma, clarifying why later, we will consider abelian subalgebras $A \subset M$ satisfying $\mathcal{Z}(A' \cap M) = A$. Note that since A is abelian, the condition $\mathcal{Z}(A' \cap M) = A$ is equivalent with the “bicommutant” property $(A' \cap M)' \cap M = A$. Also note that the composition of two partial isomorphisms was defined before Lemma 2.2.

LEMMA 2.5. *Let (M, τ) be a tracial von Neumann algebra and $A, B, C \subset M$ abelian von Neumann subalgebras. If $v \in \mathcal{Q}_M(A, B)$, $w \in \mathcal{Q}_M(B, C)$ and if $\mathcal{Z}(B' \cap M) = B$, then there exists an element $u \in \mathcal{Q}_M(A, C)$ such that $\alpha_w \circ \alpha_v = \alpha_u$.*

Proof. Choose $v \in \mathcal{Q}_M(A, B)$ and $w \in \mathcal{Q}_M(B, C)$. Note that $vbw \in \mathcal{Q}_M(A, C)$ for every $b \in B' \cap M$ and $\alpha_{vbw} = \alpha_w \circ \alpha_v|_{Aq_{vbw}}$. We claim that $\bigvee_{b \in B' \cap M} q_{vbw} = \alpha_v^{-1}(q_w p_v)$.

Denote $\alpha_v^{-1}(q_w p_v) - \bigvee_{b \in B' \cap M} q_{vbw}$ by r . We need to prove that r is zero. Since $(B' \cap M)' \cap M = B$, we have that for every $x \in M$, the projection $\text{supp}(E_B(xx^*))$ equals the projection of $L^2(M)$ onto the closed linear span of $(B' \cap M)xM \subset L^2(M)$. Since $w^*bv^*r = 0$ for every $b \in B' \cap M$, it follows that $w^*q_{v^*r} = 0$. Therefore q_w is orthogonal to q_{v^*r} . Because $q_{v^*r} = \alpha_v(r)$ and $\alpha_v(r) \leq q_w$, it follows that $\alpha_v(r) = 0$. Hence $r = 0$ and our claim that $\bigvee_{b \in B' \cap M} q_{vbw} = \alpha_v^{-1}(q_w p_v)$ is proven.

By cutting down with appropriate projections, we find $b_n \in B' \cap M$ such that the projections $q_{vb_n w}$ are orthogonal and sum up to $\alpha_v^{-1}(q_w p_v)$. In particular, the left supports, resp. right supports, of the elements $vb_n w$ are orthogonal. So we can define $u = \sum_n vb_n w$. It follows that $u \in \mathcal{Q}_M(A, C)$ and $\alpha_w \circ \alpha_v = \alpha_u$. \square

2.4. THE TYPE OF AN ERGODIC NONSINGULAR COUNTABLE EQUIVALENCE RELATION

Let \mathcal{R} be a nonsingular ergodic countable Borel equivalence relation on a standard probability space (X, μ) . Using the map $\pi : \mathcal{R} \rightarrow X : \pi(x, y) = x$, we define the measure $\mu^{(1)}$ on \mathcal{R} given by

$$\mu^{(1)}(U) = \int_X \#(U \cap \pi^{-1}(x)) d\mu(x) \quad \text{for all Borel sets } U \subset \mathcal{R}.$$

We define $\mathcal{R}^{(2)} := \{(x, y, z) \in X^3 \mid (x, y), (y, z) \in \mathcal{R}\}$. Similarly, using the map $\rho : \mathcal{R}^{(2)} \rightarrow X : \rho(x, y, z) = x$, we define the measure $\mu^{(2)}$ on $\mathcal{R}^{(2)}$ given by

$$\mu^{(2)}(V) = \int_X \#(V \cap \rho^{-1}(x)) d\mu(x) \quad \text{for all Borel sets } V \subset \mathcal{R}^{(2)}.$$

The *Radon-Nikodym 1-cocycle* of \mathcal{R} is the $\mu^{(1)}$ -a.e. uniquely defined Borel map $\omega : \mathcal{R} \rightarrow \mathbb{R}$ such that

$$\omega(\varphi(x), x) = \log\left(\frac{d\mu \circ \varphi}{d\mu}(x)\right) \quad \text{for all } \varphi \in [[\mathcal{R}]] \text{ and almost every } x \in \text{dom } \varphi.$$

Note that ω satisfies the 1-cocycle relation $\omega(x, z) = \omega(x, y) + \omega(y, z)$ for $\mu^{(2)}$ -a.e. $(x, y, z) \in \mathcal{R}^{(2)}$. One then defines the *Maharam extension* $\tilde{\mathcal{R}}$ of \mathcal{R} as the equivalence relation on $(X \times \mathbb{R}, \mu \times \exp(-t)dt)$ defined by

$$(x, t) \sim (y, s) \quad \text{if and only if } (x, y) \in \mathcal{R} \text{ and } t - s = \omega(x, y).$$

Note that $\mu \times \exp(-t)dt$ is an infinite invariant measure for $\tilde{\mathcal{R}}$. Denote the von Neumann algebra of all $\tilde{\mathcal{R}}$ -invariant functions in $L^\infty(X \times \mathbb{R})$ by $L^\infty(X \times \mathbb{R})^{\tilde{\mathcal{R}}}$. Since \mathcal{R} was assumed to be ergodic, one can easily check that the action of \mathbb{R} on $L^\infty(X \times \mathbb{R})^{\tilde{\mathcal{R}}}$ given by translation of the second variable, is also ergodic. Depending on how this action of \mathbb{R} looks like, we define as follows the type of \mathcal{R} .

- I or II, if the action is conjugate with $\mathbb{R} \curvearrowright \mathbb{R}$;

- III $_{\lambda}$ ($0 < \lambda < 1$), if the action is conjugate with $\mathbb{R} \curvearrowright \mathbb{R}/\mathbb{Z} \log(\lambda)$;
- III $_1$, if the action is on one point ;
- III $_0$, if the action is properly ergodic, i.e. is ergodic and has orbits of measure zero.

REMARK 2.6. Denote by $L(\mathcal{R})$ the von Neumann algebra associated with \mathcal{R} . Denote by φ the normal semifinite faithful state on $L(\mathcal{R})$ that is induced by μ . Finally denote by $(\sigma_t^\varphi)_{t \in \mathbb{R}}$ its modular automorphism group. There is a canonical identification $L(\tilde{\mathcal{R}}) \cong L(\mathcal{R}) \rtimes_{\sigma^\varphi} \mathbb{R}$. Under this identification, the dual action of \mathbb{R} on $L(\mathcal{R}) \rtimes_{\sigma^\varphi} \mathbb{R}$ corresponds to the action of \mathbb{R} on $L(\tilde{\mathcal{R}})$ that we defined above. Also, the center of $L(\mathcal{R}) \rtimes_{\sigma^\varphi} \mathbb{R}$ corresponds to $L^\infty(X \times \mathbb{R})^{\tilde{\mathcal{R}}}$. Altogether it follows that the type of the equivalence relation \mathcal{R} coincides with the type of the factor $L(\mathcal{R})$.

LEMMA 2.7. *Let \mathcal{R} be a nonsingular ergodic countable Borel equivalence relation on the standard probability space (X, μ) . Denote by ω its Radon-Nikodym 1-cocycle. If the essential image $\text{Im}(\omega)$ of ω equals $\log(\lambda)\mathbb{Z}$ for some $0 < \lambda < 1$ and if the kernel $\text{Ker}(\omega)$ of ω is an ergodic equivalence relation, then \mathcal{R} is of type III $_{\lambda}$.*

Proof. Since $\text{Ker}(\omega)$ is an ergodic equivalence relation on (X, μ) , we have

$$L^\infty(X \times \mathbb{R})^{\tilde{\mathcal{R}}} \subset L^\infty(X \times \mathbb{R})^{\text{Ker}(\omega)} = 1 \otimes L^\infty(\mathbb{R}) .$$

For a given $F \in L^\infty(\mathbb{R})$, we have that $1 \otimes F$ is $\tilde{\mathcal{R}}$ -invariant if and only if F is invariant under translation by the essential image of ω . So,

$$L^\infty(X \times \mathbb{R})^{\tilde{\mathcal{R}}} = 1 \otimes L^\infty(\mathbb{R}/\log(\lambda)\mathbb{Z}) .$$

□

3. EQUIVALENCE RELATIONS ASSOCIATED TO SUBALGEBRAS THAT ARE ABELIAN, BUT NOT MAXIMAL ABELIAN

Throughout this section, we fix a tracial von Neumann algebra (M, τ) with separable predual. We also fix an abelian von Neumann subalgebra $A \subset M$ satisfying $\mathcal{Z}(A' \cap M) = A$. Choose a standard probability space (X, μ) such that $A = L^\infty(X, \mu)$. For every nonsingular partial automorphism φ of (X, μ) , we denote by α_φ the corresponding partial automorphism of A .

We first prove that $Q_M(A)$ induces a nonsingular countable Borel equivalence relation $\mathcal{R}(A \subset M)$ on (X, μ) . For this, we introduce the notation

$$\mathcal{G}(A \subset M) := \{\alpha_v \mid v \in Q_M(A)\} . \tag{2}$$

PROPOSITION 3.1. *There exists a nonsingular countable Borel equivalence relation \mathcal{R} on (X, μ) with the following property: a nonsingular partial automorphism φ of X satisfies $\alpha_\varphi \in \mathcal{G}(A \subset M)$ if and only if $(x, \varphi(x)) \in \mathcal{R}$ for a.e. $x \in \text{dom}(\varphi)$.*

Moreover, \mathcal{R} is essentially unique: if a nonsingular countable Borel equivalence relation \mathcal{R}' on (X, μ) satisfies the same property, then there exists a Borel subset $X_0 \subset X$ with $\mu(X - X_0) = 0$ and $\mathcal{R}'|_{X_0} = \mathcal{R}|_{X_0}$.

We denote $\mathcal{R}(A \subset M) := \mathcal{R}$. The equivalence relation $\mathcal{R}(A \subset M)$ is ergodic if and only if $\text{QN}_M(A)''$ is a factor.

Before proving Proposition 3.1, we introduce some terminology and a lemma. To every $\alpha \in \text{PAut}(A)$ are associated the support projections $q_\alpha, p_\alpha \in A$ such that $\alpha : Aq_\alpha \rightarrow Ap_\alpha$ is a $*$ -isomorphism. Assume that $\alpha \in \text{PAut}(A)$ and $\mathcal{F} \subset \text{PAut}(A)$. We say that α is a *gluing* of elements in \mathcal{F} , if there exists a sequence of elements $\alpha_n \in \mathcal{F}$ and projections $q_n \in A$ such that $q_\alpha = \sum_n q_n$ and such that $q_n \leq q_{\alpha_n}$ and $\alpha|_{Aq_n} = \alpha_n|_{Aq_n}$ for all n .

LEMMA 3.2. *Let $\mathcal{J} \subset \text{Q}_M(A)$ and $v \in \text{Q}_M(A)$ such that $v \in \overline{\text{span}}^{\|\cdot\|_2} \mathcal{J}$. Then α_v is a gluing of elements in $\{\alpha_w \mid w \in \mathcal{J}\}$.*

Proof. By a standard maximality argument, it suffices to prove that for every nonzero projection $q \in Aq_v$, there exists a nonzero subprojection $q_0 \in Aq$ and a $w \in \mathcal{J}$ such that $q_0 \leq q_w$ and $\alpha_v|_{Aq_0} = \alpha_w|_{Aq_0}$. So fix a nonzero projection $q \in Aq_v$. It follows that $qE_A(vv^*) \neq 0$. Since $v \in \overline{\text{span}}^{\|\cdot\|_2} \mathcal{J}$, we can pick a $w \in \mathcal{J}$ such that $qE_A(vv^*) \neq 0$. Define $q_1 := \text{supp}(E_A(vv^*))$ and note that $q_1 \in q_v Aq_w = Aq_v q_w$. Also note that $qq_1 \neq 0$. For all $a \in A$, we have

$$\alpha_v^{-1}(ap_v)vv^* = vaw^* = vw^*\alpha_w^{-1}(ap_w).$$

Applying the conditional expectation onto A and using that A is abelian, we find that

$$\alpha_v^{-1}(ap_v)q_1 = \alpha_w^{-1}(ap_w)q_1 \quad \text{for all } a \in A.$$

This means that $\alpha_v|_{Aq_1} = \alpha_w|_{Aq_1}$. We put $q_0 := qq_1$. We already showed that $q_0 \neq 0$. Since $q_0 \leq q_1$, we have that $\alpha_v|_{Aq_0} = \alpha_w|_{Aq_0}$. \square

Proof of Proposition 3.1. We say that a subpseudogroup $\mathcal{G} \subset \text{PAut}(A)$ is of *countable type* if there exists a countable subset $\mathcal{J} \subset \mathcal{G}$ such that every $\alpha \in \mathcal{G}$ is a gluing of elements in \mathcal{J} . To prove the first part of the proposition, we must show that $\mathcal{G}(A \subset M)$ is a subpseudogroup of countable type of $\text{PAut}(A)$. From Lemma 2.5, it follows that $\mathcal{G}(A \subset M)$ is indeed a subpseudogroup. Since M has a separable predual, we can choose a countable $\|\cdot\|_2$ -dense subset $\mathcal{J} \subset \text{Q}_M(A)$. By Lemma 3.2, every $\alpha \in \mathcal{G}(A \subset M)$ is a gluing of elements in $\{\alpha_w \mid w \in \mathcal{J}\}$. Hence $\mathcal{G}(A \subset M)$ is of countable type. So the first part of the proposition is proven and we can essentially uniquely define the nonsingular countable Borel equivalence relation \mathcal{R} on (X, μ) .

Since $A' \cap M \subset \text{QN}_M(A)''$ and since we assumed that $(A' \cap M)' \cap M = A$, the center of $\text{QN}_M(A)''$ is a subalgebra of A . By Lemma 2.4.3, we have $\text{QN}_M(A)'' = \text{Q}_M(A)''$. Therefore,

$$\mathcal{Z}(\text{QN}_M(A)'') = \{a \in A \mid av = va \text{ for all } v \in \text{Q}_M(A)\}.$$

The right hand side equals $A^{\mathcal{R}}$, the subalgebra of \mathcal{R} -invariant functions in A . So \mathcal{R} is ergodic if and only if $\text{QN}_M(A)''$ is a factor. \square

For our application, the following theorem is crucial. It says that $\mathcal{R}(A \subset M)$ remains the same, up to stable isomorphism, if we replace A by an abelian subalgebra B that has a mutual intertwining bimodule into A .

THEOREM 3.3. *Let M be a II_1 factor with separable predual. Let $A, B \subset M$ be abelian, quasi-regular von Neumann subalgebras satisfying $\mathcal{Z}(A' \cap M) = A$ and $\mathcal{Z}(B' \cap M) = B$. If $A \prec_M B$ and $B \prec_M A$, then the equivalence relations $\mathcal{R}(A \subset M)$ and $\mathcal{R}(B \subset M)$ are stably isomorphic.*

Proof. Since A, B are quasi-regular and since $A \prec_M B$ as well as $B \prec_M A$, there exists a nonzero bifinite A - B -subbimodule of $L^2(M)$. So by Lemma 2.4.4, there exists a nonzero element $v \in \text{Q}_M(A, B)$ with corresponding $\alpha_v \in \text{PAut}(A, B)$. Using the notation in (2) and using Lemma 2.5, we find that

$$\begin{aligned} \alpha_v \circ \beta \circ \alpha_v^{-1} &\in \mathcal{G}(B \subset M) \quad \text{for all } \beta \in \mathcal{G}(A \subset M) \quad \text{and} \\ \alpha_v^{-1} \circ \gamma \circ \alpha_v &\in \mathcal{G}(A \subset M) \quad \text{for all } \gamma \in \mathcal{G}(B \subset M). \end{aligned}$$

So α_v implements a stable isomorphism between $\mathcal{R}(A \subset M)$ and $\mathcal{R}(B \subset M)$. \square

The following lemma will allow us to easily compute $\mathcal{R}(A \subset M)$ in concrete examples.

LEMMA 3.4. *Let (M, τ) be a tracial von Neumann algebra and $A \subset M$ an abelian von Neumann subalgebra satisfying $\mathcal{Z}(A' \cap M) = A$. Let $\mathcal{F} \subset M$ be a subset such that*

- $M = (\mathcal{F} \cup \mathcal{F}^* \cup (A' \cap M))''$,
- as an A - A -bimodule, $\overline{\text{span}}^{\|\cdot\|_2} A\mathcal{F}A$ is isomorphic to a direct sum of bimodules of the form ${}_A\mathcal{H}(\alpha_n)_A$ with $\alpha_n \in \text{PAut}(A)$.

Choose nonsingular partial automorphisms φ_n of (X, μ) such that $\alpha_n = \alpha_{\varphi_n}$ for all n . Up to measure zero, $\mathcal{R}(A \subset M)$ is generated by the graphs of the partial automorphisms φ_n .

Proof. We again use the notation (2). By Lemma 2.4.1, we find $v_n \in \text{Q}_M(A)$ such that $\alpha_n = \alpha_{v_n}$ and

$$\overline{\text{span}}^{\|\cdot\|_2} A\mathcal{F}A \subset \overline{\text{span}}^{\|\cdot\|_2} \{v_n(A' \cap M) \mid n \in \mathbb{N}\}. \tag{3}$$

In particular, we have $\alpha_n \in \mathcal{G}(A \subset M)$. Choose nonsingular partial automorphisms φ_n of (X, μ) such that $\alpha_n = \alpha_{\varphi_n}$ for all n .

Denote by \mathcal{R} the smallest (up to measure zero) equivalence relation on (X, μ) that contains the graphs of all the partial automorphisms φ_n . By the previous

paragraph, we know that \mathcal{R} is a subequivalence relation of $\mathcal{R}(A \subset M)$. Denote by \mathcal{J} the set of all products of elements in

$$\{v_n \mid n \in \mathbb{N}\} \cup \{v_n^* \mid n \in \mathbb{N}\} \cup (A' \cap M).$$

By construction, the graph of every α_w , $w \in \mathcal{J}$, belongs to \mathcal{R} . Combining our assumption that $M = (\mathcal{F} \cup \mathcal{F}^* \cup (A' \cap M))''$ with (3), it follows that $\text{span } \mathcal{J}$ is $\|\cdot\|_2$ -dense in $L^2(M)$. By Lemma 3.2, every $\alpha \in \mathcal{G}(A \subset M)$ is a gluing of elements in $\{\alpha_w \mid w \in \mathcal{J}\}$. So the graph of every $\alpha \in \mathcal{G}(A \subset M)$ belongs to \mathcal{R} a.e. Hence \mathcal{R} equals $\mathcal{R}(A \subset M)$ almost everywhere. \square

We finally note in the following proposition that every nonsingular countable Borel equivalence relation \mathcal{R} arises as $\mathcal{R}(A \subset M)$.

PROPOSITION 3.5. *Let \mathcal{R} be a nonsingular countable Borel equivalence relation. Then there exists a quasi-regular inclusion of an abelian von Neumann algebra A in a tracial von Neumann algebra (M, τ) satisfying $\mathcal{Z}(A' \cap M) = A$ and such that $\mathcal{R} \cong \mathcal{R}(A \subset M)$.*

Proof. Let \mathcal{R} be a nonsingular countable Borel equivalence relation on a standard probability space (X, μ) . Denote by (P, Tr) the unique hyperfinite II_∞ factor and choose a trace-scaling action $(\alpha_t)_{t \in \mathbb{R}}$ of \mathbb{R} on P . This means that $\text{Tr} \circ \alpha_t = e^{-t} \text{Tr}$. The corresponding action of \mathbb{R} on $L^2(P)$ will also be denoted by (α_t) . We denote by $\omega : \mathcal{R} \rightarrow \mathbb{R}$ the Radon-Nikodym 1-cocycle of \mathcal{R} (see Section 2.4).

In the same way as with the Maharam extension of a nonsingular group action, the equivalence relation \mathcal{R} admits a natural trace preserving action on $L^\infty(X) \overline{\otimes} P$. We denote by (\mathcal{M}, Tr) the crossed product. For completeness, we recall the construction of (\mathcal{M}, Tr) . To every $\varphi \in [[\mathcal{R}]]$, we associate the operator W_φ on $L^2(\mathcal{R}, L^2(P))$ given by

$$(W_\varphi \xi)(x, y) = \begin{cases} \alpha_{\omega(x, \varphi^{-1}(x))}(\xi(\varphi^{-1}(x), y)) & \text{if } x \in \text{dom}(\varphi^{-1}), \\ 0 & \text{otherwise,} \end{cases}$$

for every $\xi \in L^2(\mathcal{R}, L^2(P))$. One checks that $W_\varphi W_\psi = W_{\varphi \circ \psi}$ and $W_\varphi^* = W_{\varphi^{-1}}$. We represent $L^\infty(X) \overline{\otimes} P = L^\infty(X, P)$ on $L^2(\mathcal{R}, L^2(P))$ by

$$(F\xi)(x, y) = F(x)\xi(x, y) \text{ for all } \xi \in L^2(\mathcal{R}, L^2(P)) \text{ and } F \in L^\infty(X, P).$$

Note that the partial isometries W_φ , $\varphi \in [[\mathcal{R}]]$, normalize $L^\infty(X, P)$ and that

$$(W_\varphi^* F W_\varphi)(x) = \begin{cases} \alpha_{\omega(x, \varphi(x))}(F(\varphi(x))) & \text{if } x \in \text{dom } \varphi \\ 0 & \text{otherwise.} \end{cases}$$

Define \mathcal{M} as the von Neumann algebra generated by $L^\infty(X, P)$ and the partial isometries W_φ , $\varphi \in [[\mathcal{R}]]$. Denoting by $\Delta \subset \mathcal{R}$ the diagonal subset, the orthogonal projection onto $L^2(\Delta, L^2(P))$ implements a normal faithful conditional expectation $E : \mathcal{M} \rightarrow L^\infty(X) \overline{\otimes} P$ satisfying

$$E(W_\varphi) = \chi_{\{x \mid \varphi(x)=x\}} \otimes 1 \text{ for all } \varphi \in [[\mathcal{R}]].$$

The formula $\text{Tr} := (\mu \otimes \text{Tr}) \circ E$ defines a normal semifinite faithful trace on \mathcal{M} . Fix a nonzero projection $q \in P$ with $\text{Tr}(q) = 1$. Define the projection $p \in L^\infty(X) \overline{\otimes} P$ given by $p = 1 \otimes q$. Write $A := L^\infty(X)p$ and $M := p\mathcal{M}p$. Then A is a quasi-regular abelian von Neumann subalgebra of M and the restriction of Tr to M gives a normal faithful tracial state τ on M . The relative commutant $L^\infty(X)' \cap \mathcal{M}$ equals $L^\infty(X) \overline{\otimes} P$. Since P is a factor, it follows that $\mathcal{Z}(A' \cap M) = A$.

We finally prove that $\mathcal{R} \cong \mathcal{R}(A \subset M)$. Write $\mathcal{R} = \bigcup_k \text{graph}(\varphi_k)$, with $\varphi_k \in [\mathcal{R}]$. Then φ_k induces an automorphism of $L^\infty(X)$ and hence of $A = L^\infty(X)p$ that we denote by $\beta_k \in \text{Aut}(A)$. Since P is a II_∞ factor and $q \in P$ is a finite projection, we can choose partial isometries $w_n \in P$ such that $\sum_n w_n^* w_n = 1$ and $w_n w_n^* = q$ for all n . Define the elements

$$v_{n,k} := (1 \otimes w_n) W_{\varphi_k} p .$$

All $v_{n,k}$ belong to $\mathcal{Q}_M(A)$ and $\alpha_{v_{n,k}}$ equals the restriction of β_k to $A p_{n,k}$ for projections $p_{n,k} \in A$. Since the sum of all $w_n^* w_n$ equals 1, we also have that $\bigvee_n p_{n,k} = p$. Therefore the graphs of the partial automorphisms $\alpha_{v_{n,k}}$ generate an equivalence relation that is isomorphic with \mathcal{R} . To conclude the proof, we put $\mathcal{F} := \{v_{n,k} \mid n, k \in \mathbb{N}\}$ and observe that $M = (\mathcal{F} \cup \mathcal{F}^* \cup (A' \cap M))''$. By Lemma 3.4, the equivalence relation $\mathcal{R}(A \subset M)$ is generated by the graphs of the partial automorphisms $\alpha_{v_{n,k}}$. \square

4. PROOF OF THEOREM A

Throughout this section, we assume that n and m are integers satisfying $2 \leq n < |m|$. As explained in the introduction, the corresponding groups $\text{BS}(n, m)$ form a complete list of the nonamenable icc Baumslag-Solitar groups up to isomorphism.

Throughout this section, we write $M = L(\text{BS}(n, m))$ and $A = \{u_a, u_a^*\}''$. We start with the following observation.

PROPOSITION 4.1. *We have that $A \subset M$ is a quasi-regular abelian von Neumann subalgebra satisfying $\mathcal{Z}(A' \cap M) = A$. Moreover, $A' \cap M$ has no amenable direct summand.*

Proof. It is clear that $A \subset M$ is a quasi-regular abelian von Neumann subalgebra, because the element $a \in \text{BS}(n, m)$ generates an almost normal abelian subgroup of $\text{BS}(n, m)$: for every $g \in \text{BS}(n, m)$, the group $g a^{\mathbb{Z}} g^{-1} \cap a^{\mathbb{Z}}$ has finite index in $a^{\mathbb{Z}}$.

To prove that $\mathcal{Z}(A' \cap M) = A$, we define the finite index subalgebra $A_0 := \{u_a^n, u_a^{-n}\}''$ of A . We will first prove that $\mathcal{Z}(A'_0 \cap M) = A_0$. Afterwards we will show that this implies that $\mathcal{Z}(A' \cap M) = A$.

Define $G := \langle a^{\mathbb{Z}}, b^{-1} a^{\mathbb{Z}} b \rangle \subset \text{BS}(n, m)$. Then $L(G)$ is a subalgebra of $A'_0 \cap M$. So $\mathcal{Z}(A'_0 \cap M) \subset L(G)' \cap M$. Using Lemma 2.1, one can easily see that $\{g \gamma g^{-1} \mid g \in G\}$ is an infinite set for every $\gamma \in \text{BS}(n, m) - a^{\mathbb{Z}}$. Therefore $L(G)' \cap M \subset A_0$. This shows that $\mathcal{Z}(A'_0 \cap M) \subset A_0$. Since the converse inclusion is obvious, we find that $A_0 = \mathcal{Z}(A'_0 \cap M)$.

Since $A_0 \subset A$ has finite index, there exist orthogonal projections $p_j \in A$ such that $Ap_j = A_0p_j$ and $\sum_j p_j = \mathbf{1}$. But then

$$\begin{aligned} \mathcal{Z}(A' \cap M)p_j &= \mathcal{Z}((A' \cap M)p_j) = \mathcal{Z}((Ap_j)' \cap p_jMp_j) \\ &= \mathcal{Z}((A_0p_j)' \cap p_jMp_j) = \mathcal{Z}(p_j(A'_0 \cap M)p_j) \\ &= \mathcal{Z}(A'_0 \cap M)p_j = A_0p_j \subset A \end{aligned}$$

Therefore $\mathcal{Z}(A' \cap M) \subset A$. The converse inclusion being obvious, we have proven that $A = \mathcal{Z}(A' \cap M)$.

Using Lemma 2.1, it follows that G is an amalgamated free product of two copies of \mathbb{Z} over a copy of \mathbb{Z} embedded as $n\mathbb{Z}$ and $m\mathbb{Z}$ respectively. In particular, G is nonamenable and $L(G)$ has no amenable direct summand. Since $L(G) \subset A'_0 \cap M$, it follows that $A'_0 \cap M$ has no amenable direct summand either. As above, we have that

$$(A' \cap M)p_j = p_j(A'_0 \cap M)p_j$$

for all j . Hence $A' \cap M$ has no amenable direct summand. \square

We now identify the associated countable equivalence relation $\mathcal{R}(A \subset M)$.

PROPOSITION 4.2. *The equivalence relation $\mathcal{R}(A \subset M)$ is isomorphic with the unique hyperfinite ergodic countable equivalence relation of type III $_{n/|m|}$.*

Proof. Let k be the greatest common divisor of n and $|m|$. Write $n = n_0k$ and $m = m_0k$. By our assumptions on n and m , we have that $1 \leq n_0 < |m_0|$. Define the countable Borel equivalence relation $\mathcal{R}_{n,m}$ on the circle \mathbb{T} given by

$$\mathcal{R}_{n,m} := \left\{ (y, z) \in \mathbb{T} \times \mathbb{T} \mid \exists a, b \in \mathbb{N} \text{ such that } a + b > 0 \text{ and } y^{(n_0^a m_0^b k)} = z^{(m_0^a n_0^b k)} \right\}.$$

Equip \mathbb{T} with its Lebesgue measure λ and note that $\mathcal{R}_{n,m}$ is a nonsingular countable Borel equivalence relation on (\mathbb{T}, λ) .

Define $\mathcal{R}_0 := \{(y, z) \in \mathbb{T} \times \mathbb{T} \mid y^m = z^n\}$. Note that $\mathcal{R}_0 \subset \mathcal{R}_{n,m}$ and that $\mathcal{R}_{n,m}$ is the smallest equivalence relation containing \mathcal{R}_0 . Define $\pi : \mathcal{R}_0 \rightarrow \mathbb{T} : \pi(y, z) = y^m$. Note that π is $n|m|$ -to-1. Define the probability measure μ on \mathcal{R}_0 given by

$$\mu(U) = \frac{1}{n|m|} \int_{\mathbb{T}} \#(U \cap \pi^{-1}(\{x\})) d\lambda(x).$$

For all $k, l \in \mathbb{Z}$, we define the function $P_{k,l} : \mathcal{R}_0 \rightarrow \mathbb{T} : P_{k,l}(y, z) = y^k z^l$. A direct computation yields a unique unitary

$$T : L^2(\mathcal{R}_0, \mu) \rightarrow \overline{\text{span}}^{\|\cdot\|^2} Au_b A : P_{k,l} \mapsto u_a^k u_b u_a^l.$$

We turn $L^2(\mathcal{R}_0, \mu)$ into an $L^\infty(\mathbb{T})$ - $L^\infty(\mathbb{T})$ -bimodule by the formula

$$(F \cdot \xi \cdot F')(y, z) = F(y) \xi(y, z) F'(z).$$

Under the natural identification of $L^\infty(\mathbb{T})$ and A , the unitary T is A - A -bimodular.

By construction ${}_A L^2(\mathcal{R}_0, \mu)_A$ is isomorphic with a direct sum of bimodules of the form ${}_A \mathcal{H}(\alpha_j)_A$ where the union of the graphs of the partial automorphisms α_j equals \mathcal{R}_0 and hence generates the equivalence relation $\mathcal{R}_{n,m}$. Applying Lemma 3.4 to $\mathcal{F} = \{u_b\}$, we conclude that $\mathcal{R}(A \subset M) \cong \mathcal{R}_{n,m}$ up to measure zero.

As in Section 2.4, denote by $\omega : \mathcal{R}_{n,m} \rightarrow \mathbb{R}$ the Radon-Nikodym 1-cocycle. Denote by $\Lambda \subset \mathbb{T}$ the subgroup given by

$$\Lambda := \left\{ \exp\left(\frac{2\pi i s}{(n_0 m_0)^b}\right) \mid s \in \mathbb{Z}, b \in \mathbb{N} \right\}.$$

For every $z \in \Lambda$, we denote by $\alpha_z : \mathbb{T} \rightarrow \mathbb{T}$ the rotation $\alpha_z(y) = zy$. We have $\text{graph } \alpha_z \subset \mathcal{R}_{n,m}$ for all $z \in \Lambda$. Since all α_z are measure preserving, we actually have $\text{graph } \alpha_z \subset \text{Ker}(\omega)$. Since $\Lambda \subset \mathbb{T}$ is a dense subgroup, it follows that $\text{Ker}(\omega)$ is an ergodic equivalence relation. In particular, $\mathcal{R}_{n,m}$ is ergodic. A direct computation shows that $\omega(y, z) = \log(n/|m|)$ for all $(y, z) \in \mathcal{R}_0$. Since \mathcal{R}_0 generates the equivalence relation $\mathcal{R}_{n,m}$, it follows that the essential image of ω equals $\log(n/|m|)\mathbb{Z}$. Using Lemma 2.7, we conclude that $\mathcal{R}_{n,m}$ is of type III $_{n/|m|}$. By construction, $\mathcal{R}_{n,m}$ is amenable and hence, hyperfinite. \square

We are now ready to prove our main theorem.

Proof of Theorem A. Fix for $i = 1, 2$, integers $n_i, m_i \in \mathbb{Z}$ with $2 \leq n_i < |m_i|$. Put $M_i = L(\text{BS}(n_i, m_i))$ and denote by $A_i \subset M_i$ the abelian von Neumann subalgebra generated by u_a , where $a \in \text{BS}(n_i, m_i)$ is the first canonical generator. Assume that M_1 and M_2 are stably isomorphic. We must prove that

$$\frac{n_1}{|m_1|} = \frac{n_2}{|m_2|}. \tag{4}$$

Interchanging if necessary the roles of M_1 and M_2 , we can take a nonzero projection $p_1 \in A_1$ and a $*$ -isomorphism $\alpha : p_1 M_1 p_1 \rightarrow M_2$.

We claim that inside M_2 , we have $\alpha(A_1 p_1) \prec A_2$. From Proposition 4.1, we know that

$$P := \alpha(A_1 p_1)' \cap M_2 = \alpha((A_1' \cap M_1) p_1)$$

has no amenable direct summand. By Proposition 3.1 in [Ue07], the HNN extension M_2 can be viewed as the corner of an amalgamated free product of tracial von Neumann algebras. Since P has no amenable direct summand, it then follows from [CH08, Theorem 4.2] that $P' \cap M_2 \prec A_2$. So our claim that $\alpha(A_1 p_1) \prec A_2$ follows.

By symmetry, we also have the intertwining $\alpha^{-1}(A_2) \prec A_1 p_1$ inside $p_1 M_1 p_1$. Applying α , we find that $A_2 \prec \alpha(A_1 p_1)$ inside M_2 .

Having proven that inside M_2 we have the intertwining relations $\alpha(A_1 p_1) \prec A_2$ and $A_2 \prec \alpha(A_1 p_1)$, it follows from Theorem 3.3 that the equivalence relations $\mathcal{R}(A_1 p_1 \subset p_1 M_1 p_1)$ and $\mathcal{R}(A_2 \subset M_2)$ are stably isomorphic. By construction,

$\mathcal{R}(A_1 p_1 \subset p_1 M_1 p_1)$ is the restriction of $\mathcal{R}(A_1 \subset M_1)$ to the support of p_1 . So we conclude that the equivalence relations $\mathcal{R}(A_1 \subset M_1)$ and $\mathcal{R}(A_2 \subset M_2)$ are stably isomorphic. In particular, these ergodic nonsingular equivalence relations must have the same type. Using Proposition 4.2, we find (4). \square

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