The Dirac Operator with Mass $m_0 \geq 0$: Non-Existence of Zero Modes and of Threshold Eigenvalues

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Abstract. A simple global condition on the potential is given which excludes zero modes of the massless Dirac operator. As far as local conditions at infinity are concerned, it is shown that at energy zero the Dirac equation without mass term has no non-trivial L^2-solutions at infinity for potentials which are either very slowly varying or decaying at most like r^{-s} with $s \in (0, 1)$. When a mass term is present, it is proved that at the thresholds there are again no such solutions when the potential decays at most like r^{-s} with $s \in (0, 2)$. In both situations the decay rate is optimal.

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1 Introduction

In their 1986 study of the stability of matter in the relativistic setting of the Pauli operator J. Fröhlich, E.H. Lieb and M. Loss recognised that there was a restriction on the nuclear charge if and only if the three-dimensional Dirac operator with mass zero has a non-trivial kernel (see [LS], Chapters 8, 9 and the references there). An example of such a zero mode was first given by M. Loss and H.T. Yau [LY]; for many more examples see [LS], p.167. Later, the Loss-Yau example was used in a completely different setting, viz. to show the necessity of certain restrictions in analogues of Hardy and Sobolev inequalities [BEU].

An observation with remarkable technological consequences is that in certain situations of non-relativistic quantum mechanics the dynamics of wave packets in crystals can be modelled by the two-dimensional massless Dirac operator (see [FW] and the references there). When the potential is spherically symmetric, a detailed spectral analysis of the Dirac operator with mass zero was given in two and three dimensions by K.M. Schmidt [S]. In particular he showed that a variety of potentials with compact support can give rise to zero modes.
In Theorem 2.1 of the present paper we give a simple global condition on the potential which rules out zero modes of the massless Dirac operator in any dimension. Theorem 2.7 deals with a fairly large class of massless Dirac operators under conditions on the potential solely at infinity. It is shown, for example, that for energy zero there is no non-trivial solution of integrable square at infinity if the potential is very slowly varying or decaying like r^{-s} with $s \in (0, 1)$. This decay rate is in a certain sense the best possible one (see Appendix B). To rule out non-trivial L^2-solutions at infinity for the threshold energies $\pm m_0$ turns out to be more complicated. Here the asymptotic analysis of Appendix B suggests $1/r^2$-behaviour as the borderline case and Theorem 2.10 indeed permits potentials which tend to zero with a rate at most like r^{-s} with $s \in (0, 2)$. Theorem 2.10 relies on a transformation of the solutions which is intimately connected with the block-structure that can be given to the Dirac matrices. In connection with this theorem we should like to draw attention to the very interesting paper [BG] where global conditions are used.
In broad outline the proof of our Theorems 2.7 and 2.10 follows from the virial technique which was developed by Vogelsang [V] for the Dirac operator and later extended in [KOY], but basic differences in the assumptions on the potential are required in the situations considered here. At the beginning of §4 the general strategy of proof is outlined and comparison with [KOY] made, but the present paper is self-contained.

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2 Results and Examples

Given $n \geq 2$ and $N := 2^{[(n+1)/2]}$, there exist $n+1$ anti-commuting Hermitian $N \times N$ matrices $\alpha_1, \alpha_2, \ldots, \alpha_n, \beta := \alpha_{n+1}$ with square one. No specific representation of these matrices will be needed except in Appendix B. Using the notation

$$\alpha \cdot p = \sum_{j=1}^{n} \alpha_j(-i\partial_j),$$

we start by considering the differential expression

$$\tau := \alpha \cdot p + Q$$

in $L^2(\boldsymbol{R}^n)^N$. Here $Q : \boldsymbol{R}^n \to \boldsymbol{C}^{N\times N}$ is a matrix-valued function with measurable entries.
Let $\phi \in \boldsymbol{C}^N$. Then

$$u(x) := \frac{1 + i\alpha \cdot x}{(1 + |x|^2)^{n/2}}\,\phi$$

satisfies

$$\tau u = 0 \quad \text{with} \quad Q(x) = -\frac{n}{1 + r^2}. \tag{2.1}$$

This is the example of Loss and Yau [LY] mentioned in the Introduction. (Note that u is in L^2 iff $n \geq 3$.) For $n = 3$ the potential in (2.1) is in fact the first in a hierarchy of potentials with constants larger than 3, all giving rise to zero modes; see [SU].
In contrast, we have the following result.

Theorem 2.1. *Let $Q : \boldsymbol{R}^n \to \boldsymbol{C}^{N\times N}$ be measurable with*

$$\sup_{x\in\boldsymbol{R}^n} |x||Q(x)| \leq C \quad \textit{for some} \quad 0 < C < \frac{n-1}{2}.$$

Then any solution $u \in H^1_{\mathrm{loc}}(\boldsymbol{R}^n)^N \cap L^2(\boldsymbol{R}^n, r^{-1}dx)^N$ of $\tau u = 0$ is identically zero.

Remark 2.2. *a) In case Q is Hermitian, Theorem 2.1 can be rephrased as follows: The self-adjoint realisation H of τ with*

$$\int \frac{|u|^2}{r}\,dx < \infty \quad (u \in D(H)) \tag{2.2}$$

does not have the eigenvalue zero. (For the existence of such a self-adjoint realisation see [Ar] and the references therein.)
b) Suppose that V is a real-valued scalar potential with $\sup_{x\in\boldsymbol{R}^n} |x||V(x)| < \frac{n-1}{2}$. Then it needs but a small additional assumption to show by means of the virial theorem that the self-adjoint realisation H of $\alpha \cdot p + V + m_0\beta$ ($m_0 \neq 0$) with property (2.2) does not have the eigenvalues $\pm m_0$ (see, e.g., [L], Theorem 2.4 in conjunction with [We], p.335).
c) It seems to be difficult to compare our C with the size of the compactly supported potentials in [S] which produce zero modes for $n \geq 2$.

Now we replace the whole space with the exterior domain

$$E_R := \{x \in \boldsymbol{R}^n \mid |x| > R\},$$

where $R > 0$ can be arbitrarily large. On E_R we consider the Dirac equation

$$(\alpha \cdot D + Q)u = 0 \quad \text{with} \quad D := p - b. \tag{2.3}$$

We assume for simplicity that the vector potential b is in $C^1(E_R, \boldsymbol{R}^n)$. Further conditions will not be imposed on b but on the magnetic field

$$B := (\partial_i b_k - \partial_k b_i).$$

Note that B can be identified with the scalar function $\partial_1 b_2 - \partial_2 b_1$ if $n = 2$ and the vector-valued function $\operatorname{curl} b$ if $n = 3$. Solutions of (2.3) will be functions in $H^1_{\mathrm{loc}}(E_R)^N$ which satisfy (2.3), if Q is locally bounded in E_R.
The following result and its remark are essentially contained in [KOY], Example 6.1 and final remark on p.40.

THEOREM 2.3. *Let $m_0 \geq 0$, $\lambda \in \boldsymbol{R}$ and $Q := V + m_0\beta - \lambda + W$, where*

(I) $V = V^* \in C^1(E_R, \boldsymbol{C}^{N\times N})$,
(II) W is a measurable and bounded matrix function (not necessarily Hermitian).

Suppose V, W and the magnetic field B satisfy the following conditions:
a) $r^{1/2}V = o(1) = r\partial_r V$ uniformly with respect to directions;
b) there exist numbers $K \in (0, 1/2)$ and $M > 0$ such that

$$r|W| \leq K, \quad |Bx| \leq M \quad on \quad E_R.$$

Assertion: If $u \in L^2(E_R)^N$ is a solution of (2.3) for

$$|\lambda| > \sqrt{m_0^2 + M^2}/(1 - 2K),$$

then $u = 0$ on E_{R_1} for some $R_1 > R$. If $r|W| = o(1)$ or $|Bx| = o(1)$ uniformly w.r.t. directions, then $K = 0$ or $M = 0$ is permitted.

REMARK 2.4. *If $V \in C^2(R, \infty)$ is a real-valued (scalar) function, condition a) can be replaced by*

$$V(r) = o(1) = rV'(r), \quad rV''(r) = o(1) \quad as \quad r \to \infty.$$

REMARK 2.5. *Using a unique continuation result, e.g., the simple one [HP] or the more sophisticated one in [DO], one can conclude that $u = 0$ on $\boldsymbol{R}^n$.*

REMARK 2.6. *It follows immediately from Theorem 2.3 and Remark 2.5 that the potential in (2.1) cannot create a non-zero eigenvalue.*

Theorem 2.3 will now be supplemented by Theorems 2.7 and 2.10.

THEOREM 2.7. *Let $m_0 = 0, \lambda \leq 0$ and V, W as in (I), (II) of Theorem 2.3. Let $q \in C^2(R, \infty)$ be a positive bounded function with the properties*
(i) $[r(q - \lambda)]' \geq \delta_0(q - \lambda)$ for some $\delta_0 \in (0, 1)$,

(ii) $\dfrac{q'}{q^2} = o(1) = r\dfrac{q''}{q^2}$.

Suppose V, W and B satisfy the following conditions:

(H.1) $r(V - q) = O(1), \quad \partial_r V - q' = o\left(\dfrac{q}{r}\right)$;

(H.2) $r|W| \leq K$ *for some* $K \in [0, \delta_0/2)$;

(H.3) *there is a function* $a(r)$ *with* $|Bx| \leq a(r)$ *and* $\dfrac{a}{q} = o(1)$.

Assertion: Any solution $u \in L^2(E_R)^N$ of (2.3) with $Q = V + W - \lambda$ vanishes identically on E_{R_1} for some $R_1 > R$.

REMARK 2.8. *a) To prove Theorem 2.7, it will be important to observe that condition (i) implies*

$$r(q - \lambda) \geq \text{const.}\, r^{\delta_0} \quad (r > R).$$

b) If q is a negative bounded function with property (ii) and which satisfies $[r(\lambda - q)]' \geq \delta_0(\lambda - q)$ for some $\delta_0 \in (0, 1)$, then Theorem 2.7 holds for $\lambda \geq 0$.
c) In case V decays at infinity, hypothesis (H.3) demands a corresponding stronger decay of B to prevent the existence of eigenvalues. (The contrasting situation that V and B become large at infinity is considered in [MS].)

Examples. For simplicity we assume $V = q$ and $W = 0$. Let q_0 be a positive number. Then the functions

$$q = q_0\,[2 + \sin(\log\log r)], \tag{2.4}$$

$$q = q_0\,(\log r)^{-s} \quad (s > 0), \tag{2.5}$$

$$q = q_0\, r^{-s} \quad (0 < s < 1), \tag{2.6}$$

have the required properties (i), (ii). In addition, a magnetic field with the decay property (H.3) is allowed. As far as (2.5) and (2.6) are concerned, Remarks 2.4–2.5 already rule out any eigenvalue $\lambda \neq 0$. In case there is no vector potential, it follows from [S], Corollary 1 that the self-adjoint operator associated with $\tau = \alpha \cdot p + q$ in $L^2(\boldsymbol{R}^n)^N$ has purely absolutely continuous spectrum outside $[q_0, 3q_0]$.

REMARK 2.9. *More realistic potentials than (2.4) will have the property*

$$\lim_{r\to\infty} q(r) =: q_\infty \neq 0.$$

In such situations, however, it may be possible to use Theorem 2.3 or Remark 2.4 to show that

$$(\alpha \cdot D + Q - q_\infty)u = \lambda u$$

has no non-trivial solution of integrable square at infinity if $\lambda = -q_\infty$. A case in point is the potential

$$q = \frac{r}{1+r},$$

which does not obey condition (i). Assuming $|Bx| = o(1)$ uniformly w.r.t. directions, it follows from Theorem 2.3 that

$$(\alpha \cdot D + q - 1)u = \lambda u$$

has no solution $u \neq 0$ in L^2 at infinity if $\lambda \neq 0$. In particular, $\alpha \cdot D + q$ has no zero mode.

For the equation

$$(\alpha \cdot D + V + m_0\beta)u = -m_0 u, \quad D := p - b \tag{2.7}$$

our result is as follows.

Theorem 2.10. *Let $m_0 > 0$ and $\mu := \sqrt{q(q+2m_0)}$. Let $q \in C^2(R,\infty)$ be a positive function with $q = o(1)$ and the following properties:*

(i) $\dfrac{(r\mu)'}{\mu} \geq \delta_0$ *for some $\delta_0 \in (0,1)$;*

(ii) $r\dfrac{q'}{q} = O(1), \quad r\dfrac{q''}{q^{3/2}} = o(1).$

Suppose $V \in C^1(E_R, \boldsymbol{R})$ and B satisfy the following conditions:

(H.1) $r^2(V-q) = O(1), \quad \partial_r V - q' = o\Big(\dfrac{q}{r}\Big)$;

(H.2) there is a function $a(r)$ with $|Bx| \leq a(r)$ and $\dfrac{a}{\sqrt{q}} = o(1)$.

Assertion: Any solution $u \in L^2(E_R)^N$ of (2.7) vanishes identically on E_{R_1} for some $R_1 > R$.

Remark 2.11. *a) The function μ originates from a transformation in Appendix A (see (A.13)–(A.18)). Theorem 2.10 holds good for solutions of*

$$(\alpha \cdot D + V + m_0\beta)u = m_0 u,$$

if $q = o(1)$ is a negative function and $\mu := \sqrt{q(q-2m_0)}$.
b) Since

$$\frac{(r\mu)'}{\mu} = 1 + \frac{rq'}{2q} + o(1),$$

q may decay like r^{-s} with $0 < s < 2$.

3 Proof of Theorem 2.1

Since

$$\int r|\alpha \cdot pu|^2 = \int r|Qu|^2 \leq C^2 \int \frac{|u|^2}{r} < \infty,$$

we can find a sequence of functions $\{u_j\}$ in $C_0^\infty(\boldsymbol{R}^n)^N$ with

$$r^{-1/2}u_j \to r^{-1/2}u, \quad \sqrt{r}\,(\alpha \cdot p)u_j \to \sqrt{r}\,(\alpha \cdot p)u$$

in $L^2(\boldsymbol{R}^n)^N$. Let $U_j = r^{(n-1)/2}u_j$. We write $\|U_j\|$ rather than $\|U_j(r\cdot)\|$ for the norm in $L^2(S^{n-1})^N$ and similarly for the scalar product. We use the decomposition in (A.3) of Appendix A and note that the symmetric operator S with $b = 0$ has a purely discrete spectrum with

$$-\left(\boldsymbol{N}_0 + \frac{n-1}{2}\right) \cup \left(\boldsymbol{N}_0 + \frac{n-1}{2}\right)$$

as eigenvalues. Hence

$$\begin{aligned}
\int r|\alpha\cdot pu_j|^2 &= \int_0^\infty r \left\|\alpha_r\left(-i\partial_r U_j + \frac{i}{r}SU_j\right)\right\|^2 \\
&= \int_0^\infty r\left\langle -i\partial_r U_j + \frac{i}{r}SU_j, -i\partial_r U_j + \frac{i}{r}SU_j\right\rangle \\
&= \int_0^\infty r\left(\|\partial_r U_j\|^2 + \frac{1}{r^2}\|SU_j\|^2\right) + \int_0^\infty \partial_r\langle -U_j, SU_j\rangle \\
&= \int_0^\infty r\left(\|\partial_r U_j\|^2 + \frac{1}{r^2}\|SU_j\|^2\right) \\
&\geq \left(\frac{n-1}{2}\right)^2 \int_0^\infty \frac{\|U_j\|^2}{r},
\end{aligned}$$

and the assertion follows in the limit $j \to \infty$. $\square$

4 Preliminaries to the proof of Theorem 2.7

To explain the general strategy of the proof of Theorem 2.7, let u be a solution of (2.3). We multiply $U := r^{(n-1)/2}u$ by functions e^{φ}, $\varphi = \varphi(|\cdot|)$ real-valued, and $\chi = \chi(|\cdot|)$ with support in E_R and

$$0 \leq \chi \leq 1, \quad \chi = 1 \text{ on } [s, t_k], \quad \chi = 0 \text{ outside } [s-1, t_{k+1}],$$

where $\{t_k\}$ is a sequence tending to infinity as $k \to \infty$. Then $\xi := \chi e^{\varphi}U$ satisfies

$$\left[-i\alpha_r\mathcal{D}_r + i\alpha_r\left(\frac{S}{r} + \varphi'\right) + Q\right]\xi = g := -i\alpha_r\chi' e^{\varphi}U, \tag{4.1}$$

where $\mathcal{D}_r = \partial_r - i(x/r)\cdot b$. As in proofs of unique continuation results by means of Carleman inequalities, the idea is (as it was in the earlier papers ([V], [KOY]) to prove the existence of a constant $C > 0$ such that for large s and in the limit $t_k \to \infty$ the inequality

$$\int_s^\infty e^{2\varphi}\|U\|^2 \leq C\int_{s-1}^s e^{2\varphi}\|U\|^2 \tag{4.2}$$

holds. (For the precise inequality see (5.3) below.) If, for example $\varphi = (\ell/2)r^b$ is permitted for some $b > 0$, (4.2) implies

$$e^{\ell(s+1)^b}\int_{s+1}^\infty \|U\|^2 \leq Ce^{\ell s^b}\int_{s-1}^s \|U\|^2, \tag{4.3}$$

and in the limit $\ell \to \infty$ the desired conclusion $U = 0$ on E_{s+1} follows.
The present paper differs from [KOY] in three important respects. Firstly, the function q in Theorem 2.7 and 2.10 is allowed to tend to zero at infinity, while it was absolutely necessary to require $\pm q \geq$ const. > 0 in [KOY]. Secondly, in contrast to [KOY], Proposition 3.1, the virial relation (A.8) from which we set out here

$$\begin{aligned}
&\int \langle [\partial_r r(V-\lambda)]\xi, \xi\rangle \\
&= \underbrace{-\int \langle (\alpha\cdot Bx)\xi,\xi\rangle}_{I_1} + \underbrace{\int 2\mathrm{Re}\langle r\, W\xi, \mathcal{D}_r\xi\rangle}_{I_2} + \underbrace{\int 2r\varphi'\mathrm{Re}\langle -i\alpha_r\mathcal{D}_r\xi,\xi\rangle}_{I_3} \\
&\quad \underbrace{-\int 2r\mathrm{Re}\langle g, \mathcal{D}_r\xi\rangle}_{T_1}, \qquad (4.4)
\end{aligned}$$

does not contain a term involving q'/q. Such a term arose in [KOY] as it was necessary to divide ξ by $(\pm q)^{1/2}$ in order to cope with the case that the potential (and possibly a variable mass) became large at infinity. Thirdly, we use a more refined cutoff function.
Given $t_k := 2^k$ and $s < t_k$, there exists a function $\chi \in C^\infty(0,\infty)$ with

$$\chi(r) = 1 \text{ for } s \leq r \leq t_k \quad\text{and}\quad \chi(r) = 0 \text{ for } r \geq t_{k+1}$$

such that

$$0 \leq -r\chi'(r) \leq \text{const.}\frac{r}{t_{k+1}-t_k} \leq \text{const.}\frac{2^{k+1}}{2^k}$$

for $r \in [t_k, t_{k+1}]$ and all $k \in \boldsymbol{N}$. Moreover,

$$r^\ell|\chi^{(\ell)}(r)| \leq \text{const.} \quad (r\in[t_k,t_{k+1}], k\in\boldsymbol{N})$$

for $\ell = 2$ and $\ell = 3$.
Estimates of the five terms in (4.4) will lead us to inequality (5.1) below, from which an inequality of type (4.3) will eventually emerge with the help of a bootstrap argument.
We start with the left-hand side of (4.4) and write

$$\begin{aligned}
\partial_r[r(V-\lambda)] &= V - q + r(\partial_r V - q') + [r(q-\lambda)]' \\
&\geq \left[\frac{O(1/r)}{q-\lambda} + o(1)\frac{q}{q-\lambda} + \delta_0\right](q-\lambda)
\end{aligned}$$

by means of (i) and (H.1) in Theorem 2.7. Since $0 < q/(q-\lambda) \leq 1$ and $(q-\lambda)^{-1} \leq O(r^{1-\delta_0})$ at infinity in view of Remark 2.8, we have

$$\int \langle [\partial_r r(V-\lambda)]\xi,\xi\rangle \geq \int [\delta_0 + o(1)](q-\lambda)\|\xi\|^2. \qquad (4.5)$$

The four terms I_1, I_2, I_3 and T_1 on the right-hand side of (4.4) are estimated as follows.

Lemma 4.1.

$$a)\ \ I_1 := -\int \langle(\alpha\cdot Bx)\xi,\xi\rangle \le \int o(1)(q-\lambda)\|\xi\|^2,$$

$$\begin{aligned} b)\ \ I_2 &:= \int 2\mathrm{Re}\langle r\,W\xi,\mathcal{D}_r\xi\rangle \\ &\le \int \left[2K - \frac{K}{(q-\lambda)^2}\left(\frac{\varphi'}{r}+\varphi''\right)+o(1)\right](q-\lambda)\|\xi\|^2 + T_2, \end{aligned}$$

where

$$T_2 \;:=\; K\int\left\{\frac{(\chi')^2}{q-\lambda}+|\chi'|\left[\frac{\text{const.}}{r(q-\lambda)}+o(1)\right]\right\}\|e^{\varphi}U\|^2 \tag{4.6}$$

$$\le\; \text{const.}\left\{\int_{s-1}^{s} r\|e^{\varphi}U\|^2 + \int_{t_k}^{t_{k+1}} r^{-\delta_0}(q-\lambda)\|e^{\varphi}U\|^2\right\}. \tag{4.7}$$

Proof. a) follows immediately from assumption (H.3) since $\alpha\cdot Bx$ is an Hermitian matrix with square $|Bx|^2$.
To prove b), we observe

$$I_2 = \int 2\mathrm{Re}\langle r\,W\xi,\mathcal{D}_r\xi\rangle \le K\left(\int(q-\lambda)\|\xi\|^2 + \int\frac{1}{q-\lambda}\|\mathcal{D}_r\xi\|^2\right)$$

and use relations (A.9), (A.11) with

$$h=\frac{1}{q-\lambda},\quad j = r\left(\frac{h}{r}\right)' = h' - \frac{h}{r}.$$

Then

$$\begin{aligned}\int\frac{1}{q-\lambda}\|\mathcal{D}_r\xi\|^2 \;\le\;& \int\frac{1}{q-\lambda}\|g-Q\xi\|^2 - \int\frac{1}{r(q-\lambda)}\langle A\xi,\xi\rangle \\ &+\int j\,\mathrm{Im}\langle Q\xi,\alpha_r\xi\rangle + \int\left[\frac{j'}{2} - h\left(\frac{\varphi'}{r}+\varphi''\right)\right]\|\xi\|^2 \\ &+\int j\chi\chi'\,\|e^{\varphi}U\|^2, \end{aligned}\tag{4.8}$$

where $Q = V + W - \lambda$, $A = -\alpha_r(\alpha\cdot Bx)$. Before turning to the first term on the right-hand side of (4.8) we note

$$\mathrm{Im}\langle Q\xi,\alpha_r\xi\rangle = \mathrm{Im}\,\langle(V-q)\xi,\alpha_r\xi\rangle + \mathrm{Im}\,\langle W\xi,\alpha_r\xi\rangle$$

and

$$|\mathrm{Im}\,\langle(V-q)\xi,\alpha_r\xi\rangle| \le \frac{\text{const.}}{r(q-\lambda)}(q-\lambda)\|\xi\|^2 \le \frac{\text{const.}}{r^{\delta_0}}(q-\lambda)\|\xi\|^2 \tag{4.9}$$

as well as

$$|\mathrm{Im}\,\langle W\xi,\alpha_r\xi\rangle| \le \frac{\text{const.}}{r(q-\lambda)}(q-\lambda)\|\xi\|^2 \le \frac{\text{const.}}{r^{\delta_0}}(q-\lambda)\|\xi\|^2 \tag{4.10}$$

(see hypotheses (H.1), (H.2) and Remark 2.8 a). Similarly, the term

$$\begin{aligned} -2r\mathrm{Re}\langle g, Q\xi\rangle &= 2r\mathrm{Re}\,\langle i\alpha_r\chi' e^{\varphi}U, Q\chi e^{\varphi}U\rangle \\ &= 2r\mathrm{Re}\,\langle i\alpha_r\chi' e^{\varphi}U, (V-q+W)\chi e^{\varphi}U\rangle \end{aligned}$$

can be estimated by

$$|-2r\mathrm{Re}\langle g, Q\xi\rangle| \le \text{const.}|\chi'|\|e^{\varphi}U\|^2.$$

Next,

$$\begin{aligned} &\|g-Q\xi\|^2 = \|(q-\lambda+V-q+W)\xi\|^2 + \|g\|^2 - 2\mathrm{Re}\,\langle g,Q\xi\rangle \\ &= (q-\lambda)^2\|\xi\|^2 + \|(V-q+W)\xi\|^2 + 2(q-\lambda)\mathrm{Re}\,\langle \xi,(V-q+W)\xi\rangle \\ &\quad +\|g\|^2 - 2\mathrm{Re}\,\langle g,Q\xi\rangle \\ &\le [1+o(1)](q-\lambda)^2\|\xi\|^2 + \left[(\chi')^2 + \frac{\text{const.}}{r}|\chi'|\right]\|e^{\varphi}U\|^2. \end{aligned} \tag{4.11}$$

Using Remark 2.8 a) again, the second term in (4.8) can be majorised by

$$\text{const.}\int \frac{a}{r^{\delta_0}}\|\xi\|^2.$$

With hypothesis (H.2) we see that it is

$$\int o(1)(q-\lambda)\|\xi\|^2.$$

The same is true of the third term in (4.8), since (4.9) and (4.10) hold and $j = o(1)$ by Remark 2.8 a) and the first part of assumption (ii). This leaves us with

$$\frac{j'}{q-\lambda} = \frac{1}{r^2(q-\lambda)^2} + \frac{q'}{r(q-\lambda)^3} - \frac{q''}{(q-\lambda)^3} + \frac{2(q')^2}{(q-\lambda)^4},$$

which, by assumptions (ii) and Remark 2.8 a), is again $o(1)$. Collecting terms, we finally obtain (4.6) from which (4.7) follows, employing the properties of our cutoff function and Remark 2.8 a). □

Lemma 4.2. *Let $\varphi' \ge 0$ and*

$$k_\varphi := -r\varphi'\varphi'' - (\varphi')^2 + \frac{1}{2}(r\varphi'')', \tag{4.12}$$

$$c := \frac{a}{q-\lambda} + \frac{\text{const.}}{r(q-\lambda)} + \frac{1}{2}\frac{q'}{(q-\lambda)^2} + \frac{1}{2}\frac{rq''}{(q-\lambda)^2} \tag{4.13}$$

Then,

$$\begin{aligned} I_3 &:= \int 2r\varphi' \mathrm{Re}\langle -i\alpha_r \mathcal{D}_r \xi, \xi\rangle \\ &\leq \int \left[\frac{k_\varphi}{(q-\lambda)^2} + \frac{\varphi'}{q-\lambda}c + \mathrm{const.}\frac{|\varphi''|}{(q-\lambda)^2}\right](q-\lambda)\|\xi\|^2 \qquad (4.14) \\ &\quad + T_3, \end{aligned}$$

where

$$\begin{aligned} T_3 &:= \int \frac{r}{q-\lambda}[\,\varphi'(\chi')^2 + |\varphi''||\chi''|\,]\,\|e^\varphi U\|^2 \\ &\leq \mathrm{const.}\left\{\int_{s-1}^{s} r^2(\varphi' + |\varphi''|)\,\|e^\varphi U\|^2 \right. \qquad (4.15) \\ &\quad \left. + \int_{t_k}^{t_{k+1}} r^{2-2\delta_0}\left(\frac{\varphi'}{r} + |\varphi''|\right)(q-\lambda)\|e^\varphi U\|^2\right\}. \end{aligned}$$

Proof. By setting $\xi = \sqrt{q-\lambda}\, w$, I_3 can be written as

$$\begin{aligned} I_3 &= \int 2r\varphi'\mathrm{Re}\langle -i\alpha_r\mathcal{D}_r\xi, \xi\rangle = \int 2r\varphi'\mathrm{Re}\langle -i\alpha_r\mathcal{D}_r w, (q-\lambda)w\rangle \\ &= \int 2r\varphi'\mathrm{Re}\langle -i\alpha_r\mathcal{D}_r w, Qw\rangle - \int 2r\varphi'\mathrm{Re}\langle -i\alpha_r\mathcal{D}_r w, (V-q+W)w\rangle. \end{aligned}$$

Using (A.10), (A.12) with $h = r\varphi'$, $j = r\varphi''$, we obtain

$$\begin{aligned} I_3 &= \int r\varphi'(\|f\|^2 - \|\mathcal{D}_r w\|^2 - \|f + i\alpha_r\mathcal{D}_r w\|^2) - \int \varphi'\langle Aw, w\rangle \\ &+ \int \left[r\varphi'\varphi'' + \frac{1}{2}(r\varphi'')' - [r(\varphi')^2]' + \frac{1}{2}\left(\frac{r\varphi'}{q-\lambda}q'\right)' - \frac{r\varphi''}{2(q-\lambda)}q'\right]\|w\|^2 \\ &+ \int r\varphi''\mathrm{Im}\langle Qw, \alpha_r w\rangle + \int \frac{r\varphi''}{q-\lambda}\chi\chi'\|e^\varphi U\|^2 \\ &- \int 2r\varphi'\mathrm{Re}\langle -i\alpha_r\mathcal{D}_r w, (V-q+W)w\rangle. \qquad (4.16) \end{aligned}$$

The last term can simply be estimated by

$$\int r\varphi'\left(\|\mathcal{D}_r w\|^2 + \frac{\mathrm{const.}}{r^2}\|w\|^2\right).$$

Replacing ξ by w in (4.9) and (4.10), we have

$$\begin{aligned} I_3 &\leq \int \left\{k_\varphi + \varphi'\left[a + \frac{\mathrm{const.}}{r} + \frac{1}{2}\left(\frac{rq'}{q-\lambda}\right)'\right] + \mathrm{const.}|\varphi''|\right\}\|w\|^2 \\ &\quad + \int \frac{r}{q-\lambda}\,[\varphi'(\chi')^2 + |\varphi''||\chi'|]\,\|e^\varphi U\|^2, \end{aligned}$$

which leads to (4.12)–(4.14). The estimate (4.15) is again a consequence of the properties of χ and of Remark 2.8 a). □

LEMMA 4.3. *Let* $\varphi' \geq 0$. *Then*

$$\begin{aligned} T_1 &:= -\int 2r\mathrm{Re}\langle g, \mathcal{D}_r\xi\rangle \\ &\leq \text{const.}\left\{\int_{s-1}^{s} r\left[(1+\varphi')\|e^{\varphi}U\|^2 + e^{2\varphi}\|\mathcal{D}_rU\|^2\right]\right. \\ &\quad \left.+\int_{t_k}^{t_{k+1}} r^{1-\delta_0}\left(1+|\varphi''|\right)(q-\lambda)\|e^{\varphi}U\|^2\right\}. \end{aligned} \tag{4.17}$$

Proof. Let $\eta := e^{\varphi}U$. Since

$$\mathrm{Re}\langle -i\alpha_r\chi'\eta, \partial_r(\chi\eta)\rangle = \chi\chi'\,\mathrm{Re}\langle -i\alpha_r\eta, \partial_r\eta\rangle,$$

we have

$$\begin{aligned} T_1 &= -\int 2r\mathrm{Re}\langle g, \mathcal{D}_r\xi\rangle = \int 2r\mathrm{Re}\langle i\alpha_r\chi' e^{\varphi}U, \mathcal{D}_r\xi\rangle \\ &= \int 2r\chi\chi'\mathrm{Re}\langle i\alpha_r\eta, \mathcal{D}_r\eta\rangle. \end{aligned}$$

On the interval $[s-1, s]$ the integral can simply be estimated by

$$\int_{s-1}^{s} 2r\chi'\|\eta\|\|\mathcal{D}_r\eta\| \leq \int_{s-1}^{s} re^{2\varphi}\|U\|\|\mathcal{D}_rU + \varphi'U\|.$$

On $[t_k, t_{k+1}]$ we use the estimate

$$\int_{s-1}^{s} r(-\chi')(\|\eta\|^2 + \|\mathcal{D}_r\eta\|^2)$$

and observe that (A.9) and (A.11) hold with $\chi = 1$ (i.e., $g = 0$) and $\xi = \eta$, provided h and j have compact support. Hence, with $h = r(-\chi')$, $j = r(h/r)' = -r\chi''$ we have

$$\begin{aligned} &\int r(-\chi')\left[\|\mathcal{D}_r\eta\|^2 + \left\|\left(\frac{S}{r}+\varphi'\right)\eta\right\|^2\right] \\ &= \int r(-\chi')\|Q\eta\|^2 + \int \chi'\langle A\eta, \eta\rangle - \int r\chi''\mathrm{Im}\langle (V-q+W)\eta, \alpha_r\eta\rangle \\ &+ \int\left[-\frac{1}{2}(\chi''+r\chi''') + \chi'\varphi' + r\chi'\varphi''\right]\|\eta\|^2. \end{aligned}$$

The integration extends over $[t_k, t_{k+1}]$ only where $\chi'\varphi' \leq 0$. Using Remark 2.8 a) and the estimates of the derivatives of χ, the assertion follows. □

REMARK 4.4. *The constant in (4.6), (4.13), and (4.14) is the sum of the constants which occur in assumptions (H.1) and (H.2). In view of the hypotheses of Theorem 2.7 the function c in (4.13) is* $o(1)$ *at infinity. It is important that the constants in (4.7), (4.15) and (4.17) are independent of* φ *and* $t_k := 2^k$.

5 Proof of Theorem 2.7

Before proving Theorem 2.7 we prepare the following

Proposition 5.1. *Suppose $f > 0$, $g \geq 0$ are functions on $(0, \infty)$ with*

$$\int^{\infty} \frac{1}{f} = \infty, \quad \int^{\infty} g < \infty,$$

and f is continuous and non-decreasing. Let $t_k := 2^k$ $(k \in \boldsymbol{N})$. Then we have

$$\liminf_{k \to \infty} \int_{t_k}^{t_{k+1}} fg = 0$$

Proof. Assume to the contrary that there are numbers $\varepsilon_0 > 0$ and $N \in \boldsymbol{N}$ such that

$$\int_{t_k}^{t_{k+1}} fg \geq \varepsilon_0$$

for $N \leq k \in \boldsymbol{N}$. Then

$$\int_{t_N}^{\infty} g = \sum_{k=N+1}^{\infty} \int_{t_{k-1}}^{t_k} (fg) \frac{1}{f} \geq \varepsilon_0 \sum_{k=N+1}^{\infty} \frac{1}{f(t_k)} \geq \varepsilon_0 \int_{t_{N+1}}^{\infty} \frac{1}{f} = \infty$$

gives the desired contradiction.

Proof of Theorem 2.7

From (4.4) and (4.5) and Lemma 4.1–4.3 we see

$$\begin{aligned} & \int [\delta_0 - 2K + o(1)](q-\lambda) e^{2\varphi} \|\chi U\|^2 \qquad (5.1) \\ \leq \; & \int \left[\frac{k_\varphi}{(q-\lambda)^2} + \frac{\varphi'}{q-\lambda} c + \text{const.} \frac{|\varphi''|}{(q-\lambda)^2} \right. \\ & \left. - \frac{K}{(q-\lambda)^2} \left(\frac{\varphi'}{r} + \varphi'' \right) \right] (q-\lambda) e^{2\varphi} \|\chi U\|^2 \\ & + \text{const.} \int_{s-1}^{s} r^2 e^{2\varphi} [(1 + \varphi' + |\varphi''|) \|U\|^2 + \|\mathcal{D}_r U\|^2] \\ & + \text{const.} \int_{t_k}^{t_{k+1}} r^{1-\delta_0} e^{2\varphi} (1 + \varphi' + r|\varphi''|) |(q-\lambda) \|U\|^2, \end{aligned}$$

where $t_k := 2^k$.

a) We claim

$$\int_s^{\infty} r^{\ell} (q-\lambda) \|U\|^2 < \infty$$

for all $s > R$ and $\ell > 0$. Let $j \in \boldsymbol{N}$. We choose $\varphi = (j/2) \log\log r$ in (5.1) and note

$$\varphi' = \frac{j}{2r \log r}, \quad \varphi'' = -\frac{j}{4r^2 \log r}\left(1 + \frac{1}{\log r}\right),$$
$$k_\varphi = \frac{j}{4r^2 \log r}\left[1 + \frac{2}{\log r} + \frac{j+2}{(\log r)^2}\right] \le \frac{j(j-1)}{4r^2(\log r)^3} + \frac{2j}{4r^2 \log r}$$

for $r > R_0$ if R_0 is sufficiently large. Using Remark 2.8 a), the first integral on the right-hand side of (5.1) can be majorised by

$$\text{const.} \int \left\{ \frac{1}{r^{2\delta_0}} \left[\frac{j(j-1)}{(\log r)^3} + \frac{j}{(\log r)^2} + \frac{j}{(\log r)} \right] \right.$$
$$\left. + \frac{1}{r^{\delta_0}} \frac{j}{(\log r)} o(1) \right\} (\log r)^j (q - \lambda) \|\chi U\|^2.$$

The last integral on the right-hand side of (5.1) can be estimated by

$$\int_{t_k}^{t_{k+1}} r^{1-\delta_0}(1 + \text{const.}\, j)(\log r)^j (q - \lambda)\|U\|^2. \tag{5.2}$$

With $(q - \lambda)$ being bounded, $(q - \lambda)\|U\|^2$ is in $L^1(R_0, \infty)$. Thus there is a sub-sequence $\{t_{k_\ell}\}_{\ell=1}^\infty$ on which (5.2) tends to zero in view of Proposition 5.1. This proves

$$\int_{R_0}^\infty (\log r)^j (q - \lambda)\|U\|^2 < \infty.$$

Moreover, for some $c_0 > 0$ the inequality (5.1) implies

$$c_0 \int_{R_0}^\infty \sum_{j=0}^{L} \frac{(\ell \log r)^j}{j!} (q - \lambda)\|U\|^2$$
$$\le \text{const.} \int_{s-1}^\infty \left[\frac{\ell^2}{r^{2\delta_0} \log r} \sum_{j=2}^{L} \frac{(\ell \log r)^{j-2}}{(j-2)!} + \frac{\ell}{r^{\delta_0}} \sum_{j=1}^{L} \frac{(\ell \log r)^{j-1}}{(j-1)!} \right]$$
$$\cdot (q - \lambda)\|U\|^2 + \text{const.} \int_{s-1}^{s} r^2 \sum_{j=0}^{L} \frac{(\ell \log r)^j}{j!} [(1 + j)\|U\|^2 + \|\mathcal{D}_r U\|^2].$$

Since we can let $L \to \infty$, this establishes the claim.

b) Next we assert

$$\int_s^\infty e^{\ell r^b} (q - \lambda)\|U\|^2 < \infty$$

for all $s > R$, $\ell > 0$ and $b \in (0, \delta_0)$.

We insert $\varphi = (jb/2)\log r$ into (5.1) and observe

$$\varphi' = \frac{jb}{2r}, \quad \varphi'' = -\frac{jb}{2r^2}, \quad k_\varphi = \frac{jb}{4r^2}.$$

In view of Part a) there is a sub-sequence $\{t_{k_\ell}\}$ with

$$\lim_{\ell\to\infty} \int_{t_{k_\ell}}^{t_{k_\ell+1}} r^{1-\delta_0}(1+jb)r^{jb}(q-\lambda)\|U\|^2 = 0.$$

Using Remark 2.8 a) again, we see that (5.1) implies

$$c_0 \int_s^\infty r^{jb}(q-\lambda)\|U\|^2 \leq \text{const.} \int_{s-1}^\infty \left(\frac{1}{r^{2\delta_0}} + \frac{1}{r^{\delta_0}}\right) jbr^{jb}(q-\lambda)\|U\|^2$$
$$+\,\text{const.} \int_{s-1}^s r^{2+jb}[(1+jb)\|U\|^2 + \|\mathcal{D}_r U\|^2]$$

or

$$c_0 \int_s^\infty \sum_{j=0}^L \frac{(\ell r^b)^j}{j!}(q-\lambda)\|U\|^2$$
$$\leq \text{const.} \int_{s-1}^\infty \frac{\ell b}{r^{\delta_0-b}} \sum_{j=1}^L \frac{(\ell r^b)^{j-1}}{(j-1)!}(q-\lambda)\|U\|^2$$
$$+\,\text{const.} \int_{s-1}^s r^2 \sum_{j=0}^L \frac{(\ell r^b)^j}{j!}[(1+jb)\|U\|^2 + \|\mathcal{D}_r U\|^2].$$

For $b \in (0,\delta_0)$ we can again move the first term of the right-hand side to the left and let $L \to \infty$.

c) In order to show that U vanishes a.e. on E_{R_1} for some $R_1 > R$, we choose $\varphi = (\ell/2)r^b$ where $\ell > 0$ and $b \in (0,\delta_0)$. From

$$\varphi' = \frac{\ell b}{2} r^{b-1}, \quad \varphi'' = -\frac{\ell b}{2}(1-b)r^{b-2}$$

we observe $(\varphi'/r) + \varphi'' > 0$, so that the last part of the first integral on the right-hand side of (5.1) can be discarded. On account of Part b) there is a sequence $\{t_{k_\ell}\}$ on which the last integral vanishes. Finally we note

$$k_\varphi = -\frac{\ell b}{4}\left[\ell b^2 - \frac{(1-b)^2}{r^b}\right] r^{2(b-1)}.$$

With

$$X := \frac{\ell b}{2(q-\lambda)} r^{b-1}$$

we therefore have

$$-\frac{k_\varphi}{(q-\lambda)^2} - \frac{\varphi'}{q-\lambda}c(r) - \text{const.}\frac{|\varphi''|}{(q-\lambda)^2} = bX^2 - d(r)X,$$

where

$$d(r) := \frac{1-b}{r(q-\lambda)}\left(\frac{1-b}{2} + \text{const.}\right) + c(r) = o(1).$$

Hence

$$\begin{aligned}&\int_s^\infty \left[\delta_0 - 2K + o(1) - \frac{d(r)^2}{4b}\right](q-\lambda)e^{\ell r^b}\|U\|^2 \\ &\leq \text{const.}\int_{s-1}^s r^2 e^{\ell r^b}[(1+\ell b r^b)\|U\|^2 + \|\mathcal{D}_r U\|^2]. \end{aligned} \tag{5.3}$$

Now there is an $R_1 > R$ with the property that the left-hand side of (5.3) can be estimated from below by

$$\text{const.}\, e^{\ell(s+1)^b}\int_{s+1}^\infty \|U\|^2$$

for $s > R_1$. The assertion therefore follows in the limit $\ell \to \infty$.

6 Proof of Theorem 2.10

From (A.16)-(A.17) with $\lambda = -m_0$ we see

$$\mu := [(q+m_0)^2 - m_0^2]^{1/2} = \sqrt{q}\sqrt{q+2m_0}, \quad F := \left(\frac{q+2m_0}{q}\right)^{1/4}, \tag{6.1}$$

and

$$\sqrt{\mu}F = \sqrt{q+2m_0}, \quad \frac{\sqrt{\mu}}{F} = \sqrt{q}.$$

As a consequence

$$\xi := \chi e^\varphi \begin{pmatrix} FU_1 \\ (1/F)U_2 \end{pmatrix} = \mu^{-1/2}\chi e^\varphi \zeta \quad \text{with} \quad \zeta := \begin{pmatrix} \sqrt{q+2m_0}\,U_1 \\ \sqrt{q}\,U_2 \end{pmatrix}$$

solves

$$\left[-i\alpha_r\mathcal{D}_r + i\alpha_r\left(\frac{S}{r} + \varphi'\right) + Q\right]\xi = g := -\mu^{-1/2}(i\alpha_r)\chi' e^\varphi \zeta,$$

where

$$Q = \mu I_N + (V-q)\begin{pmatrix} F^{-2}I_{N/2} & 0 \\ 0 & F^2 I_{N/2} \end{pmatrix} - \frac{m_0 q'}{2q(q+2m_0)}(i\alpha_r\beta).$$

So our virial relation (A.8) now reads

$$\begin{aligned}&\int \{\langle[\partial_r(rQ)]\xi,\xi\rangle + \langle(\alpha\cdot Bx)\xi,\xi\rangle\}\\ &= \int 2r\varphi'\mathrm{Re}\langle -i\alpha_r\mathcal{D}_r\xi,\xi\rangle - \int 2r\mathrm{Re}\langle g,\mathcal{D}_r\xi\rangle. \end{aligned}\tag{6.2}$$

Before beginning our estimates we observe that the assumption (i) implies

$$r\mu \geq \text{const.}\, r^{\delta_0} \quad \text{or} \quad q^{-1/2} \leq \text{const.}\, r^{1-\delta_0}. \tag{6.3}$$

In view of (6.1) and our assumptions (ii), (H.1) we therefore have

$$\begin{aligned}\frac{1}{\mu}(rF^{-2})'(V-q) &= \frac{1}{\mu F^2}\left[1-\frac{m_0 r q'}{q(q+2m_0)}\right](V-q)\\ &= o(1)\frac{V-q}{q} = o(1),\end{aligned}$$

$$\begin{aligned}\frac{1}{\mu}(rF^{2})'(V-q) &= \frac{F^2}{\mu}\left[1+\frac{m_0 r q'}{q(q+2m_0)}\right](V-q)\\ &= O(1)\frac{V-q}{q} = o(1),\end{aligned}$$

and from

$$\left[\frac{rq'}{q(q+2m_0)}\right]' = \frac{1}{q+2m_0}\left[\frac{q'}{q}+\frac{rq''}{q} - r\left(\frac{q'}{q}\right)^2 - \frac{r(q')^2}{q(q+2m_0)}\right]$$

we find

$$\frac{1}{\mu}\left[\frac{rq'}{q(q+2m_0)}\right]' = o(1),$$

taking advantage of (6.3) again. Since

$$\int \langle(\alpha\cdot Bx)\xi,\xi\rangle \geq -\int \frac{a}{\mu}\,\mu\,\|\xi\|^2 = -\int o(1)\,\mu\,\|\xi\|^2,$$

the left-hand side of (6.2) can be estimated from below by

$$\int [\delta_0 + o(1)]\mu\|\xi\|^2.$$

LEMMA 6.1. *Let* $\varphi' \geq 0$ *and*

$$\begin{aligned} k_\varphi &:= -r\varphi'\varphi'' - (\varphi')^2 + \frac{1}{2}(r\varphi'')',\\ c &:= \frac{a}{\mu} + \frac{r(V-q)^2}{\mu q} + \frac{1}{2\mu}\left[\left(r\frac{\mu'}{\mu}\right)' + m_0\left|\left(r\frac{q'}{\mu^2}\right)'\right|\right].\end{aligned}$$

Then

$$\begin{aligned} I &:= \int 2r\varphi' \mathrm{Re}\langle -i\alpha_r \mathcal{D}_r \xi, \xi\rangle \\ &\leq \int \left(\frac{k_\varphi}{\mu^2} + \frac{\varphi'}{\mu}c + \text{const.}\,\frac{|\varphi''|}{\mu^2}\right)\mu\|\xi\|^2 + J, \end{aligned}$$

where

$$\begin{aligned} J &:= \int \frac{r}{\mu^2}[\varphi'(\chi')^2 + |\varphi''||\chi''|]\,\|e^\varphi\zeta\|^2 \\ &\leq \text{const.}\left\{\int_{s-1}^{s} r^3(\varphi' + |\varphi''|)\,\|e^\varphi\zeta\|^2 \right. \\ &\quad \left. + \int_{t_k}^{t_{k+1}} r^{2-2\delta_0}\left(\frac{\varphi'}{r} + |\varphi''|\right)\|e^\varphi\zeta\|^2\right\} \end{aligned}$$

with a constant which is independent of φ *and* $t_k := 2^k$*. (Note that the assumptions of Theorem 2.10 imply* $c = o(1)$*.)*

Proof. Since $w := \mu^{-1/2}\xi$ satisfies

$$\left[-i\alpha_r\mathcal{D}_r + i\alpha_r\left(\frac{S}{r} + \varphi'\right) + Q - i\frac{\mu'}{2\mu}\alpha_r\right]w = f := \mu^{-1/2}g,$$

we have

$$\begin{aligned} &\int j\left\langle\left(\frac{S}{r} + \varphi'\right)w, w\right\rangle + \frac{1}{2}\int\left(j' - j\frac{\mu'}{\mu}\right)\|w\|^2 \\ &+ \int j\,\mathrm{Im}\langle Qw, \alpha_r w\rangle + \int \frac{j}{\mu^2}\chi\chi'\|e^\varphi\zeta\|^2 = 0 \end{aligned}$$

as a substitute for identity (A.12).

Let

$$\begin{aligned} I_1 &:= \int 2r\varphi'\,\mathrm{Re}\left\langle -i\alpha_r\mathcal{D}_r w, \frac{m_0 q'}{2q(q+2m_0)}(i\alpha_r\beta)w\right\rangle, \\ I_2 &:= \int 2r\varphi'\,\mathrm{Re}\left\langle -i\alpha_r\mathcal{D}_r w, (V-q)\begin{pmatrix}F^{-2}w_1 \\ F^2 w_2\end{pmatrix}\right\rangle. \end{aligned}$$

Replacing q by μ in (A.10), we can write

$$\begin{aligned} I &= \int 2r\varphi'\,\mathrm{Re}\,\langle -i\alpha_r\mathcal{D}_r w, \mu w\rangle = \int 2r\varphi'\,\mathrm{Re}\,\langle -i\alpha_r\mathcal{D}_r w, Qw\rangle + I_1 + I_2 \\ &= \int r\varphi'(\|f\|^2 - \|\mathcal{D}_r w\|^2 - \|f + i\alpha_r\mathcal{D}_r w\|^2) - \int \varphi'\langle Aw, w\rangle \\ &\quad + \int\left[k_\varphi + \frac{1}{2}\left(r\varphi'\,\frac{\mu'}{\mu}\right)' - \frac{1}{2}r\varphi''\,\frac{\mu'}{\mu}\right]\|w\|^2 + \int \frac{r\varphi''}{\mu^2}\chi\chi'\|e^\varphi\zeta\|^2 \\ &\quad + \int r\varphi''\,\mathrm{Im}\langle Qw, \alpha_r w\rangle + I_1 + I_2. \end{aligned}$$

Integrating by parts in I_1, we obtain

$$\begin{aligned} I_1 &= -\int 2r\varphi' \operatorname{Re}\left\langle \partial_r w, \frac{q'}{2q(q+2m_0)}(m_0\beta)w \right\rangle \qquad (6.4) \\ &= \frac{1}{2}\int \left\langle w, \left\{ r\varphi'' \frac{q'}{q(q+2m_0)}(m_0\beta) + \varphi' \left[\frac{rq'}{q(q+2m_0)}\right]'(m_0\beta)\right\} w \right\rangle \end{aligned}$$

and note that the first term in (6.4) cancels

$$\int r\varphi'' \operatorname{Im}\langle Qw, \alpha_r w\rangle = -\frac{1}{2}\int r\varphi'' \left\langle \frac{q'}{q(q+2m_0)}(m_0\beta)w, w\right\rangle .$$

Furthermore, since

$$F^{-4}|w_1|^2 + F^4|w_2|^2 = \frac{q}{q+2m_0}|w_1|^2 + \frac{q+2m_0}{q}|w_2|^2,$$

we have

$$I_2 \le \int r\varphi' \|\mathcal{D}_r w\|^2 + \text{const.} \int r\varphi' \frac{(V-q)^2}{q}\|w\|^2.$$

Collecting terms, the assertion follows. □

Lemma 6.2. *Let $\varphi' \ge 0$. Then*

$$\begin{aligned} T &:= -\int 2r \operatorname{Re}\langle g, \mathcal{D}_r\xi\rangle \\ &\le \Big\{ \text{const.} \int_{s-1}^{s} r^2 \left[(1+\varphi')\|e^{\varphi}\zeta\|^2 + e^{2\varphi}\|\mathcal{D}_r\zeta\|^2\right] \\ &\qquad + \int_{t_k}^{t_{k+1}} r^{1-\delta_0}(1+|\varphi''|)\|e^{\varphi}\zeta\|^2 \Big\} \end{aligned}$$

with a constant which is independent of φ and t_k.

Proof. Abbreviating $\phi := e^{\varphi}\zeta$, we have

$$T = -\int 2r \operatorname{Re}\langle g, \mathcal{D}_r\xi\rangle = \int 2r\frac{\chi\chi'}{\mu}\operatorname{Re}\langle i\alpha_r\phi, \mathcal{D}_r\phi\rangle.$$

On the interval $[s-1, s]$ the integral can be estimated by

$$\int_{s-1}^{s} 2r\frac{\chi'}{\mu}\|\phi\|\,\|\mathcal{D}_r\phi\| \le \text{const.} \int_{s-1}^{s} r^{2-\delta_0} e^{2\varphi}\|\zeta\|\,\|\mathcal{D}_r\zeta + \varphi'\zeta\|,$$

using (6.3). On $[t_k, t_{k+1}]$ we majorise the integral by

$$\int_{t_k}^{t_{k+1}} \frac{r}{\mu}(-\chi')(\|\phi\|^2 + \|\mathcal{D}_r\phi\|^2) \qquad (6.5)$$

and note that it is permitted to use (A.9), (A.11) with $\chi = 1$ (i.e., $g = 0$) and $\xi = \phi$, since

$$h = \frac{r}{\mu}(-\chi'), \quad j = r\left(\frac{h}{r}\right)' = h' - \frac{h}{r}$$

have compact support. Hence on $[t_k, t_{k+1}]$

$$\begin{aligned}
&\int \frac{r}{\mu}(-\chi')\left[\|\mathcal{D}_r\phi\|^2 + \left\|\left(\frac{S}{r} + \varphi'\right)\phi\right\|^2\right] \\
&= \int \frac{r}{\mu}(-\chi')\|Q\phi\|^2 + \int \frac{\chi'}{\mu}\langle A\phi, \phi\rangle + \int \left[\frac{j'}{2} - \left(\frac{\varphi'}{r} + \varphi''\right)h\right]\|\phi\|^2 \\
&\quad + \int j\,\mathrm{Im}\langle Q\phi, \alpha_r\phi\rangle.
\end{aligned}$$

Now, $-\varphi'' h \leq \text{const.}|\varphi''|r^{1-\delta_0}$ by (6.3), while $-(\varphi'/r)h \leq 0$ on $[t_k, t_{k+1}]$. From $h' = o(1) = h''$ we conclude $j = o(1) = j'$. The integral (6.5) can therefore be estimated by

$$\text{const.}\int_{t_k}^{t_{k+1}} r^{1-\delta_0}(1 + |\varphi''|)\|\phi\|^2,$$

which concludes the proof. □

Summing up, we have

$$\begin{aligned}
&\int [\delta_0 + o(1)]\|\chi e^{\varphi}\zeta\|^2 \leq \int \left(\frac{k_\varphi}{\mu^2} + \frac{\varphi'}{\mu}c + \text{const.}\frac{|\varphi''|}{\mu^2}\right)\|\chi e^{\varphi}\zeta\|^2 \\
&+ \text{const.}\left\{\int_{s-1}^{s} r^3 e^{2\varphi}[(1 + \varphi' + |\varphi''|)\|\zeta\|^2 + \|\mathcal{D}_r\zeta\|^2]\right. \\
&\left.+ \int_{t_k}^{t_{k+1}} r^{1-\delta_0}(1 + \varphi' + r|\varphi''|)e^{2\varphi}\|\zeta\|^2\right\},
\end{aligned}$$

which does not differ from (5.1) in any essential way. The previous bootstrap argument can therefore be repeated almost verbatim, proving $\zeta = 0$ and so $u = 0$ a.e. on E_{R_1} for some $R_1 > R$. □

Appendix

A Identities in Connection with the Virial Theorem

A.1 Algebraic Relations

The principal part in

$$(\alpha \cdot D + Q)u = 0, \quad D := p - b, \quad p = -i\nabla \tag{A.1}$$

can be decomposed with the operators

$$\mathcal{D}_r := \partial_r - i\frac{x}{r}\cdot b, \quad \alpha_r := \alpha \cdot \frac{x}{r},$$
$$S := \frac{n-1}{2} - \sum_{1\leq j<k\leq n} i\alpha_j\alpha_k(x_jD_k - x_kD_j) \tag{A.2}$$

as follows:

$$\alpha \cdot D = \alpha_r\left(-ir^{(1-n)/2}\mathcal{D}_r r^{(n-1)/2} + \frac{i}{r}S\right). \tag{A.3}$$

S is a symmetric operator in $L^2(S^{n-1})^N$ which commutes with every operator which solely depends on the radial variable r; it anticommutes with α_r. In two dimensions we have

$$S = \frac{1}{2} - i\sigma_1\sigma_2(x_1D_2 - x_2D_1) = \frac{1}{2} + \sigma_3(x_1D_2 - x_2D_1),$$

where σ_1, σ_2, σ_3 are the Pauli matrices. For $n = 3$ it is convenient to define $\sigma := (-i\alpha_2\alpha_3, -i\alpha_3\alpha_1, -i\alpha_1\alpha_2)$. Then

$$S = 1 + \sigma \cdot L \quad \text{with} \quad L := x \times D,$$

but the operator $K := \beta S$ is also used instead of S.
We notice that $\alpha \cdot Bx$ anticommutes with α_r, since B is skew-symmetric. A longer but completely elementary calculation shows

$$A := [\mathcal{D}_r, S] = -i\alpha_r(\alpha \cdot Bx).$$

Since $\alpha_r^2 = 1$, this implies $A^2 = |Bx|^2$ and

$$[\mathcal{D}_r, i\alpha_r S] = \alpha \cdot Bx. \tag{A.4}$$

Furthermore,

$$T := (\beta + \eta\, \alpha \cdot Bx)(1 + \zeta\, i\alpha_r)$$

is an Hermitian $N \times N$ matrix with square

$$T^2 = (1 + \eta^2|Bx|^2)(1 + \zeta^2),$$

if η, $\zeta \in \boldsymbol{R}$.
Finally we note that the $(n+1)$ Dirac matrices can be given the following block structure :

$$\alpha_j = \begin{pmatrix} 0 & a_j \\ a_j^* & 0 \end{pmatrix} \quad (j = 1, 2, \cdots, n), \quad \beta = \begin{pmatrix} I_{N/2} & 0 \\ 0 & I_{N/2} \end{pmatrix}. \tag{A.5}$$

The a_j are $(N/2) \times (N/2)$ matrices (Hermitian if n is odd) which satisfy the following commutation relations :

$$a_ja_k^* + a_ka_j^* = 2\delta_{jk}I_{N/2}, \quad a_j^*a_k + a_k^*a_j = 2\delta_{jk}I_{N/2}.$$

For $n = 2$, 3 this is of course well-known; the Pauli matrices take the role of the α_j if $n = 2$ and of the a_j if $n = 3$. For general n see [KY].

A.2 Analytic Tools

Let φ, χ, q, h and j be smooth real-valued functions which depend only on r; we assume that χ has compact support and q is positive. When u is a solution of (A.1), then (A.3) implies that

$$\xi := \chi e^{\varphi} U \quad \text{with} \quad U := r^{(n-1)/2} u$$

and

$$w := q^{-(1/2)} \xi$$

are solutions of

$$\left[-i\alpha_r \mathcal{D}_r + i\alpha_r \left(\frac{S}{r} + \varphi' \right) + Q \right] \xi = g := -i\alpha_r \chi' e^{\varphi} U, \tag{A.6}$$

$$\left[-i\alpha_r \mathcal{D}_r + i\alpha_r \left(\frac{S}{r} + \varphi' \right) + Q - i\alpha_r \frac{q'}{2q} \right] w = f := q^{-(1/2)} g. \tag{A.7}$$

Splitting Q into an Hermitian part Q_1 and $Q_2 := Q - Q_1$, the following virial relation holds :

$$\int \langle [\partial_r (r Q_1] \xi, \xi \rangle = - \int \langle (\alpha \cdot Bx) \xi, \xi \rangle \tag{A.8}$$
$$+ \int 2 \operatorname{Re} \langle r\, Q_2 \xi, \mathcal{D}_r \xi \rangle + \int 2r\varphi' \operatorname{Re} \langle -i\alpha_r \mathcal{D}_r \xi, \xi \rangle - \int 2r \operatorname{Re} \langle g, \mathcal{D}_r \xi \rangle.$$

Norm and scalar product in $L^2(S^{n-1})^N$ are denoted by $\| \cdot \|$ and $\langle \cdot, \cdot \rangle$, respectively. We write $\|\xi\|$ rather than $\|\xi(r\cdot)\|$ and similarly for the scalar product. Integration is over $(0, \infty)$.

Starting from

$$\int 2r \operatorname{Re} \langle g, \mathcal{D}_r \xi \rangle$$

(A.8) can immediately be verified, using (A.6) and (A.4) as well as an integration by parts in the term containing Q_1 (see [KOY], Proposition 3.1 and the remark on p. 40). In case χ can be replaced by 1, the virial theorem in its familiar form follows from (A.8) by setting $\varphi = 0$, $Q_2 = 0$ and observing that g is zero.

As a consequence of (A.6) – (A.7) we note the following energy relations:

$$\int h \left[\|\mathcal{D}_r \xi\|^2 + \left\| \left(\frac{S}{r} + \varphi' \right) \xi \right\|^2 \right] = \int h \|g - Q\xi\|^2 \tag{A.9}$$
$$- \int \frac{h}{r} \langle A\xi, \xi \rangle - \int r \left(\frac{h}{r} \right)' \left\langle \frac{S}{r} \xi, \xi \right\rangle - \int (h\varphi')' \|\xi\|^2,$$

$$\int 2h \operatorname{Re}\langle -i\alpha_r \mathcal{D}_r w, Qw\rangle \tag{A.10}$$
$$= \int h(\|f\|^2 - \|\mathcal{D}_r w\|^2 - \|f + i\alpha_r \mathcal{D}_r w\|^2) - \int \frac{h}{r}\langle Aw, w\rangle$$
$$- \int r\left(\frac{h}{r}\right)' \left\langle \frac{S}{r} w, w\right\rangle + \int \left[\left(h\frac{q'}{2q}\right)' - (h\varphi')'\right]\|w\|^2.$$

(A.9) follows from

$$\int h\|i\alpha_r(g - Q\xi)\|^2 = \int h\left\|\mathcal{D}_r\xi - \left(\frac{S}{r} + \varphi'\right)\xi\right\|^2$$

by undoing the square on the right-hand side and integrating by parts (cf. [KOY], Proposition 3.3 and the remark on p.40). Similarly, (A.10) can be proved by inserting into the left-hand side the expression for Qw which arises from (A.7) (cf. [KOY], Proposition 3.2 and the remark on p.40).
Since S is unbounded from above and from below, the corresponding term on the right-hand side of (A.9)–(A.10) has to be eliminated. This can be done with the help of the auxiliary identities

$$\int j\left\langle \left(\frac{S}{r} + \varphi'\right)\xi, \xi\right\rangle + \frac{1}{2}\int j'\|\xi\|^2 + \int j \operatorname{Im}\langle Q\xi, \alpha_r\xi\rangle$$
$$+ \int j\chi\chi'\|e^{\varphi}U\|^2 = 0, \tag{A.11}$$

$$\int j\left\langle \left(\frac{S}{r} + \varphi'\right)w, w\right\rangle + \frac{1}{2}\int \left(j' - j\frac{q'}{q}\right)\|w\|^2 + \int j \operatorname{Im}\langle Qw, \alpha_r w\rangle$$
$$+ \int \frac{j}{q}\chi\chi'\|e^{\varphi}U\|^2 = 0. \tag{A.12}$$

(A.11), for example, results at once from

$$\int \operatorname{Im}\left\langle i\alpha_r\left(\frac{S}{r} + \varphi'\right)\xi, j\alpha_r\xi\right\rangle,$$

inserting (A.6) and integrating by parts (cf. [KOY], p.23).
Let $F = F(r) > 0$ be a smooth function, $m_0 > 0$ and $\lambda \in \boldsymbol{R}$. When V is a scalar function and $Q = V + m_0\beta - \lambda$, it may be advantageous to split a solution u of (A.1) into two vectors u_1, u_2 with $N/2$ components and use the block structure (A.5) of the Dirac matrices jointly with the transformation

$$\zeta := \begin{pmatrix} F\,U_1 \\ (1/F)\,U_2 \end{pmatrix}, \quad U_j = r^{(n-1)/2}u_j \tag{A.13}$$

(cf. [KOY], p.37). Then $\alpha_r = \begin{pmatrix} 0 & a_r \\ a_r^* & 0 \end{pmatrix}$ where $a_r := \sum_{j=1}^n (x_j/r) a_j$. With a smooth function $q = q(r)$ and

$$P := \frac{F'}{F} i\alpha_r\beta + \begin{pmatrix} (1/F^2)\,(V - q + q + m_0 - \lambda) I_{N/2} & 0 \\ 0 & F^2(V - q + q - m_0 - \lambda) I_{N/2} \end{pmatrix}, \tag{A.14}$$

we then have

$$\left(-\alpha_r \mathcal{D}_r + \frac{i}{r}\alpha_r S + P\right)\zeta = 0. \tag{A.15}$$

If, for example, $q - m_0 - \lambda > 0$, requiring

$$\frac{1}{F^2}(q + m_0 - \lambda) = \mu = F^2(q - m_0 - \lambda), \tag{A.16}$$

leads to

$$\mu = [(q-\lambda)^2 - m_0^2]^{1/2}, \quad F = \left(\frac{q + m_0 - \lambda}{q - m_0 - \lambda}\right)^{1/4} \tag{A.17}$$

Since

$$\frac{F'}{F} = -\frac{m_0}{2}\frac{q'}{\mu^2},$$

the potential (A.14) becomes

$$P = \mu I_N + (V - q)\begin{pmatrix} F^{-2} I_{N/2} & 0 \\ 0 & F^2 I_{N/2} \end{pmatrix} - \frac{m_0}{2}\frac{q'}{\mu^2} i\alpha_r\beta. \tag{A.18}$$

B Asymptotic Behaviour of Solutions

In case Q in

$$(-i\alpha\cdot\nabla + Q)u = 0 \tag{B.1}$$

is $Q = m_0\beta + \lambda - q$ and q is a rotationally symmetric scalar function, it suffices to discuss the ordinary differential equation

$$u' = \begin{pmatrix} -(k/r) & -q + m_0 + \lambda \\ q + m_0 - \lambda & (k/r) \end{pmatrix} u, \tag{B.2}$$

where k is an eigenvalue of the angular momentum operator S in (A.2) with $b = 0$. k is an integer or half-integer such that $|k| \geq (n-1)/2$. (For general n, the reduction of (B.1) to (B.2) can be found in [KY].)

B.1 Case $m_0 = 0 = \lambda$

In the new variables

$$t = \log r, \quad v(t) = u(e^t) \tag{B.3}$$

(B.2) reads

$$v' = \left\{ \begin{pmatrix} -k & 0 \\ 0 & k \end{pmatrix} + \begin{pmatrix} 0 & -e^t\, q(e^t) \\ e^t\, q(e^t) & 0 \end{pmatrix} \right\} v. \tag{B.4}$$

If $e^t\, q(e^t) \to 0$ as $t \to \infty$, then (B.4) has a fundamental system of solutions $v_\pm$ with the property

$$\lim_{t\to\infty} \frac{1}{t} \log |v_\pm(t)| = \pm k.$$

(We refer to the references in [AKS] for this theorem which goes back to Perron, Lettenmeyer and Hartman-Wintner.) Hence (B.2) has a solution which is in L^2 at infinity if $rq(r) \to 0$ as $r \to \infty$ and $2|k| > 1$. Note that (B.4) has a fundamental system of solutions

$$v_\pm(t) = e^{\pm\sqrt{k^2-c^2}\, t} v_\pm^0$$

if $q = c/r$. Hence (B.2) has an L^2-solution at infinity if $|k| > [(1/2) + c^2]^{1/2}$.

B.2 Case $\lambda = -m_0 < 0$

In this case (B.2) reads

$$u' = \left\{ \begin{pmatrix} 0 & 0 \\ 2m_0 & 0 \end{pmatrix} + \begin{pmatrix} -(k/r) & -q \\ q & (k/r) \end{pmatrix} \right\} u =: (J+R)u. \tag{B.5}$$

The Jordan matrix J can be removed by observing that

$$\phi := \begin{pmatrix} 1 & 0 \\ \sigma & 1 \end{pmatrix} \quad \text{with} \quad \sigma := 2m_0\, r$$

has the properties $\phi' = J = J\phi$. Hence $z := \phi^{-1}u$ satisfies

$$\phi' z + \phi z' = (J+R)\, \phi z \quad \text{or} \quad z' = \phi^{-1} R\, \phi z$$

(see [Ea], p.43 for this trick). Since the asymptotically leading term in $\phi^{-1}R\phi$ still has the double eigenvalue zero, a second transformation is required. With

$$D := \begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix}, \quad w := D^{-1} z$$

we obtain

$$\begin{aligned} w' &= [(D^{-1})' D + (\phi D)^{-1} R \phi D] w \\ &= \left\{ \begin{pmatrix} -k & 0 \\ 2k & k-1 \end{pmatrix} + \begin{pmatrix} -\sigma r q & -\sigma r q \\ \sigma r q + rq/\sigma & -\sigma r q \end{pmatrix} \right\} \frac{1}{r}\, w. \end{aligned} \tag{B.6}$$

The constant matrix in (B.6) has the eigenvalues

$$\mu_\pm := -\frac{1}{2} \pm \sqrt{\frac{1}{4} + k(k-1)}.$$

So, if $k \neq 0$, 1, and if $r^2\, q \to 0$ $r \to \infty$, (B.6) has a fundamental system of solutions $w_\pm$ with

$$|w_\pm(r)| = r^{\,\mu_\pm + o(1)} \quad \text{as} \quad r \to \infty.$$

Since $u = \phi D w$ and $|\phi D| \leq \text{const.} r$, (B.5) has a solution which is of integrable square at infinity if $2\mu_- + 3 < 0$, i.e., $|k - (1/2)| > 1$.
For $q = c/r^2$ system (B.6) reads

$$w' = \left\{ A + \begin{pmatrix} 0 & 0 \\ \frac{c}{2m_0} & 0 \end{pmatrix} \frac{1}{r^2} \right\} \frac{1}{r}\, w, \tag{B.7}$$

where

$$A := \begin{pmatrix} -(k+\sigma_0) & -\sigma_0 \\ 2k + \sigma_0 & k - 1 - \sigma_0 \end{pmatrix}$$

and $\sigma_0 := 2mc$. Introducing new variables as in (B.3), (B.7) becomes

$$\tilde{w}'(t) = \left\{ A + e^{-2t} \begin{pmatrix} 0 & 0 \\ \frac{c}{2m_0} & 0 \end{pmatrix} \right\} \tilde{w}(t). \tag{B.8}$$

Since the eigenvalues of A are

$$\mu_\pm := -\frac{1}{2}(1 + 2\sigma_0) \pm \left[\frac{1}{4} + k(k - 1 - 2\sigma_0) - \sigma_0^2 \right]^{1/2},$$

it follows from the Levinson theorem ([Ea], Theorem 1.8.1) that (B.8) has a solution which is in L^2 at infinity if $|k|$ is sufficiently large. The same is therefore true of (B.5)

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