Equivariant Embeddings of Commutative Linear Algebraic Groups of Corank One

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Abstract. Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, $\mathbb{G}_m = (\mathbb{K} \setminus \{0\}, \times)$ be its multiplicative group, and $\mathbb{G}_a = (\mathbb{K}, +)$ be its additive group. Consider a commutative linear algebraic group $\mathbb{G} = (\mathbb{G}_m)^r \times \mathbb{G}_a$. We study equivariant $\mathbb{G}$-embeddings, i.e. normal $\mathbb{G}$-varieties X containing $\mathbb{G}$ as an open orbit. We prove that X is a toric variety and all such actions of $\mathbb{G}$ on X correspond to Demazure roots of the fan of X. In these terms, the orbit structure of a $\mathbb{G}$-variety X is described.

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1. Introduction

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, $\mathbb{G}_m = (\mathbb{K} \setminus \{0\}, \times)$ be its multiplicative group, and $\mathbb{G}_a = (\mathbb{K}, +)$ be its additive group. It is well known that any connected commutative linear algebraic group $\mathbb{G}$ over $\mathbb{K}$ is isomorphic to $(\mathbb{G}_m)^r \times (\mathbb{G}_a)^s$ with some non-negative integers r and s, see [20, Theorem 15.5]. We say that r is the *rank* of the group $\mathbb{G}$ and s is the *corank* of $\mathbb{G}$.

The aim of this paper is to study equivariant embeddings of commutative linear algebraic groups. Let us recall that an *equivariant embedding* of an algebraic group G is a pair (X, x), where X is an algebraic variety equipped with a regular action $G \times X \to X$ and $x \in X$ is a point with the trivial stabilizer such that the orbit Gx is open and dense in X. We assume that the variety X is normal. If X is supposed to be complete, we speak about *equivariant compactifications* of G. For the study of compactifications of reductive groups, see e.g. [26]. More generally, equivariant embeddings of homogeneous spaces of

reductive groups is a popular object starting from early 1970th. Recent survey of results in this field may be found in [27].
Let us return to the case $\mathbb{G} = (\mathbb{G}_m)^r \times (\mathbb{G}_a)^s$. If $s = 0$ then $\mathbb{G}$ is a torus and we come to the famous theory of toric varieties, see [13], [23], [18], [12]. Another extreme $r = 0$ corresponds to embeddings of a commutative unipotent (=vector) group. This case is also studied actively during last decades, see [19], [8], [3], [16], [14]. The next natural step is to study the mixed case $r > 0$ and $s > 0$ and to combine advantages of both torus and additive group actions.
The present paper deals with the case $s = 1$, i.e. from now on $\mathbb{G}$ is a connected commutative linear algebraic group of corank one. In other words, $\mathbb{G} = (\mathbb{G}_m)^{n-1} \times \mathbb{G}_a$, where $n = \dim X$.
Let X be a toric variety with the acting torus $\mathbb{T}$. Consider an action $\mathbb{G}_a \times X \to X$ normalized by $\mathbb{T}$. Then $\mathbb{T}$ acts on $\mathbb{G}_a$ by conjugation with some character e. Such a character is called a *Demazure root* of X. If $T = \mathrm{Ker}(e)$, then the group $\mathbb{G} := T \times \mathbb{G}_a$ acts on X with an open orbit, and X is a $\mathbb{G}$-embedding, see Proposition 6. Our main result (Theorem 2) states that all $\mathbb{G}$-embeddings can be realized this way. To this end we prove that for any $\mathbb{G}$-embedding X the $(\mathbb{G}_m)^{n-1}$-action on X can be extended to an action of a bigger torus $\mathbb{T}$ which normalizes the $\mathbb{G}_a$-action and X is toric with respect to $\mathbb{T}$.
This result can not be generalized to groups of corank two; examples of non-toric surfaces which are equivariant compactifications of $\mathbb{G}_a^2$ can be found in [14]. Similar examples are constructed in [14], [15] for semidirect products $\mathbb{G}_m \rightthreetimes \mathbb{G}_a$. Such groups can be considered as non-commutative groups of corank one.
If two toric varieties are isomorphic as abstract varieties, then they are isomorphic as toric varieties [10, Theorem 4.1]. This shows that the structure of a torus embedding on a toric variety is unique up to isomorphism. A structure of a $\mathbb{G}$-embedding on a given variety may be non-unique, see Examples 2, 4. Such structures are given by Demazure roots and thus the number of structures is finite if X is complete, and it is at most countable for arbitrary X. At the same time, $\mathbb{G}_a^6$-embeddings into $\mathbb{P}^6$ admit a non-trivial moduli space [19, Example 3.6].
The paper is organized as follows. Section 2 contains preliminaries on torus actions on affine varieties. We recall basic facts on affine toric varieties and introduce a description of affine T-varieties in terms of proper polyhedral divisors due to Altmann and Hausen [1]. A correspondence between $\mathbb{G}_a$-actions on X normalized by T and homogeneous locally nilpotent derivations (LNDs) of the algebra $\mathbb{K}[X]$ is explained. We define Demazure roots of a cone and use them to describe homogeneous LNDs on $\mathbb{K}[X]$, where X is toric. Also we give a description of homogeneous LNDs of horizontal type on algebras with grading of complexity one obtained by Liendo [22].
In Section 3 we show that if X is a normal affine T-variety of complexity one and the algebra $\mathbb{K}[X]$ admits a homogeneous LND of degree zero, then X is toric with an acting torus $\mathbb{T}$, T is a subtorus of $\mathbb{T}$, and $\mathbb{T}$ normalizes the corresponding $\mathbb{G}_a$-action. This gives the result for affine $\mathbb{G}$-embeddings. Moreover, Proposition 3 provides an explicit description of affine $\mathbb{G}$-embeddings.

Section 4 deals with compactifications of $\mathbb{G}$. Here we use the Cox construction and a lifting of the action of $\mathbb{G}$ to the total coordinate space $\overline{X}$ of X to deduce the result from the affine case.

In Section 5 we recall basic facts on toric varieties and introduce the notion of a Demazure root of a fan following Demazure [13]. The action of the corresponding one-parameter subgroup on the toric variety is also described there.

Let Σ be a fan and e be a Demazure root of Σ. In Section 6 we define a $\mathbb{G}$-embedding associated to the pair (Σ, e) and study the $\mathbb{G}$-orbit structure of X. It turns out that the number of $\mathbb{G}$-orbits on X is finite.

Finally, in Section 7 we prove that any $\mathbb{G}$-embedding is associated with some pair (Σ, e). The idea is to reduce the general case to the complete one via equivariant compactification. At the end several explicit examples of $\mathbb{G}$-embeddings are given.

Some results of this paper appeared in preprint [5]. They form a part of the Ph.D. thesis of the second author [21].

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2. $\mathbb{G}_a$-actions on affine T-varieties

Let X be an irreducible affine variety with an effective action of an algebraic torus T, M be the character lattice of T, N be the lattice of one-parameter subgroups of T, and $A = \mathbb{K}[X]$ be the algebra of regular functions on X. It is well known that there is a bijective correspondence between effective T-actions on X and effective M-gradings on A. In fact, the algebra A is graded by a semigroup of lattice points in some convex polyhedral cone $\omega \subseteq M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q}$. So we have

$$A = \bigoplus_{m \in \omega_M} A_m \chi^m,$$

where $\omega_M = \omega \cap M$ and χ^m is the character corresponding to m.

A derivation ∂ on an algebra A is said to be *locally nilpotent* (LND) if for each $f \in A$ there exists $n \in \mathbb{N}$ such that $\partial^n(f) = 0$. For any LND ∂ on A the map $\varphi_\partial : \mathbb{G}_a \times A \to A$, $\varphi_\partial(s, f) = \exp(s\partial)(f)$, defines a structure of a rational $\mathbb{G}_a$-algebra on A. This induces a regular action $\mathbb{G}_a \times X \to X$, where $X = \operatorname{Spec} A$. In fact, any regular $\mathbb{G}_a$-action on X arises this way, see [17, Section 1.5]. A derivation ∂ on A is said to be *homogeneous* if it respects the M-grading. If $f, h \in A \setminus \ker \partial$ are homogeneous, then $\partial(fh) = f\partial(h) + \partial(f)h$ is homogeneous too and $\deg \partial(f) - \deg f = \deg \partial(h) - \deg h$. So any homogeneous derivation ∂ has a well defined *degree* given as $\deg \partial = \deg \partial(f) - \deg f$ for any homogeneous $f \in A \setminus \ker \partial$. It is easy to see that an LND on A is homogeneous if and only if the corresponding $\mathbb{G}_a$-action is normalized by the torus T in the automorphism group $\operatorname{Aut}(X)$, cf. [17, Section 3.7].

Any derivation on $\mathbb{K}[X]$ extends to a derivation on the field of fractions $\mathbb{K}(X)$ by the Leibniz rule. A homogeneous LND ∂ on $\mathbb{K}[X]$ is said to be of *fiber type* if $\partial(\mathbb{K}(X)^T) = 0$ and of *horizontal type* otherwise. In other words, ∂ is of fiber type if and only if the general orbits of corresponding $\mathbb{G}_a$-action on X are contained in the closures of T-orbits.

Let X be an affine toric variety, i. e. a normal affine variety with a generically transitive action of a torus T. In this case

$$A = \bigoplus_{m \in \omega_M} \mathbb{K}\chi^m = \mathbb{K}[\omega_M]$$

is the semigroup algebra. Recall that for given cone $\omega \subset M_{\mathbb{Q}}$, its *dual cone* is defined by

$$\sigma = \{n \in N_{\mathbb{Q}} \mid \langle n, p \rangle \geqslant 0 \ \ \forall p \in \omega\},$$

where $\langle , \rangle$ is the pairing between dual lattices N and M. Let $\sigma(1)$ be the set of rays of a cone σ and n_ρ be the primitive lattice vector on the ray ρ. For $\rho \in \sigma(1)$ we set

$$S_\rho := \{e \in M \mid \langle n_\rho, e \rangle = -1 \text{ and } \langle n_{\rho'}, e \rangle \geqslant 0 \ \ \forall \rho' \in \sigma(1), \, \rho' \neq \rho\}.$$

One easily checks that the set S_ρ is infinite for each $\rho \in \sigma(1)$. The elements of the set $\mathfrak{R} := \bigsqcup_\rho S_\rho$ are called the *Demazure roots* of σ. Let $e \in S_\rho$. Then ρ is called the *distinguished ray* of the root e. One can define the homogeneous LND on the algebra A by the rule

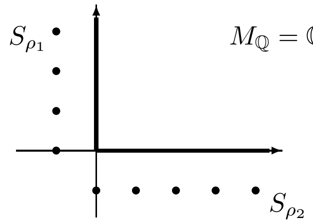
$$\partial_e(\chi^m) = \langle n_\rho, m \rangle \chi^{m+e}.$$

In fact, every homogeneous LND on A has a form $\alpha\partial_e$ for some $\alpha \in \mathbb{K}$, $e \in \mathfrak{R}$, see [22, Theorem 2.7]. In other words, $\mathbb{G}_a$-actions on X normalized by the acting torus are in bijection with Demazure roots of the cone σ.

Clearly, all homogeneous LNDs on a toric variety are of fiber type.

Example 1. Consider $X = \mathbb{A}^k$ with the standard action of the torus $(\mathbb{K}^\times)^k$. It is a toric variety with the cone $\sigma = \mathbb{Q}^k_{\geqslant 0}$ having rays $\rho_1 = \langle (1, 0, \ldots, 0) \rangle_{\mathbb{Q}_{\geqslant 0}}, \ldots, \rho_k = \langle (0, 0, \ldots, 0, 1) \rangle_{\mathbb{Q}_{\geqslant 0}}$. The dual cone ω is $\mathbb{Q}^k_{\geqslant 0}$ as well. In this case

$$S_{\rho_i} = \{(c_1, \ldots, c_{i-1}, -1, c_{i+1}, \ldots, c_k) \mid c_j \in \mathbb{Z}_{\geqslant 0}\}.$$



Denote $x_1 = \chi^{(1,0,\ldots,0)}, \ldots, x_k = \chi^{(0,\ldots,0,1)}$. Then $\mathbb{K}[X] = \mathbb{K}[x_1, \ldots, x_k]$. It is easy to see that the homogeneous LND corresponding to the root $e = (c_1, \ldots, c_k) \in S_{\rho_i}$ is

$$\partial_e = x_1^{c_1} \ldots x_{i-1}^{c_{i-1}} x_{i+1}^{c_{i+1}} \ldots x_k^{c_k} \frac{\partial}{\partial x_i}.$$

This LND gives rise to the $\mathbb{G}_a$-action

$$x_i \mapsto x_i + s x_1^{c_1} \dots x_{i-1}^{c_{i-1}} x_{i+1}^{c_{i+1}} \dots x_k^{c_k}, \quad x_j \mapsto x_j, \quad j \neq i, \quad s \in \mathbb{G}_a.$$

Let us recall that the *complexity* of an action of a torus T on an irreducible variety X is the codimension of a general T-orbit on X, or, equivalently, the transcendence degree of the field of rational invariants $\mathbb{K}(X)^T$ over $\mathbb{K}$. In particular, actions of complexity zero are precisely actions with an open T-orbit.

Now we recall a description of normal affine T-varieties of complexity one in terms of proper polyhedral divisors. Let N and M be two mutually dual lattices with the pairing denoted by $\langle , \rangle$, σ be a strongly convex cone in $N_{\mathbb{Q}}$, and $\omega \subseteq M_{\mathbb{Q}}$ be the dual cone. A polyhedron $\Delta \subseteq N_{\mathbb{Q}}$, which can be decomposed as Minkowski sum of a bounded polyhedron and the cone σ, is called *σ-tailed*. Let C be a smooth curve. A *σ-polyhedral* divisor on C is a formal sum

$$\mathfrak{D} = \sum_{z \in C} \Delta_z \cdot z,$$

where Δ_z are the σ-tailed polyhedra and only finite number of them are not equal to σ. The divisor $\mathfrak{D}$ is *trivial*, if $\Delta_z = \sigma$ for all $z \in C$.

The finite set $\operatorname{Supp} \mathfrak{D} := \{z \in C \mid \Delta_z \neq \sigma\}$ is called the *support* of $\mathfrak{D}$. For every $m \in \omega_M$ we can obtain the $\mathbb{Q}$-divisor $\mathfrak{D}(m) = \sum_{z \in C} h_z(m) \cdot z$, where $h_z(m) := \min_{p \in \Delta_z} \langle p, m \rangle$. So a σ-polyhedral divisor is just a piecewise-linear function from ω_M to the group of $\mathbb{Q}$-divisors on C. One can define the M-graded algebra

$$A[C, \mathfrak{D}] = \bigoplus_{m \in \omega_M} A_m \chi^m,$$

where

$$A_m = H^0(C, \mathfrak{D}(m)) := \{f \in \mathbb{K}(X) \mid \operatorname{div} f + \mathfrak{D}(m) \geqslant 0\},$$

where the multiplication of homogeneous elements is given as in $\mathbb{K}(X)$.

A σ-polyhedral divisor on smooth curve C is called *proper* if either C is affine, or C is projective and the polyhedron $\deg \mathfrak{D} := \sum_{z \in C} \Delta_z$ is a proper subset of σ.

The next theorem expresses the main results of [1] specialized to the case of torus actions of complexity one.

Theorem 1. (1) *Let C be a smooth curve and $\mathfrak{D}$ a proper σ-polyhedral divisor on C. Then the M-graded algebra $A[C, \mathfrak{D}]$ is a normal finitely generated effectively graded* $(\operatorname{rk} M + 1)$*-dimensional domain. Conversely, for each normal finitely generated domain A with a grading of complexity one there exist a smooth curve C and a proper σ-polyhedral divisor $\mathfrak{D}$ on C such that A is isomorphic to $A[C, \mathfrak{D}]$.*

(2) *The M-graded domains* $\operatorname{Spec} A[C, \mathfrak{D}]$ *and* $\operatorname{Spec} A[C, \mathfrak{D}']$ *are isomorphic if and only if for every $z \in C$ there exists a lattice vector $v_z \in N$ such that*

$$\mathfrak{D} = \mathfrak{D}' + \sum_z (v_z + \sigma) \cdot z,$$

and for all $m \in \omega_M$ *the divisor* $\sum_z \langle v_z, m\rangle \cdot z$ *is principal.*

The following result is obtained in [1, Section 11].

Proposition 1. *Let* $\mathfrak{D}$ *be a proper* σ*-polyhedral divisor on a smooth curve* C*,* $X = \operatorname{Spec} A[C,\mathfrak{D}]$*, and* $T \times X \to X$ *be the corresponding torus action. Then this action can be realized as a subtorus action an a toric variety if and only if either* $C = \mathbb{A}^1$ *and* $\mathfrak{D}$ *can be chosen supported in at most one point, or* $C = \mathbb{P}^1$ *and* $\mathfrak{D}$ *can be chosen supported in at most two points.*

Also we need a description of homogeneous LNDs of horizontal type for a T-variety X of complexity one from [22]. Below we follow the approach given in [7]. We have $\mathbb{K}[X] = A[C,\mathfrak{D}]$ for some C and $\mathfrak{D}$. It turns out that C is isomorphic to $\mathbb{A}^1$ or $\mathbb{P}^1$ whenever there exists a homogeneous LND of horizontal type on $A[C,\mathfrak{D}]$, see [22, Lemma 3.15].
Let C be $\mathbb{A}^1$ or $\mathbb{P}^1$, $\mathfrak{D} = \sum_{z\in C} \Delta_z \cdot z$ a σ-polyhedral divisor on C, $z_0 \in C$, $z_\infty \in C\backslash\{z_0\}$, and v_z a vertex of Δ_z for every $z \in C$. Put $C' = C$ if $C = \mathbb{A}^1$ and $C' = C\backslash\{z_\infty\}$ if $C = \mathbb{P}^1$. A collection $\widetilde{\mathfrak{D}} = \{\mathfrak{D}, z_0; v_z, \forall z \in C\}$ if $C = \mathbb{A}^1$ and $\widetilde{\mathfrak{D}} = \{\mathfrak{D}, z_0, z_\infty; v_z, \forall z \in C'\}$ if $C = \mathbb{P}^1$ is called a *colored* σ-polyhedral divisor on C if the following conditions hold:

($*$) $v_{\deg} := \sum_{z\in C'} v_z$ is a vertex of $\deg \mathfrak{D}|_{C'} := \sum_{z\in C'} \Delta_z$;

($**$) $v_z \in N$ for all $z \in C'$, $z \neq z_0$.

Let $\widetilde{\mathfrak{D}}$ be a colored σ-polyhedral divisor on C and $\delta \subseteq N_{\mathbb{Q}}$ be the cone generated by $\deg \mathfrak{D}|_{C'} - v_{\deg}$. Denote by $\widetilde{\sigma} \subseteq (N \oplus \mathbb{Z})_{\mathbb{Q}}$ the cone generated by $(\delta, 0)$ and $(v_{z_0}, 1)$ if $C = \mathbb{A}^1$, and by $(\delta, 0)$, $(v_{z_0}, 1)$ and $(\Delta_{z_\infty} + v_{\deg} - v_{z_0} + \delta, -1)$ if $C = \mathbb{P}^1$. By definition, put d the minimal positive integer such that $d \cdot v_{z_0} \in N$. A pair $(\widetilde{\mathfrak{D}}, e)$, where $e \in M$, is said to be *coherent* if

- (i) there exists $s \in \mathbb{Z}$ such that $\widetilde{e} = (e, s) \in M \oplus \mathbb{Z}$ is a Demazure root of the cone $\widetilde{\sigma}$ with distinguished ray $\widetilde{\rho} = (d \cdot v_{z_0}, d)$;
- (ii) $\langle v, e\rangle \geqslant 1 + \langle v_z, e\rangle$ for all $z \in C'\backslash\{z_0\}$ and all vertices $v \neq v_z$ of the polyhedron Δ_z;
- (iii) $d \cdot \langle v, e\rangle \geqslant 1 + \langle v_{z_0}, e\rangle$ for all vertices $v \neq v_{z_0}$ of the polyhedron Δ_{z_0};
- (iv) if $Y = \mathbb{P}^1$, then $d \cdot \langle v, e\rangle \geqslant -1 - d \cdot \sum_{z\in Y'} \langle v_z, e\rangle$ for all vertices v of the polyhedron Δ_{z_∞}.

It follows from [7, Theorem 1.10] that homogeneous LNDs of horizontal type on $A[C,\mathfrak{D}]$ are in bijection with the coherent pairs $(\widetilde{\mathfrak{D}}, e)$. Namely, let $(\widetilde{\mathfrak{D}}, e)$ be a coherent pair. Without loss of generality we may assume that $z_0 = 0$, $z_\infty = \infty$ if $C = \mathbb{P}^1$, and $v_z = 0 \in N$ for all $z \in C'\backslash\{z_0\}$. Let $\mathbb{K}(C) = \mathbb{K}(t)$. Then the homogeneous LND of horizontal type corresponding to $(\widetilde{\mathfrak{D}}, e)$ is given by

$$(1) \qquad \partial(\chi^m \cdot t^r) = d(\langle v_0, m\rangle + r)\chi^{m+e} \cdot t^{r+s} \quad \text{for all } m \in M, r \in \mathbb{Z}.$$

In particular, the vector e is the degree of the derivation ∂.

3. The affine case

Let (X,x) be an equivariant embedding of the group $\mathbb{G} = (\mathbb{G}_m)^{n-1} \times \mathbb{G}_a$, where $n = \dim X$. In this section we assume that X is normal and affine. Let us denote the subgroup $(\mathbb{G}_m)^{n-1}$ of $\mathbb{G}$ by T. Since the action of T on X is effective, it has complexity one and defines an effective grading of the algebra $\mathbb{K}[X]$ by the lattice M. In particular, the graded algebra $\mathbb{K}[X]$ has the form $A[C,\mathfrak{D}]$ for some smooth curve C and some proper σ-polyhedral divisor on C, where σ is a cone in $N_{\mathbb{Q}}$.
Since the action of the subgroup $\mathbb{G}_a$ commutes with T-action on X, the corresponding homogeneous LND on $\mathbb{K}[X]$ has degree zero. Moreover, the group $\mathbb{G}$ acts on X with an open orbit. It implies that the $\mathbb{G}_a$-action on X is of horizontal type, and hence either $C = \mathbb{A}^1$ or $C = \mathbb{P}^1$.

Proposition 2. *Let $X = \operatorname{Spec} A[C,\mathfrak{D}]$ be a T-variety of complexity one. Suppose that there exists a homogeneous LND of horizontal type and of degree zero on $A[C,\mathfrak{D}]$. Then*

(1) *if $C = \mathbb{A}^1$, then one can assume (via Theorem 1) that $\mathfrak{D}$ is a trivial σ-polyhedral divisor;*
(2) *if $C = \mathbb{P}^1$, then one can choose $\mathfrak{D} = \Delta_\infty \cdot [\infty]$, where $\Delta_\infty \subsetneq \sigma$ is some σ-tailed polyhedron.*

Proof. Let $(\widetilde{\mathfrak{D}}, 0)$ be the coherent pair corresponding to the homogeneous LND of horizontal type. Without loss of generality we may assume that $z_0 = 0$ and $z_\infty = \infty$ if $C = \mathbb{P}^1$. By definition of a coherent pair, there exists $s \in \mathbb{Z}$ such that $(0,s)$ is a Demazure root of the cone $\widetilde{\sigma}$ with distinguished ray (dv_0, d). It implies that $s = -1$, $d = 1$, and hence $v_0 \in N$. Further, the inequality $\langle v, 0\rangle \geqslant 1 + \langle v_z, 0\rangle$ should be satisfied for every $z \in C'$ and every vertex $v \neq v_z$ of Δ_z. It means that each polyhedron Δ_z, where $z \in C'$, has only one vertex v_z. Replacing σ-polyhedral divisor $\mathfrak{D}$ with $\mathfrak{D}' = \mathfrak{D} + \sum_{z\in C'}(-v_z + \sigma)\cdot z$ and using Theorem 1, we obtain the assertion. The condition $\Delta_\infty \subsetneq \sigma$ follows from the fact that $\mathfrak{D}$ is a proper σ-polyhedral divisor. □

Corollary 1. *Under the conditions of Proposition 2 the variety X is toric with T being a subtorus of the acting torus $\mathbb{T}$.*

Proof. It follows immediately from Propositions 1 and 2. □

The next proposition is a specification of Corollary 1. In particular, it shows that the $\mathbb{G}_a$-action on X is normalized by the acting torus $\mathbb{T}$.

Proposition 3. *Under the conditions of Proposition 2,*

(1) *if $C = \mathbb{A}^1$, then $X \cong Y \times \mathbb{A}^1$, where Y is the toric variety corresponding to the cone σ and $\mathbb{G}_a$ acts on $\mathbb{A}^1$ by translations;*
(2) *if $C = \mathbb{P}^1$, then X is the toric variety with the cone $\widetilde{\sigma} \subset N \oplus \mathbb{Z}$ generated by $(\sigma, 0), (\Delta_\infty, -1)$ and $(0,1)$. The $\mathbb{G}_a$-action on X is given by Demazure root $\widetilde{e} = (0,-1) \in M \oplus \mathbb{Z}$ of the cone $\widetilde{\sigma}$.*

Proof. Let $\mathbb{K}(C) = \mathbb{K}(t)$. If $C = \mathbb{A}^1$ then $\mathfrak{D}$ is trivial and

$$A[C, \mathfrak{D}] = \bigoplus_{m \in \omega_M} \mathbb{K}[t] \cdot \chi^m = \mathbb{K}[\omega_M] \otimes \mathbb{K}[t] = \mathbb{K}[Y] \otimes \mathbb{K}[t].$$

Hence $X \cong Y \times \mathbb{A}^1$. Applying formula (1), we obtain that the homogeneous LND is given by

$$\partial(\chi^m \cdot t^r) = r\chi^m \cdot t^{r-1} \tag{2}$$

for all $m \in \omega_M$ and $r \in \mathbb{Z}_{\geqslant 0}$. Thus $\mathbb{G}_a$ acts on $Y \times \mathbb{A}^1$ as $(y,t) \mapsto (y, t+s)$.
If $C = \mathbb{P}^1$ then $\mathfrak{D} = \Delta_\infty \cdot [\infty]$ and we obtain

$$A[C, \mathfrak{D}] = \bigoplus_{m \in \omega_M} \bigoplus_{r=0}^{h_\infty(m)} \mathbb{K}\chi^m \cdot t^r = \bigoplus_{(m,r) \in \widetilde{\omega}_{\widetilde{M}}} \mathbb{K}\chi^m \cdot t^r = \mathbb{K}[\widetilde{\omega}_{\widetilde{M}}],$$

where $\widetilde{M} = M \oplus \mathbb{Z}$ and $\widetilde{\omega} \subset \widetilde{M}_{\mathbb{Q}}$ is the cone dual to $\widetilde{\sigma}$. So we see that $A[C, \mathfrak{D}]$ is a semigroup algebra and X is a toric variety with the cone $\widetilde{\sigma}$. In this case formula (2) gives the LND corresponding to the Demazure root $\widetilde{e} = (0, -1)$. □

4. The complete case

In this section we study equivariant compactifications of the group $\mathbb{G}$. First we briefly recall the main ingredients of the Cox construction, see [4, Chapter 1] for more details.

Let X be a normal variety with finitely generated divisor class group $\mathrm{Cl}(X)$ and only constant invertible regular functions.

Suppose that $\mathrm{Cl}(X)$ is free. Denote by $\mathrm{WDiv}(X)$ the group of Weil divisors on X and fix a subgroup $K \subseteq \mathrm{WDiv}(X)$ which maps onto $\mathrm{Cl}(X)$ isomorphically. The *Cox ring* of the variety X is defined as

$$R(X) = \bigoplus_{D \in K} H^0(X, D),$$

where $H^0(X, D) = \{f \in \mathbb{K}(X) \mid \operatorname{div} f + D \geqslant 0\}$ and multiplication on homogeneous components coincides with multiplication in $\mathbb{K}(X)$ and extends to $R(X)$ by linearity.

If $\mathrm{Cl}(X)$ has torsion, we choose a finitely generated subgroup $K \subseteq \mathrm{WDiv}(X)$ that projects to $\mathrm{Cl}(X)$ surjectively. Denote by $K_0 \subset K$ the kernel of this projection. Take compatible bases $D_1, \ldots, D_s$ and $D_1^0 = d_1 D_1, \ldots, D_r^0 = d_r D_r$ in K and K_0 respectively. Let us choose the set of rational functions $\mathcal{F} = \{F_D \in \mathbb{K}(X)^\times : D \in K_0\}$ such that $\mathrm{div}(F_D) = D$ and $F_{D+D'} = F_D F_{D'}$. Suppose that $D, D' \in K$ and $D - D' \in K_0$. The map $f \mapsto F_{D-D'} f$ is an isomorphism of the vector spaces $H^0(X, D)$ and $H^0(X, D')$. The linear span of the elements $f - F_{D-D'} f$ over all D, D' with $D - D' \in K_0$ and all $f \in H^0(X, D)$ is an ideal $I(K, \mathcal{F})$ of the graded ring $T_K(X) := \bigoplus_{D \in K} H^0(X, D)$. The Cox ring of the variety X is given by

$$R(X) = T_K(X)/I(K, \mathcal{F}).$$

This construction does not depend on the choice of K and $\mathcal{F}$, see [2, Lemma 3.1 and Proposition 3.2] and [4, Proposition 1.4.2.2].
Suppose that the Cox ring $R(X)$ is finitely generated. Then $\overline{X} := \operatorname{Spec} R(X)$ is a normal affine variety with an action of the quasitorus $H_X := \operatorname{Spec} \mathbb{K}[\operatorname{Cl}(X)]$. There is an open H_X-invariant subset $\widehat{X} \subseteq \overline{X}$ such that the complement $\overline{X} \backslash \widehat{X}$ is of codimension at least two in $\overline{X}$, there exists a good quotient $p_X \colon \widehat{X} \to \widehat{X}/\!/H_X$, and the quotient space $\widehat{X}/\!/H_X$ is isomorphic to X, see [4, Construction 1.6.3.1]. So we have the following diagram

$$\begin{array}{ccc} \widehat{X} & \xrightarrow{\ i\ } & \overline{X} = \operatorname{Spec} R(X) \\ \big\downarrow {\scriptstyle /\!/H_X} & & \\ X & & \end{array}$$

Let us return to equivariant compactifications of $\mathbb{G}$.

Proposition 4. *Let $\mathbb{G} = T \times \mathbb{G}_a$ and X be a normal compactification of $\mathbb{G}$. Then the T-action on X can be extended to an action of a bigger torus $\mathbb{T}$ such that $\mathbb{T}$ normalizes $\mathbb{G}_a$ and X is a toric variety with the acting torus $\mathbb{T}$.*

Proof. The variety X is rational with torus action of complexity one. By [4, Theorem 4.3.1.5], the divisor class group $\operatorname{Cl}(X)$ and the Cox ring $R(X)$ are finitely generated.
There exists a finite epimorphism $\epsilon \colon \mathbb{G}' \to \mathbb{G}$ of connected linear algebraic groups and an action $\mathbb{G}' \times \widehat{X} \to \widehat{X}$ which commutes with the quasitorus H_X and $p_X(g' \cdot \widehat{x}) = \epsilon(g') \cdot p_X(\widehat{x})$ for all $g' \in \mathbb{G}'$ and $\widehat{x} \in \widehat{X}$, see [4, Theorem 4.2.3.1]. The group $\mathbb{G}'$ has a form $T' \times \mathbb{G}_a$, where ϵ defines a finite epimorphism of tori $T' \to T$ and is identical on $\mathbb{G}_a$.
Since $\overline{X} = \operatorname{Spec} \mathbb{K}[\widehat{X}]$, the action of $\mathbb{G}'$ extends to the affine variety $\overline{X}$. This variety is an embedding of the group $(T' H_X^0) \times \mathbb{G}_a$. By Proposition 3, it is toric with an acting torus $\overline{\mathbb{T}}$ normalizing the $\mathbb{G}_a$-action and $T' H_X^0$ is a subtorus of $\overline{\mathbb{T}}$. Since X is complete, [25, Corollary 2.5] implies that the subset $\widehat{X}$ is invariant under the torus $\overline{\mathbb{T}}$. By [4, Lemma 4.2.1.3], the action of $\overline{\mathbb{T}}$ descends to an action of the torus $\mathbb{T} := \overline{\mathbb{T}}/H_X^0$ on X. Here $\mathbb{T}$ normalizes $\mathbb{G}_a$, its action extends the action of T on X, and X is toric with respect to $\mathbb{T}$. □

5. Toric varieties and Demazure roots

We keep notations of Section 2. Let X be a toric variety of dimension n with an acting torus $\mathbb{T}$ and Σ be the corresponding fan of convex polyhedral cones in the space $N_{\mathbb{Q}}$, see [18] or [12] for details.
As before, let $\Sigma(1)$ be the set of rays of the fan Σ and n_ρ be the primitive lattice vector on the ray ρ. For $\rho \in \Sigma(1)$ we consider the set S_ρ of all vectors $e \in M$ such that

(1) $\langle n_\rho, e\rangle = -1$ and $\langle n_{\rho'}, e\rangle \geqslant 0 \ \ \forall \rho' \in \sigma(1),\ \rho' \neq \rho$;
(2) if σ is a cone of Σ and $\langle v, e\rangle = 0$ for all $v \in \sigma$, then the cone generated by σ and ρ is in Σ as well.

Note that condition (1) implies condition (2) if Σ is a maximal fan with support $|\Sigma|$. This is the case if X is affine or complete.
The elements of the set $\mathfrak{R} := \bigsqcup_{\rho} S_\rho$ are called the *Demazure roots* of the fan Σ, cf. [13, Définition 4] and [23, Section 3.4]. Again elements $e \in \mathfrak{R}$ are in bijection with $\mathbb{G}_a$-actions on X normalized by the acting torus, see [13, Théoreme 3] and [23, Proposition 3.14]. If X is affine, the $\mathbb{G}_a$-action given by a Demazure root e coincides with the action given by the locally nilpotent derivation ∂_e of the algebra $\mathbb{K}[X]$ as defined in Section 2. Let us denote the image under this action of the group $\mathbb{G}_a$ in $\mathrm{Aut}(X)$ by H_e. Thus H_e is a one-parameter unipotent subgroup normalized by $\mathbb{T}$ in $\mathrm{Aut}(X)$.
We recall basic facts from toric geometry. There is a bijection between cones $\sigma \in \Sigma$ and $\mathbb{T}$-orbits $\mathcal{O}_\sigma$ on X such that $\sigma_1 \subseteq \sigma_2$ if and only if $\mathcal{O}_{\sigma_2} \subseteq \overline{\mathcal{O}_{\sigma_1}}$. Here $\dim \mathcal{O}_\sigma = n - \dim\langle\sigma\rangle$. Moreover, each cone $\sigma \in \Sigma$ defines an open affine $\mathbb{T}$-invariant subset U_σ on X such that $\mathcal{O}_\sigma$ is a unique closed $\mathbb{T}$-orbit on U_σ and $\sigma_1 \subseteq \sigma_2$ if and only if $U_{\sigma_1} \subseteq U_{\sigma_2}$.
Let ρ_e be the distinguished ray corresponding to a root e, n_e be the primitive lattice vector on ρ_e, and R_e be the one-parameter subgroup of $\mathbb{T}$ corresponding to n_e.
Our aim is to describe the action of H_e on X.

Proposition 5. *For every point $x \in X \setminus X^{H_e}$ the orbit $H_e x$ meets exactly two $\mathbb{T}$-orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ on X, where $\dim \mathcal{O}_1 = \dim \mathcal{O}_2 + 1$. The intersection $\mathcal{O}_2 \cap H_e x$ consists of a single point, while*

$$\mathcal{O}_1 \cap H_e x = R_e y \quad \textit{for any} \quad y \in \mathcal{O}_1 \cap H_e x.$$

Proof. It follows from the proof of [23, Proposition 3.14] that the affine charts U_σ, where $\sigma \in \Sigma$ is a cone containing ρ_e, are H_e-invariant, and the complement of their union is contained in X^{H_e}, cf. [9, Lemma 2.4]. This reduces the proof to the case X is affine. Then the assertion is proved in [6, Proposition 2.1]. □

A pair of $\mathbb{T}$-orbits $(\mathcal{O}_1, \mathcal{O}_2)$ on X is said to be *H_e-connected* if $H_e x \subseteq \mathcal{O}_1 \cup \mathcal{O}_2$ for some $x \in X \setminus X^{H_e}$. By Proposition 5, $\mathcal{O}_2 \subseteq \overline{\mathcal{O}_1}$ for such a pair (up to permutation) and $\dim \mathcal{O}_1 = \dim \mathcal{O}_2 + 1$. Since the torus normalizes the subgroup H_e, any point of $\mathcal{O}_1 \cup \mathcal{O}_2$ can actually serve as a point x.

Lemma 1. *A pair of $\mathbb{T}$-orbits $(\mathcal{O}_{\sigma_1}, \mathcal{O}_{\sigma_2})$ is H_e-connected if and only if $e|_{\sigma_2} \le 0$ and σ_1 is a facet of σ_2 given by the equation $\langle v, e\rangle = 0$.*

Proof. The proof again reduces to the affine case, where the assertion is [6, Lemma 2.2]. □

6. The orbit structure

We keep notations of the previous section. Let us begin with a construction mentioned in the Introduction. Let X be a toric variety with the acting torus $\mathbb{T}$. Consider a non-trivial action $\mathbb{G}_a \times X \to X$ normalized by $\mathbb{T}$ and thus represented by a Demazure root e of the fan Σ of X. Then $\mathbb{T}$ acts on $\mathbb{G}_a$ by conjugation with the character e and the semidirect product $\mathbb{T} \ltimes \mathbb{G}_a$ acts on X as well. Let $T = \operatorname{Ker}(e) \subseteq \mathbb{T}$ and consider the group $\mathbb{G} := T \times \mathbb{G}_a$.

Proposition 6. *The variety X is an embedding of $\mathbb{G}$.*

Proof. Take a point $x \in X$ whose stabilizers in $\mathbb{T}$ and $\mathbb{G}_a$ are trivial. It suffices to show that the stabilizer of x in $\mathbb{G}$ is trivial. To this end, note that by the Jordan decomposition [20, Theorem 15.3] any subgroup of $T \times \mathbb{G}_a$ is a product of subgroups in T and $\mathbb{G}_a$ respectively. □

Remark 1. The $\mathbb{G}$-embedding of Proposition 6 is defined by the pair (Σ, e).

Since $\langle n_e, e\rangle = -1$, we have $\mathbb{T} = T \times R_e$.

Lemma 2. *Any $(T \times \mathbb{G}_a)$-invariant subset in X is also $\mathbb{T}$-invariant.*

Proof. Note that an orbit $\mathbb{T}x$ does not coincide with the orbit Tx if and only if the stabilizer of x in $\mathbb{T}$ is contained in T. For $x \in \mathcal{O}_\sigma$ this condition is equivalent to $e|_\sigma = 0$. It shows that for every $x \in X^{\mathbb{G}_a}$ we have $\mathbb{T}x = Tx$. If $x \in X \setminus X^{\mathbb{G}_a}$, then by Proposition 5 the orbit $\mathbb{G}_a x$ is invariant under R_e. This proves that any orbit of $(T \times \mathbb{G}_a)$ is R_e- and T-invariant, thus the assertion. □

Proposition 7. *Let X be a $\mathbb{G}$-embedding given by a pair (Σ, e). Then any $\mathbb{G}$-orbit on X is either a union $\mathcal{O}_1 \cup \mathcal{O}_2$ of two $\mathbb{T}$-orbits on X or a unique $\mathbb{T}$-orbit; the first possibility occurs if and only if the pair $(\mathcal{O}_1, \mathcal{O}_2)$ is H_e-connected. In particular, the number of $\mathbb{G}$-orbits on X is finite.*

Proof. The assertion follows directly from Lemma 2 and Proposition 5. □

Proposition 8. *Let X be a $\mathbb{G}$-embedding given by a pair (Σ, e). Then the stabilizer of any point $x \in X$ in $\mathbb{G}$ is connected and the closure of any $\mathbb{G}$-orbit on X is a (normal) toric variety. If X is smooth, then the closure of any $\mathbb{G}$-orbit is smooth.*

Proof. The stabilizer of a point x in $\mathbb{G}$ is the direct product of stabilizers in T and in $\mathbb{G}_a$. An algebraic subgroup of $\mathbb{G}_a$ is either $\{0\}$ or $\mathbb{G}_a$ itself, while the stabilizer in T is the kernel of the (primitive) character e restricted to the (connected) stabilizer of x in $\mathbb{T}$. Thus the stabilizer of x in $\mathbb{G}$ is connected.
Proposition 7 shows that any $\mathbb{G}$-orbit on X contains an open $\mathbb{T}$-orbit, and thus the closure of a $\mathbb{G}$-orbit coincides with the closure of some $\mathbb{T}$-orbit. Now the last two assertions follow from [18, Section 3.1]. □

Remark 2. If X contains l torus invariant prime divisors, then the number of $\mathbb{G}$-invariant prime divisors on X is $l - 1$. On a toric variety, the closure of any torus orbit is an intersection of torus invariant prime divisors. In contrast, not every $\mathbb{G}$-orbit closure on X is an intersection of $\mathbb{G}$-invariant prime divisors, see Example 3.

7. The general case

We are going to show that every $\mathbb{G}$-embedding can be realized as in Proposition 6.

Theorem 2. *Let $\mathbb{G} = T \times \mathbb{G}_a$ and X be a normal equivariant $\mathbb{G}$-embedding. Then the T-action on X can be extended to an action of a bigger torus $\mathbb{T}$ such that $\mathbb{T}$ normalizes $\mathbb{G}_a$ and X is a toric variety with the acting torus $\mathbb{T}$. In particular, every $\mathbb{G}$-embedding comes from a pair (Σ, e), where Σ is a fan and e is a Demazure root of Σ.*

Proof. We begin with a classical result of Sumihiro. Let X be a normal variety with a regular action $G \times X \to X$ of a linear algebraic group G. By [24, Theorem 3], there exists a normal complete G-variety $\mathbf{X}$ such that X can be embedded equivariantly as an open subset of $\mathbf{X}$. In other words, $\mathbf{X}$ is an equivariant compactification of X.
Let X be a normal embedding of $\mathbb{G}$ and $\mathbf{X}$ be an equivariant compactification of X. By Proposition 4, the T-action on $\mathbf{X}$ can be extended to an action of a bigger torus $\mathbb{T}$ such that $\mathbb{T}$ normalizes $\mathbb{G}_a$ and $\mathbf{X}$ is a toric variety with the acting torus $\mathbb{T}$. Since the subset $X \subseteq \mathbf{X}$ is $(T \times \mathbb{G}_a)$-invariant, it is invariant under $\mathbb{T}$, see Lemma 2. This provides the desired structure of a toric variety on X. □

Proposition 9. *A complete toric variety X admits a structure of a $\mathbb{G}$-embedding if and only if $\mathrm{Aut}(X)^0 \neq \mathbb{T}$.*

Proof. The variety X admits a structure of a $\mathbb{G}$-embedding if and only if $\mathrm{Aut}(X)^0$ contains at least one root subgroup. It is well known that the group $\mathrm{Aut}(X)^0$ is generated by $\mathbb{T}$ and root subgroups [13, Proposition 11], [23, Section 3.4], [11, Corollary 4.7]. □

Consider two structures of a $\mathbb{G}$-embedding on a variety X. We say that such structures are *equivalent*, if there is an automorphism of X sending one structure to the other. Since the structure of a toric variety on X is unique up to automorphism, we may assume that our two structures share the same acting torus $\mathbb{T}$ and the same fan Σ, and are given by two roots e, e' of Σ. Then the structures are equivalent if and only if e can be sent to e' by an automorphism of the torus $\mathbb{T}$. This leads to the following result.

Proposition 10. *Two structures of a $\mathbb{G}$-embedding given by pairs (Σ, e) and (Σ, e') are equivalent if and only if there is an automorphism ϕ of the lattice N which preserves the fan Σ and such that the induced automorphism ϕ^* of the dual lattice M sends e to e'.*

Let us finish with explicit examples of $\mathbb{G}$-embeddings into a given variety.

Example 2. We find all structures of $\mathbb{G}$-embeddings on $\mathbb{A}^2$. The cone of $\mathbb{A}^2$ as a toric variety is $\mathbb{Q}^2_{\geqslant 0}$. The set of Demazure roots of $\mathbb{Q}^2_{\geqslant 0}$ is

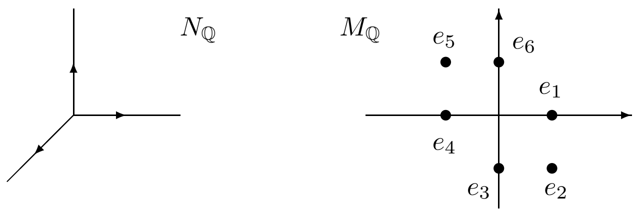
$$\mathfrak{R} = \{(-1, k) \mid k \in \mathbb{Z}_{\geqslant 0}\} \sqcup \{(k, -1) \mid k \in \mathbb{Z}_{\geqslant 0}\},$$

see Example 1. The $\mathbb{G}$-action on $\mathbb{A}^2$ corresponding to the root $(-1,k)$ is given by

$$(t,s)\circ(x_1,x_2)=(t^kx_1+st^kx_2^k,tx_2), \tag{3}$$

where $(x_1,x_2)\in\mathbb{A}^2$, $s\in\mathbb{G}_a$, and $t\in\mathbb{K}^\times$. If $k\neq 0$, then there is a line of $\mathbb{G}_a$-fixed points and the stabilizer of a non-zero point on this line is a cyclic group of order k. If $k=0$, then there is no $\mathbb{G}_a$-fixed point. So formula (3) gives non-equivalent $\mathbb{G}$-actions for different k. With $k\neq 0$ we have three $\mathbb{G}$-orbits on $\mathbb{A}^2$, while for $k=0$ there are two $\mathbb{G}$-orbits.
Note that $\mathbb{G}$-actions defined by the roots $(k,-1)$ and $(-1,k)$ are equivalent via the automorphism $x_1\leftrightarrow x_2$ of $\mathbb{A}^2$.

Example 3. Let $X=\mathbb{P}^2$. It is a complete toric variety with a fan Σ generated by the vectors $(1,0)$, $(0,1)$ and $(-1,-1)$:



The set of Demazure roots is

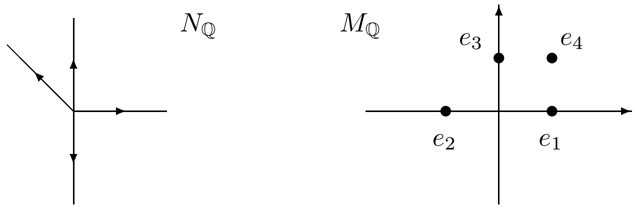
$$\mathfrak{R}=\{e_1=(1,0),e_2=(1,-1),e_3=(0,-1),$$
$$e_4=(-1,0),e_5=(-1,1),e_6=(0,1)\}.$$

We see that for any i and j there exists isomorphism of the fan Σ sending e_i to e_j. So any $\mathbb{G}$-embedding into $\mathbb{P}^2$ is equivalent to

$$(t,s)\circ[z_0:z_1:z_2]=[tz_0+stz_1:tz_1:z_2].$$

This time seven $\mathbb{T}$-orbits glue to five $\mathbb{G}$-orbits.

Example 4. Consider the Hirzebruch surface $\mathbb{F}_1$. The corresponding complete fan Σ is generated by the vectors $(1,0)$, $(0,1)$, $(0,-1)$, and $(-1,1)$:



The set of Demazure roots is

$$\mathfrak{R}=\{e_1=(1,0),e_2=(-1,0),e_3=(0,1),e_4=(1,1)\}.$$

By an automorphism, we can send e_1 to e_2 and e_3 to e_4. For the first equivalence class we have six $\mathbb{G}$-orbits, while in the second one the number of $\mathbb{G}$-orbits is seven.

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