

p -ADIC L-FUNCTIONS OF AUTOMORPHIC FORMS
AND EXCEPTIONAL ZEROS

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ABSTRACT. We construct p -adic L-functions for automorphic representations of GL_2 of a number field F , and show that the corresponding p -adic L-function of a modular elliptic curve E over F has an extra zero at the central point for each prime above p at which E has split multiplicative reduction, a part of the exceptional zero conjecture.

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INTRODUCTION

Let F be a number field (with adèle ring \mathbb{A}_F), and p a prime number. Let $\pi = \bigotimes_v \pi_v$ be an automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$. Attached to π is the complex L-function $L(s, \pi)$, $s \in \mathbb{C}$, of Jacquet-Langlands [JL70]. Under certain conditions on π , we can also define a p -adic L-function $L_p(s, \pi)$ of π , with $s \in \mathbb{Z}_p$. It is related to $L(s, \pi)$ by the *interpolation property*: For every character $\chi : \mathcal{G}_p \rightarrow \mathbb{C}^*$ of finite order we have

$$L_p(0, \pi \otimes \chi) = \tau(\chi) \prod_{\mathfrak{p}|p} e(\pi_{\mathfrak{p}}, \chi_{\mathfrak{p}}) \cdot L\left(\frac{1}{2}, \pi \otimes \chi\right),$$

where $e(\pi_{\mathfrak{p}}, \chi_{\mathfrak{p}})$ is a certain Euler factor (see theorem 4.12 for its definition) and $\tau(\chi)$ is the Gauss sum of χ .

$L_p(s, \pi)$ was defined by Haran [Har87] in the case where π has trivial central character and $\pi_{\mathfrak{p}}$ is an ordinary spherical principal series representation for all $\mathfrak{p}|p$. For a totally real field F , Spieß [Sp14] has given a new construction of $L_p(s, \pi)$ that also allows for $\pi_{\mathfrak{p}}$ to be a special (Steinberg) representation for some $\mathfrak{p}|p$. In this article, we generalize Spieß' construction of $L_p(s, \pi)$ to

automorphic representations π of GL_2 over any number field, with arbitrary central character, and show that L_p has the conjectured number of exceptional zeros at the central point. We assume that π is ordinary at all primes $\mathfrak{p}|p$ (cf. definition 2.3), that π_v is discrete of weight 2 at all real infinite places v , and is the principal series representation $\sigma(|\cdot|^{1/2}, |\cdot|^{-1/2})$ at the complex places.

We define a p -adic measure μ_π , which heuristically is the image under the global reciprocity map of a product of certain local distributions $\mu_{\pi_{\mathfrak{p}}}$ on $F_{\mathfrak{p}}^*$ attached to $\pi_{\mathfrak{p}}$ for $\mathfrak{p}|p$ and a Whittaker function times the Haar measure on the group of p -ideles $\mathbb{I}^p = \prod'_{v|p} F_v^*$.

Then we can define the p -adic L-function of π as an integral with respect to μ_π over the Galois group \mathcal{G}_p of the maximal abelian extension that is unramified outside p and ∞ ; it is naturally a t -variable function, where t is the \mathbb{Z}_p -rank of \mathcal{G}_p :

$$L_p(\underline{s}, \pi) := L_p(s_1, \dots, s_t, \pi) := \int_{\mathcal{G}_p} \prod_{i=1}^t \exp_p(s_i \ell_i(\gamma)) \mu_\pi(d\gamma)$$

for $s_1, \dots, s_t \in \mathbb{Z}_p$, where the ℓ_i are \mathbb{Z}_p -valued homomorphisms corresponding to the t independent \mathbb{Z}_p -extensions of F (cf. section 4.7 for their definition).

For a modular elliptic curve E over F corresponding to π (i.e. the local L-factors of the Hasse-Weil L-function $L(E, s)$ and of the automorphic L-function $L(s - \frac{1}{2}, \pi)$ coincide at all places v of F), our construction allows us to define the p -adic L-function of E as $L_p(E, \underline{s}) := L_p(\underline{s}, \pi)$. The condition that π be ordinary at all $\mathfrak{p}|p$ means that E must have good ordinary or multiplicative reduction at all places $\mathfrak{p}|p$ of F .

The *exceptional zero conjecture* (formulated by Mazur, Tate and Teitelbaum [MTT86] for $F = \mathbb{Q}$, and by Hida [Hi09] for totally real F) states that

$$\mathrm{ord}_{s=0} L_p(E, s) \geq n, \tag{1}$$

where n is the number of $\mathfrak{p}|p$ at which E has split multiplicative reduction, and gives an explicit formula for the value of the n -th derivative $L_p^{(n)}(E, 0)$ as a multiple of certain L-invariants times $L(E, 1)$. The conjecture was proved in the case $F = \mathbb{Q}$ by Greenberg and Stevens [GS93] and independently by Kato, Kurihara and Tsuji, and for totally real fields F by Spieß [Sp14]. In this article, we prove (1) for all number fields F .

The structure of this article is as follows: In chapter 2, we describe the local distributions $\mu_{\pi_{\mathfrak{p}}}$ on $F_{\mathfrak{p}}^*$; they are the image of a Whittaker functional under a map δ on the dual of $\pi_{\mathfrak{p}}$. For constructing δ , we describe $\pi_{\mathfrak{p}}$ in terms of what we call the “Bruhat-Tits graph” of $F_{\mathfrak{p}}^2$: the directed graph whose vertices (resp. edges) are the lattices of $F_{\mathfrak{p}}^2$ (resp. inclusions between lattices). Roughly speaking, it is a covering of the (directed) Bruhat-Tits tree of $\mathrm{GL}_2(F_{\mathfrak{p}})$ with fibres $\cong \mathbb{Z}$. When $\pi_{\mathfrak{p}}$ is the Steinberg representation, $\mu_{\mathfrak{p}}$ can actually be extended to all of $F_{\mathfrak{p}}$.

In chapter 3, we attach a p -adic distribution μ_ϕ to any map $\phi(U, x^p)$ of an open compact subset $U \subseteq F_p^* := \prod_{\mathfrak{p}|p} F_{\mathfrak{p}}^*$ and an idele $x^p \in \mathbb{I}^p$ (satisfying certain

conditions). Integrating ϕ over all the infinite places, we get a cohomology class $\kappa_\phi \in H^d(F^{*f}, \mathcal{D}_f(\mathbb{C}))$ (where $d = r + s - 1$ is the rank of the group of units of F , $F^{*f} \cong F^*/\mu_F$ is a maximal torsion-free subgroup of F^* , and $\mathcal{D}_f(\mathbb{C})$ is a space of distributions on the finite ideles of F). We show that μ_ϕ can be described solely in terms of κ_ϕ , and μ_ϕ is a (vector-valued) p -adic measure if κ_ϕ is “integral”, i.e. if it lies in the image of $H^d(F^{*f}, \mathcal{D}_f(R))$, for a Dedekind ring R consisting of “ p -adic integers”.

In chapter 4, we define a map ϕ_π by

$$\phi_\pi(U, x^p) := \sum_{\zeta \in F^*} \mu_{\pi_p}(\zeta U) W^p \begin{pmatrix} \zeta x^p & 0 \\ 0 & 1 \end{pmatrix}$$

($U \subseteq F_p^*$ compact open, $x^p \in \mathbb{I}^p$). ϕ_π satisfies the conditions of chapter 3, and we show that $\kappa_\pi := \kappa_{\phi_\pi}$ is integral by “lifting” the map $\phi_\pi \mapsto \kappa_\pi$ to a function mapping an automorphic form to a cohomology class in $H^d(\mathrm{GL}_2(F)^+, \mathcal{A}_f)$, for a certain space of functions \mathcal{A}_f . (Here $\mathrm{GL}_2(F)^+$ is the subgroup of $M \in \mathrm{GL}_2(F)$ with totally positive determinant.) For this, we associate to each automorphic form φ a harmonic form ω_φ on a generalized upper-half space \mathcal{H}_∞ , which we can integrate between any two cusps in $\mathbb{P}^1(F)$.

Then we can define the p -adic L-function $L_p(\underline{s}, \pi) := L_p(\underline{s}, \kappa_\pi)$ as above, with $\mu_\pi := \mu_{\phi_\pi}$. By a result of Harder [Ha87], $H^d(\mathrm{GL}_2(F)^+, \mathcal{A}_f)_\pi$ is one-dimensional, which implies that $L_p(\underline{s}, \pi)$ has values in a one-dimensional \mathbb{C}_p -vector space. Finally, we formulate an exceptional zero conjecture (conjecture 4.15) for all number fields F , and show that $L_p(\underline{s}, \pi)$ satisfies (1).

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1 PRELIMINARIES

Let \mathcal{X} be a totally disconnected locally compact topological space, R a topological Hausdorff ring. We denote by $C(\mathcal{X}, R)$ the ring of continuous maps $\mathcal{X} \rightarrow R$, and let $C_c(\mathcal{X}, R) \subseteq C(\mathcal{X}, R)$ be the subring of compactly supported maps. When R has the discrete topology, we also write $C^0(\mathcal{X}, R) := C(\mathcal{X}, R)$, $C_c^0(\mathcal{X}, R) := C_c(\mathcal{X}, R)$.

We denote by $\mathfrak{Co}(\mathcal{X})$ the set of all compact open subsets of \mathcal{X} , and for an R -module M we denote by $\mathrm{Dist}(\mathcal{X}, M)$ the R -module of M -valued distributions on \mathcal{X} , i.e. the set of maps $\mu : \mathfrak{Co}(\mathcal{X}) \rightarrow M$ such that $\mu(\bigcup_{i=1}^n U_i) = \sum_{i=1}^n \mu(U_i)$ for any pairwise disjoint sets $U_i \in \mathfrak{Co}(\mathcal{X})$.

For an open set $H \subseteq \mathcal{X}$, we let $1_H \in C(\mathcal{X}, R)$ be the R -valued indicator function of H on \mathcal{X} .

Throughout this paper, we fix a prime p and embeddings $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$. Let \mathcal{O} denote the valuation ring of $\overline{\mathbb{Q}}$ with respect to the p -adic valuation induced by ι_p .

We write $G := \mathrm{GL}_2$ throughout the article, and let B denote the Borel subgroup of upper triangular matrices, T the maximal torus (consisting of all diagonal matrices), and Z the center of G .

For a number field F , we let $G(F)^+ \subseteq G(F)$ and $B(F)^+ \subseteq B(F)$ denote the corresponding subgroups of matrices with totally positive determinant, i.e. $\sigma(\det(g))$ is positive for each real embedding $\sigma : F \hookrightarrow \mathbb{R}$. (If F is totally complex, this is an empty condition, so we have $G(F)^+ = G(F)$, $B(F)^+ = B(F)$ in this case.) Similarly, we define $G(\mathbb{R})^+$ and $G(\mathbb{C})^+ = G(\mathbb{C})$.

1.1 p -ADIC MEASURES

DEFINITION 1.1. Let \mathcal{X} be a compact totally disconnected topological space. For a distribution $\mu : \mathfrak{C}\mathfrak{o}(\mathcal{X}) \rightarrow \mathbb{C}$, consider the extension of μ to the \mathbb{C}_p -linear map $C^0(\mathcal{X}, \mathbb{C}_p) \rightarrow \mathbb{C}_p \otimes_{\mathbb{Q}} \mathbb{C}$, $f \mapsto \int f d\mu$. If its image is a finitely-generated \mathbb{C}_p -vector space, μ is called a p -adic measure.

We denote the space of p -adic measures on \mathcal{X} by $\mathrm{Dist}^b(\mathcal{X}, \mathbb{C}) \subseteq \mathrm{Dist}(\mathcal{X}, \mathbb{C})$. It is easily seen that μ is a p -adic measure if and only if the image of μ , considered as a map $C^0(\mathcal{X}, \mathbb{Z}) \rightarrow \mathbb{C}$, is contained in a finitely generated $\overline{\mathcal{O}}$ -module. A p -adic measure can be integrated against any continuous function $f \in C(\mathcal{X}, \mathbb{C}_p)$.

2 LOCAL RESULTS

For this chapter, let F be a finite extension of \mathbb{Q}_p , \mathcal{O}_F its ring of integers, ϖ its uniformizer and $\mathfrak{p} = (\varpi)$ the maximal ideal. Let q be the cardinality of $\mathcal{O}_F/\mathfrak{p}$, and set $U := U^{(0)} := \mathcal{O}_F^\times$, $U^{(n)} := 1 + \mathfrak{p}^n \subseteq U$ for $n \geq 1$.

We fix an additive character $\psi : F \rightarrow \overline{\mathbb{Q}}^*$ with $\ker \psi \supseteq \mathcal{O}_F$ and $\mathfrak{p}^{-1} \not\subseteq \ker \psi$.¹ We let $|\cdot|$ be the absolute value on F^* (normalized by $|\varpi| = q^{-1}$), $\mathrm{ord} = \mathrm{ord}_\varpi$ the additive valuation, and dx the Haar measure on F normalized by $\int_{\mathcal{O}_F} dx = 1$. We define a (Haar) measure on F^* by $d^\times x := \frac{q}{q-1} \frac{dx}{|x|}$ (so $\int_{\mathcal{O}_F^\times} d^\times x = 1$).

2.1 GAUSS SUMS

Recall that the *conductor* of a character $\chi : F^* \rightarrow \mathbb{C}^*$ is by definition the largest ideal \mathfrak{p}^n , $n \geq 0$, such that $\ker \chi \supseteq U^{(n)}$, and that χ is *unramified* if its conductor is $\mathfrak{p}^0 = \mathcal{O}_F$.

DEFINITION 2.1. Let $\chi : F^* \rightarrow \mathbb{C}^*$ be a quasi-character with conductor \mathfrak{p}^f . The *Gauss sum* of χ (with respect to ψ) is defined by

$$\tau(\chi) := [U : U^{(f)}] \int_{\varpi^{-f}U} \psi(x)\chi(x)d^\times x.$$

¹Note that there is in general no ψ such that $\ker(\psi) = \mathcal{O}_F$, since $\mathfrak{p}^{-1}/\mathcal{O}_F$ has more than p points of order p if $F|\mathbb{Q}_p$ has inertia index > 1 .

For a locally constant function $g : F^* \rightarrow \mathbb{C}$, we define

$$\int_{F^*} g(x)dx := \lim_{n \rightarrow \infty} \int_{x \in F^*, -n \leq \text{ord}(x) \leq n} g(x)dx,$$

whenever that limit exists.

LEMMA 2.2. *Let $\chi : F^* \rightarrow \mathbb{C}^*$ be a quasi-character with conductor \mathfrak{p}^f . For $f = 0$, assume $|\chi(\varpi)| < q$. Then we have*

$$\int_{F^*} \chi(x)\psi(x)dx = \begin{cases} \frac{1-\chi(\varpi)^{-1}}{1-\chi(\varpi)^{-1}q^{-1}} & \text{if } f = 0 \\ \tau(\chi) & \text{if } f > 0. \end{cases}$$

(Cf. [Sp14], lemma 3.4.)

2.2 TAMELY RAMIFIED REPRESENTATIONS OF $GL_2(F)$

For an ideal $\mathfrak{a} \subset \mathcal{O}_F$, let $K_0(\mathfrak{a}) \subseteq G(\mathcal{O}_F)$ be the subgroup of matrices congruent to an upper triangular matrix modulo \mathfrak{a} .

Let $\pi : GL_2(F) \rightarrow GL(V)$ be an irreducible admissible infinite-dimensional representation on a \mathbb{C} -vector space V , with central quasicharacter χ . It is well-known (e.g [Ge75], Thm. 4.24) that there exists a maximal ideal $\mathfrak{c}(\pi) = \mathfrak{c} \subset \mathcal{O}_F$, the conductor of π , such that the space $V^{K_0(\mathfrak{c}) \cdot \chi} = \{v \in V | \pi(g)v = \chi(a)v \ \forall g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(\mathfrak{c})\}$ is non-zero (and in fact one-dimensional). A representation π is called *tamely ramified* if its conductor divides \mathfrak{p} .

If π is tamely ramified, then π is the spherical resp. special representation $\pi(\chi_1, \chi_2)$ (in the notation of [Ge75] or [Sp14]):

If the conductor is \mathcal{O}_F , π is (by definition) spherical and thus a principal series representation $\pi(\chi_1, \chi_2)$ for two unramified quasi-characters χ_1 and χ_2 with $\chi_1\chi_2^{-1} \neq |\cdot|^{\pm 1}$ ([Bu98], Thm. 4.6.4).

If the conductor is \mathfrak{p} , then $\pi = \pi(\chi_1, \chi_2)$ with $\chi_1\chi_2^{-1} = |\cdot|^{\pm 1}$.

For $\alpha \in \mathbb{C}^*$, we define a character $\chi_\alpha : F^* \rightarrow \mathbb{C}^*$ by $\chi_\alpha(x) := \alpha^{\text{ord}(x)}$.

So let now $\pi = \pi(\chi_1, \chi_2)$ be a tamely ramified irreducible admissible infinite-dimensional representation of $GL_2(F)$; in the special case, we assume χ_1 and χ_2 to be ordered such that $\chi_1 = |\cdot| \chi_2$.

Set $\alpha_i := \chi_i(\varpi)\sqrt{q} \in \mathbb{C}^*$ for $i = 1, 2$. (We also write $\pi = \pi_{\alpha_1, \alpha_2}$ sometimes.)

Set $a := \alpha_1 + \alpha_2$, $\nu := \alpha_1\alpha_2/q$. Define a distribution $\mu_{\alpha_1, \nu} := \mu_{\alpha_1/\nu} := \psi(x)\chi_{\alpha_1/\nu}(x)dx$ on F^* .

For later use, we will need the following condition on the α_i :

DEFINITION 2.3. Let $\pi = \pi_{\alpha_1, \alpha_2}$ be tamely ramified. π is called *ordinary* if a and ν both lie in $\overline{\mathcal{O}}^*$ (i.e. they are p -adic units in $\overline{\mathbb{Q}}$). Equivalently, this means that either $\alpha_1 \in \overline{\mathcal{O}}^*$ and $\alpha_2 \in q\overline{\mathcal{O}}^*$, or vice versa.

PROPOSITION 2.4. *Let $\chi : F^* \rightarrow \mathbb{C}^*$ be a quasi-character with conductor \mathfrak{p}^f ; for $f = 0$, assume $|\chi(\varpi)| < |\alpha_2|$. Then the integral $\int_{F^*} \chi(x)\mu_{\alpha_1/\nu}(dx)$ converges*

and we have

$$\int_{F^*} \chi(x) \mu_{\alpha_1/\nu}(dx) = e(\alpha_1, \alpha_2, \chi) \tau(\chi) L(\frac{1}{2}, \pi \otimes \chi),$$

where

$$e(\alpha_1, \alpha_2, \chi) = \begin{cases} \frac{(1-\alpha_1\chi(\varpi)q^{-1})(1-\alpha_2\chi(\varpi)^{-1}q^{-1})(1-\alpha_2\chi(\varpi)q^{-1})}{(1-\chi(\varpi)\alpha_2^{-1})}, & f = 0 \text{ and } \pi \text{ spherical,} \\ \frac{(1-\alpha_1\chi(\varpi)q^{-1})(1-\alpha_2\chi(\varpi)^{-1}q^{-1})}{(1-\chi(\varpi)\alpha_2^{-1})}, & f = 0 \text{ and } \pi \text{ special,} \\ \left(\frac{\alpha_1}{\nu}\right)^{-f} = \left(\frac{\alpha_2}{q}\right)^f, & f > 0, \end{cases}$$

and where we assume the right-hand side to be continuously extended to the potential removable singularities at $\chi(\varpi) = q/\alpha_1$ or $= q/\alpha_2$.

Proof. This follows immediately from lemma 2.2 and the definition of the (Jacquet-Langlands) L-function. \square

2.3 THE BRUHAT-TITS GRAPH

Let $\tilde{\mathcal{V}}$ denote the set of lattices (i.e. submodules isomorphic to \mathcal{O}_F^2) in F^2 , and let $\tilde{\mathcal{E}}$ be the set of all inclusion maps between two lattices; for such a map $e : v_1 \hookrightarrow v_2$ in $\tilde{\mathcal{E}}$, we define $o(e) := v_1, t(e) := v_2$. Then the pair $\tilde{\mathcal{T}} := (\tilde{\mathcal{V}}, \tilde{\mathcal{E}})$ is naturally a directed graph, connected, with no directed cycles (specifically, $\tilde{\mathcal{E}}$ induces a partial ordering on $\tilde{\mathcal{V}}$). For each $v \in \tilde{\mathcal{V}}$, there are exactly $q + 1$ edges beginning (resp. ending) in v , each.

Recall that the *Bruhat-Tits tree* $\mathcal{T} = (\mathcal{V}, \vec{\mathcal{E}})$ of $G(F)$ is the directed graph whose vertices are homothety classes of lattices of F^2 (i.e. $\mathcal{V} = \tilde{\mathcal{V}}/\sim$, where $v \sim \varpi^i v$ for all $i \in \mathbb{Z}$), and the directed edges $\vec{e} \in \vec{\mathcal{E}}$ are homothety classes of inclusions of lattices. We can define maps $o, t : \vec{\mathcal{E}} \rightarrow \mathcal{V}$ analogously. For each edge $\vec{e} \in \vec{\mathcal{E}}$, there is an opposite edge $\vec{e}' \in \vec{\mathcal{E}}$ with $o(\vec{e}') = t(\vec{e}), t(\vec{e}') = o(\vec{e})$; and the undirected graph underlying \mathcal{T} is simply connected. We have a natural “projection map” $\pi : \tilde{\mathcal{T}} \rightarrow \mathcal{T}$, mapping each lattice and each homomorphism to its homothety class. Choosing a (set-theoretic) section $s : \mathcal{V} \rightarrow \tilde{\mathcal{V}}$, we get a bijection $\mathcal{V} \times \mathbb{Z} \xrightarrow{\cong} \tilde{\mathcal{V}}$ via $(v, i) \mapsto \varpi^i s(v)$.

The group $G(F)$ operates on \mathcal{V} via its standard action on F^2 , i.e. $gv = \{gx | x \in v\}$ for $g \in G(F)$, and on $\tilde{\mathcal{E}}$ by mapping $e : v_1 \rightarrow v_2$ to the inclusion map $ge : gv_1 \rightarrow gv_2$. The stabilizer of the standard vertex $v_0 := \mathcal{O}_F^2$ is $G(\mathcal{O}_F)$.

For a directed edge $\vec{e} \in \vec{\mathcal{E}}$ of the Bruhat-Tits tree \mathcal{T} , we define $U(\vec{e})$ to be the set of ends of \vec{e} (cf. [Se80]/[Sp14]); it is a compact open subset of $\mathbb{P}^1(F)$, and we have $gU(\vec{e}) = U(g\vec{e})$ for all $g \in G(F)$.

For $n \in \mathbb{Z}$, we set $v_n := \mathcal{O}_F \oplus \mathfrak{p}^n \in \tilde{\mathcal{V}}$, and denote by e_n the edge from v_{n+1} to v_n ; the “decreasing” sequence $(\pi(e_{-n}))_{n \in \mathbb{Z}}$ is the geodesic from ∞ to 0. (The geodesic from 0 to ∞ traverses the $\pi(v_n)$ in the natural order of $n \in \mathbb{Z}$.) We have $U(\pi(e_n)) = \mathfrak{p}^{-n}$ for each n .

On \mathcal{T} , we have the height function $h : \mathcal{V} \rightarrow \mathbb{Z}$ (cf. [BL95]) defined as follows: The geodesic ray from $v \in \mathcal{V}$ to ∞ must contain some $\pi(v_n)$ ($n \in \mathbb{Z}$), since

it has non-empty intersection with $A := \{\pi(v_n) | n \in \mathbb{Z}\}$; we define $h(v) := n - d(v, \pi(v_n))$ for any such v_n . This is easily seen to be well-defined, and satisfies $h(\pi(v_n)) = n$ for all $n \in \mathbb{Z}$. We have the following lemma:

LEMMA 2.5. (a) For all $\bar{e} \in \mathcal{E}$, we have

$$h(t(\bar{e})) = \begin{cases} h(o(\bar{e})) + 1 & \text{if } \infty \in U(\bar{e}), \\ h(o(\bar{e})) - 1 & \text{otherwise.} \end{cases}$$

(b) For $a \in F^*$, $b \in F$, $\bar{v} \in \mathcal{V}$ we have

$$h\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \bar{v}\right) = h(\bar{v}) - \text{ord}_{\varpi}(a).$$

(Cf. [Sp14], lemma 3.6)

Let R be a ring, M an R -module. We let $C(\tilde{\mathcal{V}}, M)$ be the R -module of maps $\phi : \tilde{\mathcal{V}} \rightarrow M$, and $C(\tilde{\mathcal{E}}, M)$ the R -module of maps $\tilde{\mathcal{E}} \rightarrow M$. Both are $G(F)$ -modules via $(g\phi)(v) := \phi(g^{-1}v)$, $(gc)(e) := c(g^{-1}e)$.

We let $\mathcal{C}_c(\tilde{\mathcal{V}}, M) \subseteq C(\tilde{\mathcal{V}}, M)$ and $\mathcal{C}_c(\tilde{\mathcal{E}}, M) \subseteq C(\tilde{\mathcal{E}}, M)$ be the $(G(F)$ -stable) submodules of maps with compact support, i.e. maps that are zero outside a finite set. We get pairings

$$\langle -, - \rangle : \mathcal{C}_c(\tilde{\mathcal{V}}, R) \times C(\tilde{\mathcal{V}}, M) \rightarrow M, \quad \langle \phi_1, \phi_2 \rangle := \sum_{v \in \tilde{\mathcal{V}}} \phi_1(v) \phi_2(v) \quad (2)$$

and

$$\langle -, - \rangle : \mathcal{C}_c(\tilde{\mathcal{E}}, R) \times C(\tilde{\mathcal{E}}, M) \rightarrow M, \quad \langle c_1, c_2 \rangle := \sum_{e \in \tilde{\mathcal{E}}} c_1(e) c_2(e). \quad (3)$$

We define Hecke operators $T, N : \mathcal{C}(\tilde{\mathcal{V}}, M) \rightarrow \mathcal{C}(\tilde{\mathcal{V}}, M)$ by

$$T\phi(v) = \sum_{t(e)=v} \phi(o(e)) \quad \text{and} \quad N\phi := \varpi\phi \quad (\text{i.e. } N\phi(v) = \phi(\varpi^{-1}v))$$

for all $v \in \tilde{\mathcal{V}}$. These restrict to operators on $\mathcal{C}_c(\tilde{\mathcal{V}}, R)$, which we sometimes denote by T_c and N_c for emphasis. With respect to (2), T_c is adjoint to TN , and N_c is adjoint to its inverse operator $N^{-1} : \mathcal{C}_c(\tilde{\mathcal{V}}, R) \rightarrow \mathcal{C}_c(\tilde{\mathcal{V}}, R)$.

T and N obviously commute, and we have the following Hecke structure theorem on compactly supported functions on $\tilde{\mathcal{V}}$ (an analogue of [BL95], Thm. 10):

THEOREM 2.6. $\mathcal{C}_c(\tilde{\mathcal{V}}, R)$ is a free $R[T, N^{\pm 1}]$ -module (where $R[T, N^{\pm 1}]$ is the ring of Laurent polynomials in N over the polynomial ring $R[T]$, with N and T commuting).

Proof. Fix a vertex $v_0 \in \tilde{\mathcal{V}}$. For each $n \geq 0$, let C_n be the set of vertices $v \in \tilde{\mathcal{V}}$ such that there is a directed path of length n from v_0 to v in $\tilde{\mathcal{V}}$, and such that $d(\pi(v_0), \pi(v)) = n$ in the Bruhat-Tits tree \mathcal{T} . So $C_0 = \{v_0\}$, and C_n is a lift of the "circle of radius n around v_0 " in \mathcal{T} , in the parlance of [BL95].

One easily sees that $\bigcup_{n=0}^\infty C_n$ is a complete set of representatives for the projection map $\pi : \tilde{\mathcal{V}} \rightarrow \mathcal{V}$; specifically, for $n > 1$ and a given $v \in C_{n-1}$, C_n contains exactly q elements adjacent to v in $\tilde{\mathcal{V}}$; and we can write $\tilde{\mathcal{V}}$ as a disjoint union $\bigcup_{j \in \mathbb{Z}} \bigcup_{n=0}^\infty N^j(C_n)$.

We further define $V_0 := \{v_0\}$ and choose subsets $V_n \subseteq C_n$ as follows: We let V_1 be any subset of cardinality q . For $n > 1$, we choose $q - 1$ out of the q elements of C_n adjacent to v' , for every $v' \in C_{n-1}$, and let V_n be the union of these elements for all $v' \in C_{n-1}$. Finally, we set

$$H_{n,j} := \{ \phi \in C_c(\tilde{\mathcal{V}}, R) \mid \text{Supp}(\phi) \subseteq \bigcup_{i=0}^n N^j(C_i) \} \quad \text{for each } n \geq 0, j \in \mathbb{Z},$$

$H_n := \bigcup_{j \in \mathbb{Z}} H_{n,j}$, and $H_{-1} := H_{-1,j} := \{0\}$. (For ease of notation, we identify $v \in \tilde{\mathcal{V}}$ with its indicator function $1_{\{v\}} \in C_c(\tilde{\mathcal{V}}, R)$ in this proof.)

Define $T' : C_c(\tilde{\mathcal{V}}, R) \rightarrow C_c(\tilde{\mathcal{V}}, R)$ by

$$T'(\phi)(v) := \sum_{\substack{t(e)=(v), \\ o(e) \in N^j(C_n)}} \phi(o(e)) \quad \text{for each } v \in N^j(C_{n-1}), j \in \mathbb{Z};$$

T' can be seen as the "restriction to one layer" $\bigcup_{n=0}^\infty N^j(C_n)$ of T . We have $T'(v) \equiv T(v) \pmod{H_{n-1}}$ for each $v \in H_n$, since the "missing summand" of T' lies in H_{n-1} .

We claim that for each $n \geq 0$, the set $X_{n,j} := \bigcup_{i=0}^n N^j T^{n-i}(V_i)$ is an R -basis for $H_{n,j}/H_{n-1,j}$. By the above congruence, we can replace T by T' in the definition of $X_{n,j}$.

The claim is clear for $n = 0$. So let $n \geq 1$, and assume the claim to be true for all $n' \leq n$. For each $v \in C_{n-1}$, the q points in C_n adjacent to v are generated by the $q - 1$ of these points lying in V_n , plus $T'v$ (which just sums up these q points). By induction hypothesis, v is generated by $X_{n-1,0}$, and thus (taking the union over all v), C_n is generated by $T'(X_{n-1,0}) \cup V_n = X_{n,0}$. Since the cardinality of $X_{n,0}$ equals the R -rank of $H_{n,0}/H_{n-1,0}$ (both are equal to $(q + 1)q^{n-1}$), $X_{n,0}$ is in fact an R -basis.

Analogously, we see that $H_{n,j}/H_{n-1,j}$ has $N^j(X_{n,0}) = X_{n,j}$ as a basis, for each $j \in \mathbb{Z}$.

From the claim, it follows that $\bigcup_{j \in \mathbb{Z}} X_{n,j}$ is an R -basis of H_n/H_{n-1} for each n , and that $V := \bigcup_{n=0}^\infty V_n$ is an $R[T, N^{\pm 1}]$ -basis of $C_c(\tilde{\mathcal{V}}, R)$. □

For $a \in R$ and $\nu \in R^*$, we let $\tilde{\mathcal{B}}_{a,\nu}(F, R)$ be the "common cokernel" of $T - a$ and $N - \nu$ in $C_c(\tilde{\mathcal{V}}, R)$, namely $\tilde{\mathcal{B}}_{a,\nu}(F, R) := C_c(\tilde{\mathcal{V}}, R) / (\text{Im}(T - a) + \text{Im}(N - \nu))$;

dually, we define $\tilde{\mathcal{B}}^{a,\nu}(F, M) := \ker(T - a) \cap \ker(N - \nu) \subseteq C(\tilde{\mathcal{V}}, M)$.

For a lattice $v \in \tilde{\mathcal{V}}$, we define a valuation ord_v on F as follows: For $w \in F^2$, the set $\{x \in F | xw \in v\}$ is some fractional ideal $\varpi^m \mathcal{O}_F \subseteq F$ ($m \in \mathbb{Z}$); we set $\text{ord}_v(w) := m$. This map can also be given explicitly as follows: Let λ_1, λ_2 be a basis of v . We can write any $w \in F^2$ as $w = x_1\lambda_1 + x_2\lambda_2$; then we have $\text{ord}_v(w) = \min\{\text{ord}_\varpi(x_1), \text{ord}_\varpi(x_2)\}$. This gives a "valuation" map on F^2 , as one easily checks. We restrict it to $F \cong F \times \{0\} \hookrightarrow F^2$ to get a valuation ord_v on F , and consider especially the value at $e_1 := (1, 0)$.

LEMMA 2.7. *Let $\alpha, \nu \in R^*$, and put $a := \alpha + q\nu/\alpha$. Define a map $\varrho = \varrho_{\alpha,\nu} : \tilde{\mathcal{V}} \rightarrow R$ by $\varrho(v) := \alpha^{h(\pi(v))} \nu^{-\text{ord}_v(e_1)}$. Then $\varrho \in \tilde{\mathcal{B}}^{a,\nu}(F, R)$.*

Proof. One easily sees that $(v \mapsto \nu^{-\text{ord}_v(e_1)}) \in \ker(N - \nu)$. It remains to show that $\varrho \in \ker(T - a)$:

We have the Iwasawa decomposition $G(F) = B(F)G(\mathcal{O}_F) = \left\{ \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \right\} Z(F)G(\mathcal{O}_F)$; thus every vertex in $\tilde{\mathcal{V}}$ can be written as $\varpi^i v$ with $v = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} v_0$, with $i \in \mathbb{Z}, a \in F^*, b \in F$.

Now the lattice $v = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} v_0$ is generated by the vectors $\lambda_1 = \begin{pmatrix} a \\ 0 \end{pmatrix}$ and $\lambda_2 = \begin{pmatrix} b \\ 1 \end{pmatrix} \in \mathcal{O}_F^2$, so $e_1 = a^{-1}\lambda_1$ and thus $\text{ord}_v(e_1) = \text{ord}_\varpi(a^{-1}) = -\text{ord}_\varpi(a)$. The $q + 1$ neighbouring vertices v' for which there exists an $e \in \tilde{\mathcal{E}}$ with $o(e) = v', t(e) = v$ are given by $N_i v, i \in \{\infty\} \cup \mathcal{O}_F/\mathfrak{p}$, with $N_\infty := \begin{pmatrix} 1 & 0 \\ 0 & \varpi \end{pmatrix}$, and $N_i := \begin{pmatrix} \varpi & i \\ 0 & 1 \end{pmatrix}$ where $i \in \mathcal{O}_F$ runs through a complete set of representatives mod ϖ . By lemma 2.5, $h(\pi(N_\infty v)) = h(\pi(v)) + 1$ and $h(\pi(N_i v)) = h(\pi(v)) - 1$ for $i \neq \infty$. By considering the basis $\{N_i \lambda_1, N_i \lambda_2\}$ of $N_i v$ for each N_i , we see that $\text{ord}_{N_\infty v}(e_1) = \text{ord}_v(e_1)$ and $\text{ord}_{N_i v}(e_1) = \text{ord}_v(e_1) - 1$ for $i \neq \infty$. Thus we have

$$\begin{aligned} (T\varrho)(v) &= \sum_{t(e)=v} \alpha^{h(\pi(o(e)))} \nu^{-\text{ord}_{o(e)}(e_1)} \\ &= \alpha^{h(\pi(v))+1} \nu^{-\text{ord}_v e_1} + q \cdot \alpha^{h(\pi(v))-1} \nu^{1-\text{ord}_v(e_1)} \\ &= (\alpha + q\alpha^{-1}\nu) \alpha^{h(\pi(v))} \nu^{-\text{ord}_v e_1} = a\varrho(v), \end{aligned}$$

and also $(T\varrho)(\varpi^i v) = (TN^{-i}\varrho)(v) = N^{-i}(a\varrho)(v) = a\varrho(\varpi^i v)$ for a general $\varpi^i v \in \tilde{\mathcal{V}}$, which shows that $\varrho \in \ker(T - a)$. □

If $a^2 \neq \nu(q + 1)^2$ (the "spherical case"), we put $\mathcal{B}_{a,\nu}(F, R) := \tilde{\mathcal{B}}_{a,\nu}(F, R)$ and $\mathcal{B}^{a,\nu}(F, M) := \tilde{\mathcal{B}}^{a,\nu}(F, M)$.

In the "special case" $a^2 = \nu(q + 1)^2$, we need to assume that the polynomial $X^2 - a\nu X + q\nu^{-1} \in R[X]$ has a zero $\alpha' \in R$. Then the map $\varrho := \varrho_{\alpha',\nu} \in C(\tilde{\mathcal{V}}, R)$ defined as above lies in $\tilde{\mathcal{B}}^{a\nu,\nu^{-1}}(F, R) = \ker(TN - a) \cap \ker(N^{-1} - \nu)$ by Lemma 2.7, since $a\nu = \alpha' + q\nu^{-1}/\alpha'$. In other words, the kernel of the map

$\langle \cdot, \varrho \rangle : C_c(\tilde{\mathcal{V}}, R) \rightarrow R$ contains $\text{Im}(T - a) + \text{Im}(N - \nu)$; and we define

$$\mathcal{B}_{a,\nu}(F, R) := \ker(\langle \cdot, \varrho \rangle) / (\text{Im}(T - a) + \text{Im}(N - \nu))$$

to be the quotient; evidently, it is an R -submodule of codimension 1 of $\tilde{\mathcal{B}}_{a,\nu}(F, R)$. Dually, $T - a$ and $N - \nu$ both map the submodule $\varrho M = \{\varrho \cdot m, m \in M\}$ of $C(\tilde{\mathcal{V}}, M)$ to zero and thus induce endomorphisms on $C(\tilde{\mathcal{V}}, M)/\varrho M$; we define $\mathcal{B}^{a,\nu}(F, M)$ to be the intersection of their kernels.

In the special case, since $\nu = \alpha^2$, Lemma 2.7 states that $\varrho(gv_0) = \chi_\alpha(ad)\varrho(v_0) = \chi_\alpha(\det g)\varrho(v_0)$ for all $g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in B(F)$, and thus for all $g \in G(F)$ by the Iwasawa decomposition, since $G(\mathcal{O}_F)$ fixes v_0 and lies in the kernel of $\chi_\alpha \circ \det$. By the multiplicity of \det , we have $(g^{-1}\varrho)(v) = \varrho(gv) = \chi_\alpha(\det g)\varrho(v)$ for all $g \in G(F)$, $v \in \tilde{\mathcal{V}}$. So $\phi \in \ker\langle \cdot, \varrho \rangle$ implies $\langle g\phi, \varrho \rangle = \langle \phi, g^{-1}\varrho \rangle = \chi_\alpha(\det g)\langle \phi, \varrho \rangle = 0$, i.e. $\ker\langle \cdot, \varrho \rangle$ and thus $\mathcal{B}_{a,\nu}(F, R)$ are $G(F)$ -modules.

By the adjointness properties of the Hecke operators T and N , we have pairings $\text{coker}(T_c - a) \times \ker(TN - a) \rightarrow M$ and $\text{coker}(N_c - \nu) \times \ker(N^{-1} - \nu) \rightarrow M$, which "combine" to give a pairing

$$\langle -, - \rangle : \mathcal{B}_{a,\nu}(F, R) \times \mathcal{B}^{a\nu,\nu^{-1}}(F, M) \rightarrow M$$

(since $\ker(TN - a) \cap \ker(N^{-1} - \nu) = \ker(T - a\nu) \cap \ker(N - \nu^{-1})$), and a corresponding isomorphism $\mathcal{B}^{a\nu,\nu^{-1}}(F, M) \xrightarrow{\cong} \text{Hom}(\mathcal{B}_{a,\nu}(F, R), M)$.

DEFINITION 2.8. Let G be a totally disconnected locally compact group, $H \subseteq G$ an open subgroup. For a smooth $R[H]$ -module M , we define the (compactly) induced G -representation of M , denoted $\text{Ind}_H^G M$, to be the space of maps $f : G \rightarrow M$ such that $f(hg) = f(g)$ for all $g \in G, h \in H$, and such that f has compact support modulo H . We let G act on $\text{Ind}_H^G M$ via $g \cdot f(x) := f(xg)$. (We can also write $\text{Ind}_H^G M = R[G] \otimes_{R[H]} M$, cf. [Br82], III.5.)

We further define $\text{Coind}_H^G M := \text{Hom}_{R[H]}(R[G], M)$. Finally, for an $R[G]$ -module N , we write $\text{res}_H^G N$ for its underlying $R[H]$ -module ("restriction").

By Theorem 2.6, $T_c - a$ (as well as $N_c - \nu$) is injective, and the induced map

$$N_c - \nu : \text{coker}(T_c - a) = C_c(\tilde{\mathcal{V}}, R) / \text{Im}(T_c - a) \rightarrow \text{coker}(T_c - a)$$

(of $R[T, N^{\pm 1}]/(T - a) = R[N^{\pm 1}]$ -modules) is also injective. Now since $G(F)$ acts transitively on $\tilde{\mathcal{V}}$, with the stabilizer of $v_0 := \mathcal{O}_F^2$ being $K := G(\mathcal{O}_F)$, we have an isomorphism $C_c(\tilde{\mathcal{V}}, R) \cong \text{Ind}_K^{G(F)} R$. Thus we have exact sequences

$$0 \rightarrow \text{Ind}_K^{G(F)} R \xrightarrow{T-a} \text{Ind}_K^{G(F)} R \rightarrow \text{coker}(T_c - a) \rightarrow 0 \tag{4}$$

and (for a, ν in the spherical case)

$$0 \rightarrow \text{coker}(T_c - a) \xrightarrow{N-\nu} \text{coker}(T_c - a) \rightarrow \mathcal{B}_{a,\nu}(F, R) \rightarrow 0, \tag{5}$$

with all entries being free R -modules. Applying $\text{Hom}_R(\cdot, M)$ to them, we get:

LEMMA 2.9. *We have exact sequences of R -modules*

$$0 \rightarrow \ker(TN - a) \rightarrow \operatorname{Coind}_K^{G(F)} M \xrightarrow{T-a} \operatorname{Coind}_K^{G(F)} M \rightarrow 0$$

and, if $\mathcal{B}_{a,\nu}(F, M)$ is spherical (i.e. $a^2 \neq \nu(q + 1)^2$),

$$0 \rightarrow \mathcal{B}^{a\nu, \nu^{-1}}(F, M) \rightarrow \ker(TN - a) \xrightarrow{N-\nu} \ker(TN - a) \rightarrow 0.$$

For the special case, we have to work a bit more to get similar exact sequences: By [Sp14], eq. (22), for the representation $St^-(F, R) := \mathcal{B}_{-(q+1),1}(F, R)$ (i.e. $\nu = 1, \alpha = -1$) with trivial central character, we have an exact sequence of G -modules

$$0 \rightarrow \operatorname{Ind}_{KZ}^G R \rightarrow \operatorname{Ind}_{K'Z}^G R \rightarrow St^-(F, R) \rightarrow 0, \tag{6}$$

where $K' = \langle W \rangle K_0(\mathfrak{p})$ is the subgroup of KZ generated by $W := \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix}$ and the subgroup $K_0(\mathfrak{p}) \subseteq K$ of matrices that are upper-triangular modulo \mathfrak{p} . (Since $W^2 \in Z$, $K_0(\mathfrak{p})Z$ is a subgroup of K' of order 2.) Now any special representation (π, V) can be written as $\pi = \chi \otimes St^-$ for some character $\chi = \chi_Z$ (cf. the proof of lemma 2.12 below), and is obviously G -isomorphic to the representation $\pi \otimes (\chi \circ \det)$ acting on the space $V \otimes_R R(\chi \circ \det)$, where $R(\chi \circ \det)$ is the ring R with G -module structure given via $gr = \chi(\det(g))r$ for $g \in G, r \in R$. Tensoring (6) with $R(\chi \circ \det)$ over R gives an exact sequence of G -modules

$$0 \rightarrow \operatorname{Ind}_{KZ}^G \chi \rightarrow \operatorname{Ind}_{K'Z}^G \chi \rightarrow V \rightarrow 0. \tag{7}$$

It is easily seen that $R(\chi \circ \det)$ fits into another exact sequence of G -modules

$$0 \rightarrow \operatorname{Ind}_H^G R \xrightarrow{\begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix} - \chi(\varpi) \operatorname{id}} \operatorname{Ind}_H^G R \xrightarrow{\psi} R(\chi \circ \det) \rightarrow 0,$$

where $H := \{g \in G \mid \det g \in \mathcal{O}_F^\times\}$ is a normal subgroup containing K , $\begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix} (f)(g) := f(\begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix}^{-1} g)$ for $f \in \operatorname{Ind}_H^G R = \{f : G \rightarrow R \mid f(Hg) = f(g) \text{ for all } g \in G\}$, $g \in G$, is the natural operation of G , and where ψ is the G -equivariant map defined by $1_U \mapsto 1$.

Now since $H \subseteq G$ is a normal subgroup, we have $\operatorname{Ind}_H^G R \cong R[G/H]$ as G -modules (in fact $G/H \cong \mathbb{Z}$ as an abstract group). Let $X \subseteq G$ be a subgroup such that the natural inclusion $X/(X \cap H) \hookrightarrow G/H$ has finite cokernel; let $g_i H, i = 1, \dots, n$ be a set of representatives of that cokernel. Then we have a (non-canonical) X -isomorphism $\bigoplus_{i=0}^n \operatorname{Ind}_{X \cap H}^X \rightarrow \operatorname{Ind}_H^G R$ defined via $(1_{(X \cap H)x})_i \mapsto 1_{Hxg_i}$ for each $i = 1, \dots, n$ (cf. [Br82], III (5.4)).

Using this isomorphism and the “tensor identity” $\operatorname{Ind}_H^G M \otimes N \cong \operatorname{Ind}_H^G (M \otimes \operatorname{res}_H^G N)$ for any groups $H \subseteq G$, H -module M and G -module N ([Br82] III.5, Ex. 2), we have

$$\begin{aligned} \operatorname{Ind}_{KZ}^G R \otimes_R \operatorname{Ind}_H^G R &\cong \operatorname{Ind}_{KZ}^G (\operatorname{res}_{KZ}^G (\operatorname{Ind}_H^G R)) \\ &= \operatorname{Ind}_{KZ}^G ((\operatorname{Ind}_{KZ \cap H}^{KZ} R)^2) \\ &= (\operatorname{Ind}_{KZ}^G (\operatorname{Ind}_K^{KZ} R))^2 = (\operatorname{Ind}_K^G R)^2 \end{aligned}$$

(since $KZ/KZ \cap H \hookrightarrow G/H$ has index 2), and similarly

$$\text{Ind}_{K'Z}^G R \otimes_R \text{Ind}_H^G R \cong (\text{Ind}_{K'}^G R)^2.$$

Thus, we can resolve the first and second term of (7) into exact sequences

$$\begin{aligned} 0 \rightarrow (\text{Ind}_K^G R)^2 &\rightarrow (\text{Ind}_K^G R)^2 \rightarrow \text{Ind}_{KZ}^G \chi \rightarrow 0, \\ 0 \rightarrow (\text{Ind}_{K'}^G R)^2 &\rightarrow (\text{Ind}_{K'}^G R)^2 \rightarrow \text{Ind}_{(W)K_0(\mathfrak{p})Z}^G \chi \rightarrow 0. \end{aligned}$$

Dualizing (7) and these by taking $\text{Hom}(\cdot, M)$ for an R -module M , we get a “resolution” of $\mathcal{B}^{a\nu, \nu^{-1}}(F, M)$ in terms of coinduced modules:

LEMMA 2.10. *We have exact sequences*

$$\begin{aligned} 0 \rightarrow \mathcal{B}^{a\nu, \nu^{-1}}(F, M) &\rightarrow \text{Coind}_{K'Z}^G M(\chi) \rightarrow \text{Coind}_{KZ}^G M(\chi) \rightarrow 0, \\ 0 \rightarrow \text{Coind}_{KZ}^G M(\chi) &\rightarrow (\text{Coind}_K^G R)^2 \rightarrow (\text{Coind}_K^G R)^2 \rightarrow 0, \\ 0 \rightarrow \text{Coind}_{K'Z}^G M(\chi) &\rightarrow (\text{Coind}_{K'}^G R)^2 \rightarrow (\text{Coind}_{K'}^G R)^2 \rightarrow 0 \end{aligned}$$

for all special $\mathcal{B}_{a, \nu}(F, R)$ (i.e. $a^2 = \nu(q + 1)^2$), where $\chi = \chi_Z$ is the central character.

It is easily seen that the above arguments could be modified to get a similar set of exact sequences in the spherical case as well (replacing K' by K everywhere), in addition to that given in lemma 2.9; but we will not need this.

2.4 DISTRIBUTIONS ON THE BRUHAT-TITS GRAPH

For $\varrho \in C(\tilde{\mathcal{V}}, R)$ we define R -linear maps

$$\tilde{\delta}_\varrho : C(\tilde{\mathcal{E}}, M) \rightarrow C(\tilde{\mathcal{V}}, M), \quad \tilde{\delta}_\varrho(c)(v) := \sum_{v=t(e)} \varrho(o(e))c(e) - \sum_{v=o(e)} \varrho(t(e))c(e),$$

$$\tilde{\delta}^\varrho : C(\tilde{\mathcal{V}}, M) \rightarrow C(\tilde{\mathcal{E}}, M), \quad \tilde{\delta}^\varrho(\phi)(e) := \varrho(o(e))\phi(t(e)) - \varrho(t(e))\phi(o(e)).$$

One easily checks that these are adjoint with respect to the pairings (2) and (3), i.e. we have $\langle \tilde{\delta}_\varrho(c), \phi \rangle = \langle c, \tilde{\delta}^\varrho(\phi) \rangle$ for all $c \in C_c(\tilde{\mathcal{E}}, R)$, $\phi \in C(\tilde{\mathcal{V}}, M)$. We denote the maps corresponding to $\varrho \equiv 1$ by $\delta := \tilde{\delta}_1$, $\delta^* := \tilde{\delta}^1$.

For each ϱ , the map $\tilde{\delta}_\varrho$ fits into an exact sequence

$$C_c(\tilde{\mathcal{E}}, R) \xrightarrow{\tilde{\delta}_\varrho} C_c(\tilde{\mathcal{V}}, R) \xrightarrow{\langle \cdot, \varrho \rangle} R \rightarrow 0$$

but it is not injective in general: e.g. for $\varrho \equiv 1$, the map $\tilde{\mathcal{E}} \rightarrow R$ symbolized by

$$\begin{array}{ccc} \cdot & \xrightarrow{-1} & \cdot \\ \downarrow 1 & & \downarrow -1 \\ \cdot & \xrightarrow{1} & \cdot \end{array}$$

(and zero outside the square) lies in $\ker \delta$.

The restriction $\delta^*|_{C_c(\tilde{\mathcal{V}}, R)}$ to compactly supported maps is injective since $\tilde{\mathcal{T}}$ has no directed circles, and we have a surjective map

$$\text{coker}(\delta^* : C_c(\tilde{\mathcal{V}}, R) \rightarrow C_c(\tilde{\mathcal{E}}, R)) \rightarrow C^0(\mathbb{P}^1(F), R)/R, \quad c \mapsto \sum_{e \in \tilde{\mathcal{E}}} c(e)1_{U(\pi(e))}$$

(which corresponds to an isomorphism of the similar map on the Bruhat-Tits tree \mathcal{T}). Its kernel is generated by the functions $1_{\{e\}} - 1_{\{e'\}}$ for $e, e' \in \tilde{\mathcal{E}}$ with $\pi(e) = \pi(e')$.

For $\varrho_1, \varrho_2 \in C(\tilde{\mathcal{V}}, R)$ and $\phi \in C(\tilde{\mathcal{V}}, M)$ it is easily checked that

$$(\tilde{\delta}_{\varrho_1} \circ \tilde{\delta}^{\varrho_2})(\phi) = (T + TN)(\varrho_1 \cdot \varrho_2) \cdot \phi - \varrho_2 \cdot (T + TN)(\varrho_1 \cdot \phi).$$

For $a' \in R$ and $\varrho \in \ker(T + TN - a')$, applying this equality for $\varrho_1 = \varrho$ and $\varrho_2 = 1$ shows that $\tilde{\delta}_\varrho$ maps $\text{Im } \delta^*$ into $\text{Im}(T + TN - a')$, so we get an R -linear map

$$\tilde{\delta}_\varrho : \text{coker}(\delta^* : C_c(\tilde{\mathcal{V}}, R) \rightarrow C_c(\tilde{\mathcal{E}}, R)) \rightarrow \text{coker}(T_c + T_c N_c - a').$$

Let now again $\alpha, \nu \in R^*$, and $a := \alpha + q\nu/\alpha$. We let $\varrho := \varrho_{\alpha, \nu} \in \tilde{\mathcal{B}}^{a, \nu}(F, R)$ as defined in lemma 2.7, and write $\tilde{\delta}_{\alpha, \nu} := \tilde{\delta}_\varrho$. Since $\tilde{\delta}_{\alpha, \nu}$ maps $1_{\{e\}} - 1_{\{\varpi e\}}$ into $\text{Im}(R - \nu)$, it induces a map

$$\tilde{\delta}_{\alpha, \nu} : C^0(\mathbb{P}^1(F), R)/R \rightarrow \mathcal{B}_{a, \nu}(F, R)$$

(same name by abuse of notation) via the commutative diagram

$$\begin{array}{ccc} \text{coker } \delta^* & \xrightarrow{\tilde{\delta}_{\alpha, \nu}} & \text{coker}(T_c + T_c N_c - a') \\ \downarrow & & \downarrow \text{mod } (N - \nu) \\ C^0(\mathbb{P}^1(F), R)/R & \xrightarrow{\tilde{\delta}_{\alpha, \nu}} & \mathcal{B}_{a, \nu}(F, R) \end{array}$$

with $a' := a(1 + \nu)$, since $\varrho \in \mathcal{B}^{a, \nu}(F, R) \subseteq \ker(T + TN - a')$.

LEMMA 2.11. *We have $\varrho(gv) = \chi_\alpha(d/a')\chi_\nu(a')\varrho(v)$, and thus*

$$\tilde{\delta}_{\alpha, \nu}(gf) = \chi_\alpha(d/a')\chi_\nu(a')g\tilde{\delta}_{\alpha, \nu}(f),$$

for all $v \in \tilde{\mathcal{V}}$, $f \in C^0(\mathbb{P}^1(F), R)/R$ and $g = \begin{pmatrix} a' & b \\ 0 & d \end{pmatrix} \in B(F)$.

Proof. (a) Using lemma 2.5(b) and the fact that $\text{ord}_{g\nu}(e_1) = -\text{ord}_\varpi(a') + \text{ord}_\nu(e_1)$, we have

$$\varrho\left(\begin{pmatrix} a' & b \\ 0 & d \end{pmatrix} v\right) = \alpha^{h(v) - \text{ord}_\varpi(a'/d)} \nu^{\text{ord}_\varpi(a') - \text{ord}_\nu(e_1)} = \chi_\alpha(d/a')\chi_\nu(a')\varrho(v)$$

for all $v \in \tilde{\mathcal{V}}$. For f and g as in the assertion, we thus have

$$\begin{aligned} \tilde{\delta}_{\alpha,\nu}(gf)(v) &= \sum_{v=t(e)} \varrho(o(e))f(g^{-1}e) - \sum_{v=o(e)} \varrho(t(e))f(g^{-1}e) \\ &= \sum_{g^{-1}v=t(e)} \varrho(o(ge))f(e) - \sum_{g^{-1}v=o(e)} \varrho(t(ge))f(e) \\ &= \chi_{\alpha}(d/a')\chi_{\nu}(a')\varrho(v) \left(\sum_{g^{-1}v=t(e)} \varrho(o(e))f(e) - \sum_{g^{-1}v=o(e)} \varrho(t(e))f(e) \right) \\ &= \chi_{\alpha}(d/a')\chi_{\nu}(a')g\tilde{\delta}_{\alpha,\nu}(f)(v). \end{aligned}$$

□

We define a function $\delta_{\alpha,\nu} : C_c(F^*, R) \rightarrow \mathcal{B}_{\alpha,\nu}(F, R)$ as follows: For $f \in C_c(F^*, R)$, we let $\psi_0(f) \in C_c(\mathbb{P}^1(F), R)$ be the extension of $x \mapsto \chi_{\alpha}(x)\chi_{\nu}(x)^{-1}f(x)$ by zero to $\mathbb{P}^1(F)$. We set $\delta_{\alpha,\nu} := \tilde{\delta}_{\alpha,\nu} \circ \psi_0$. If $\alpha = \nu$, we can define $\delta_{\alpha,\nu}$ on all functions in $C_c(F, R)$.

We let F^* operate on $C_c(F, R)$ by $(tf)(x) := f(t^{-1}x)$; this induces an action of the group $T^1(F) := \{ \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \mid t \in F^* \}$, which we identify with F^* in the obvious way. With respect to it, we have

$$\psi_0(tf)(x) = \chi_{\alpha}(t)\chi_{\nu}(t)^{-1}t\psi_0(f)(x)$$

and

$$\tilde{\delta}_{\alpha,\nu}(tf) = \chi_{\alpha}^{-1}(t)\chi_{\nu}(t)t\tilde{\delta}_{\alpha,\nu}(f),$$

so $\delta_{\alpha,\nu}$ is $T^1(F)$ -equivariant.

For an R -module M , we define an F^* -action on $\text{Dist}(F^*, M)$ by $\int fd(t\mu) := t \int (t^{-1}f)d\mu$. Let $H \subseteq G(F)$ be a subgroup, and M an $R[H]$ -module. We define an H -action on $\mathcal{B}^{\alpha,\nu^{-1}}(F, M)$ by requiring $\langle \phi, h\lambda \rangle = h \cdot \langle h^{-1}\phi, \lambda \rangle$ for all $\phi \in \mathcal{B}_{\alpha,\nu}(F, M)$, $\lambda \in \mathcal{B}^{\alpha,\nu^{-1}}(F, M)$, $h \in H$. With respect to these two actions, we get a $T^1(F) \cap H$ -equivariant mapping

$$\delta^{\alpha,\nu} : \mathcal{B}^{\alpha,\nu^{-1}}(F, M) \rightarrow \text{Dist}(F^*, M), \quad \delta^{\alpha,\nu}(\lambda) := \langle \delta_{\alpha,\nu}(\cdot), \lambda \rangle$$

dual to $\delta_{\alpha,\nu}$.

2.5 LOCAL DISTRIBUTIONS

Now consider the case $R = \mathbb{C}$. Let $\chi_1, \chi_2 : F^* \rightarrow \mathbb{C}^*$ be two unramified characters. We consider (χ_1, χ_2) as a character on the torus $T(F)$ of $\text{GL}_2(F)$, which induces a character χ on $B(F)$ by

$$\chi \begin{pmatrix} t_1 & u \\ 0 & t_2 \end{pmatrix} := \chi_1(t_1)\chi_2(t_2).$$

Put $\alpha_i := \chi_i(\varpi)\sqrt{q} \in \mathbb{C}^*$ for $i = 1, 2$. Set $\nu := \chi_1(\varpi)\chi_2(\varpi) = \alpha_1\alpha_2q^{-1} \in \mathbb{C}^*$, and $a := \alpha_1 + \alpha_2 = \alpha_i + q\nu/\alpha_i$ for either i . When a and ν are given by the α_i

like this, we will often write $\mathcal{B}_{\alpha_1, \alpha_2}(F, R) := \mathcal{B}_{a, \nu}(F, R)$ and $\mathcal{B}^{\alpha_1, \alpha_2}(F, M) := \mathcal{B}^{a\nu, \nu^{-1}}(F, M)$ (!) for its dual. In the special case $a^2 = \nu(q + 1)^2$, we assume the χ_i to be sorted such that $\chi_1 = |\cdot| \chi_2$.

Let $\mathcal{B}(\chi_1, \chi_2)$ denote the space of continuous maps $\phi : G(F) \rightarrow \mathbb{C}$ such that

$$\phi \left(\begin{pmatrix} t_1 & u \\ 0 & t_2 \end{pmatrix} g \right) = \chi_{\alpha_1}(t_1) \chi_{\alpha_2}(t_2) |t_1| \phi(g) \tag{8}$$

for all $t_1, t_2 \in F^*, u \in F, g \in G(F)$. $G(F)$ operates canonically on $\mathcal{B}(\chi_1, \chi_2)$ by right translation (cf. [Bu98], Ch. 4.5). If $\chi_1 \chi_2^{-1} \neq |\cdot|^{\pm 1}$, $\mathcal{B}(\chi_1, \chi_2)$ is a model of the spherical representation $\pi(\chi_1, \chi_2)$; if $\chi_1 \chi_2^{-1} = |\cdot|^{\pm 1}$, the special representation $\pi(\chi_1, \chi_2)$ can be given as an irreducible subquotient of codimension 1 of $\mathcal{B}(\chi_1, \chi_2)$.²

LEMMA 2.12. *We have a G -equivariant isomorphism $\tilde{\mathcal{B}}_{a, \nu}(F, \mathbb{C}) \cong \mathcal{B}(\chi_1, \chi_2)$. It induces an isomorphism $\mathcal{B}_{a, \nu}(F, \mathbb{C}) \cong \pi(\chi_1, \chi_2)$ both for spherical and special representations.*

Proof. We choose a ‘‘central’’ unramified character $\chi_Z : F^* \rightarrow \mathbb{C}$ satisfying $\chi_Z^2(\varpi) = \nu$; then we have $\chi_1 = \chi_Z \chi_0^{-1}, \chi_2 = \chi_Z \chi_0$ for some unramified character χ_0 . We set $a' := \sqrt{q} (\chi_0(\varpi)^{-1} + \chi_0(\varpi))$, which satisfies $a = \chi_Z(\varpi) a'$. For a representation (π, V) of $G(F)$, by [Bu98], Ex. 4.5.9, we can define another representation $\chi_Z \otimes \pi$ on V via

$$(g, v) \mapsto \chi_Z(\det(g)) \pi(g)v \quad \text{for all } g \in G(F), v \in V,$$

and with this definition we have $\mathcal{B}(\chi_1, \chi_2) \cong \chi_Z \otimes \mathcal{B}(\chi_0^{-1}, \chi_0)$. Since $\mathcal{B}(\chi_0^{-1}, \chi_0)$ has trivial central character, [BL95], Thm. 20 (as quoted in [Sp14]) states that $\mathcal{B}(\chi_0^{-1}, \chi_0) \cong \mathcal{B}_{a', 1}(F, \mathbb{C}) \cong \text{Ind}_{KZ}^{G(F)} R / \text{Im}(T - a')$.

Define a G -linear map $\phi : \text{Ind}_K^G R \rightarrow \chi_Z \otimes \text{Ind}_{KZ}^G R$ by $1_K \mapsto (\chi_Z \circ \det) \cdot 1_{KZ}$. Since 1_K (resp. $(\chi_Z \circ \det) \cdot 1_{KZ}$) generates $\text{Ind}_K^G R$ (resp. $\chi_Z \otimes \text{Ind}_{KZ}^G R$) as a $\mathbb{C}[G]$ -module, ϕ is well-defined and surjective.

ϕ maps $N1_K = \begin{pmatrix} \varpi & 0 \\ 0 & \varpi \end{pmatrix} 1_K$ to

$$\begin{pmatrix} \varpi & 0 \\ 0 & \varpi \end{pmatrix} ((\chi_Z \circ \det) \cdot 1_{KZ}) = \chi_Z(\varpi)^2 \cdot ((\chi_Z \circ \det) \cdot 1_{KZ}) = \nu \cdot \phi(1_K).$$

Thus $\text{Im}(N - \nu) \subseteq \ker \phi$, and in fact the two are equal, since the preimage of the space of functions of support in a coset KZg ($g \in G(F)$) under ϕ is exactly the space generated by the $1_{Kzg}, z \in Z(F) = Z(\mathcal{O}_F) \{ \begin{pmatrix} \varpi & 0 \\ 0 & \varpi \end{pmatrix} \}^{\mathbb{Z}}$.

Furthermore, ϕ maps $T1_K = \sum_{i \in \mathcal{O}_F / (\varpi) \cup \{\infty\}} N_i 1_K$ (with the N_i as in Lemma 2.7) to

$$\sum_i \chi_Z(\det(N_i)) \cdot ((\chi_Z \circ \det) \cdot N_i 1_{KZ}) = \chi_Z(\varpi) \cdot (\chi_Z \circ \det) T1_{KZ}$$

(since $\det(N_i) = \varpi$ for all i), and thus $\text{Im}(T - a)$ is mapped to $\text{Im}(\chi_Z(\varpi) T - a) = \text{Im}(\chi_Z(\varpi)(T - a')) = \text{Im}(T - a')$.

²Note that [Bu98] denotes this special representation by $\sigma(\chi_1, \chi_2)$, not by $\pi(\chi_1, \chi_2)$.

Putting everything together, we thus have G -isomorphisms

$$\begin{aligned} C_c(\tilde{V}, \mathbb{C})/(\operatorname{Im}(T - a) + \operatorname{Im}(N - \nu)) &\cong \operatorname{Ind}_K^G R/(\operatorname{Im}(T - a) + \operatorname{Im}(N - \nu)) \\ &\cong \chi_Z \otimes (\operatorname{Ind}_{KZ}^G R/\operatorname{Im}(T - a')) \\ &\cong \chi_Z \otimes \mathcal{B}(\chi_0^{-1}, \chi_0) \cong \mathcal{B}(\chi_1, \chi_2). \end{aligned}$$

Thus, $\mathcal{B}_{a,\nu}(F, \mathbb{C})$ is isomorphic to the spherical principal series representation $\pi(\chi_1, \chi_2)$ for $a^2 \neq \nu(q + 1)^2$.

In the special case, $\mathcal{B}_{a,\nu}(F, \mathbb{C})$ is a G -invariant subspace of $\tilde{\mathcal{B}}_{a,\nu}(F, \mathbb{C})$ of codimension 1, so it must be mapped under the isomorphism to the unique G -invariant subspace of $\mathcal{B}(\chi_1, \chi_2)$ of codimension 1 (in fact, the unique infinite-dimensional irreducible G -invariant subspace, by [Bu98], Thm. 4.5.1), which is the special representation $\pi(\chi_1, \chi_2)$. \square

By [Bu98], section 4.4, there exists thus for all pairs a, ν a *Whittaker functional* λ on $\mathcal{B}_{a,\nu}(F, \mathbb{C})$, i.e. a nontrivial linear map $\lambda : \mathcal{B}_{a,\nu}(F, \mathbb{C}) \rightarrow \mathbb{C}$ such that $\lambda\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \phi\right) = \psi(x)\lambda(\phi)$. It is unique up to scalar multiples.

From it, we furthermore get a *Whittaker model* $\mathcal{W}_{a,\nu}$ of $\mathcal{B}_{a,\nu}(F, \mathbb{C})$:

$$\mathcal{W}_{a,\nu} := \{W_\xi : GL_2(F) \rightarrow \mathbb{C} \mid \xi \in \mathcal{B}_{a,\nu}(F, \mathbb{C})\},$$

where $W_\xi(g) := \lambda(g \cdot \xi)$ for all $g \in GL_2(F)$. (see e.g. [Bu98], Ch. 3, eq. (5.6).) Now write $\alpha := \alpha_1$ for short. Recall the distribution $\mu_{\alpha,\nu} = \psi(x)\chi_{\alpha/\nu}(x)dx \in \operatorname{Dist}(F^*, \mathbb{C})$. For $\alpha = \nu$, it extends to a distribution on F . We have the following generalization of [Sp14], Prop. 3.10:

PROPOSITION 2.13. (a) *There exists a unique Whittaker functional $\lambda = \lambda_{a,\nu}$ on $\mathcal{B}_{a,\nu}(F, \mathbb{C})$ such that $\delta^{\alpha,\nu}(\lambda) = \mu_{\alpha,\nu}$.*

(b) *For every $f \in C_c(F^*, \mathbb{C})$, there exists $W = W_f \in \mathcal{W}_{a,\nu}$ such that*

$$\int_{F^*} (af)(x)\mu_{\alpha,\nu}(dx) = W_f \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}.$$

If $\alpha = \nu$, then for every $f \in C_c(F, \mathbb{C})$, there exists $W_f \in \mathcal{W}_{a,\nu}$ such that

$$\int_F (af)(x)\mu_{\alpha,\nu}(dx) = W_f \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}.$$

(c) *Let $H \subseteq U = \mathcal{O}_F^\times$ be an open subgroup, and write $W_H := W_{1_H}$. For every $f \in C_c^0(F^*, \mathbb{C})^H$ we have*

$$\int_{F^*} f(x)\mu_{\alpha,\nu}(dx) = [U : H] \int_{F^*} f(x)W_H \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} d^\times x.$$

Proof. (a) By [Sp14], we have a Whittaker functional of the Steinberg representation given by the composite

$$\operatorname{St}(F, \mathbb{C}) := C^0(\mathbb{P}^1(F), \mathbb{C})/\mathbb{C} \xrightarrow{\cong} C_c(F, \mathbb{C}) \xrightarrow{\Lambda} \mathbb{C}, \tag{9}$$

where the first map is the F -equivariant isomorphism

$$C^0(\mathbb{P}^1(F), \mathbb{C})/\mathbb{C} \rightarrow C_c(F, \mathbb{C}), \quad \phi \mapsto f(x) := \phi(x) - \phi(\infty),$$

(with F acting on $C_c(F, \mathbb{C})$ by $(x \cdot f)(y) := f(y - x)$, and on $C^0(\mathbb{P}^1(F), \mathbb{C})/\mathbb{C}$ by $x\phi := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \phi$), and the second is

$$\Lambda : C_c(F, \mathbb{C}) \rightarrow \mathbb{C}, \quad f \mapsto \int_F f(x)\psi(x)dx.$$

Let now $\lambda : \mathcal{B}_{a,\nu}(F, \mathbb{C}) \rightarrow \mathbb{C}$ be a Whittaker functional of $\mathcal{B}_{a,\nu}(F, \mathbb{C})$. By lemma 2.11, for $u = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in B(F)$,

$$(\lambda \circ \tilde{\delta}_{\alpha,\nu})(u\phi) = \lambda(u\tilde{\delta}_{\alpha,\nu}(\phi)) = \psi(x)\lambda(\tilde{\delta}_{\alpha,\nu}(\phi)),$$

so $\lambda \circ \tilde{\delta}_{\alpha,\nu}$ is a Whittaker functional if it is not zero.

To describe the image of $\tilde{\delta}_{\alpha,\nu}$, consider the commutative diagram

$$\begin{array}{ccccc} C_c(\tilde{\mathcal{E}}, R) & \xrightarrow{\tilde{\delta}_{\alpha,\nu}} & C_c(\tilde{\mathcal{V}}, R) & & \\ \downarrow (10) & & \downarrow \phi \mapsto \phi \cdot \varrho & & \\ C_c(\tilde{\mathcal{E}}, R) & \xrightarrow{\delta} & C_c(\tilde{\mathcal{V}}, R) & \xrightarrow{\langle \cdot, 1 \rangle} & R \longrightarrow 0 \end{array}$$

where the vertical maps are defined by

$$C_c(\tilde{\mathcal{E}}, R) \rightarrow C_c(\tilde{\mathcal{E}}, R), \quad c \mapsto (e \mapsto c(e)\varrho(o(e))\varrho(t(e))) \tag{10}$$

resp. by mapping ϕ to $v \mapsto \phi(v)\varrho(v)$; both are obviously isomorphisms.

Since the lower row is exact, we have $\text{Im } \delta = \ker \langle \cdot, 1 \rangle =: C_c^0(\tilde{\mathcal{V}}, R)$ and thus $\text{Im } \tilde{\delta}_{\alpha,\nu} = \varrho^{-1} \cdot C_c^0(\tilde{\mathcal{V}}, R)$.

Since $\lambda \neq 0$ and $\mathcal{B}_{a,\nu}(F, \mathbb{C})$ is generated by (the equivalence classes of) the $1_{\{v\}}$, $v \in \tilde{\mathcal{V}}$, there exists a $v \in \tilde{\mathcal{V}}$ such that $\lambda(1_{\{v\}}) \neq 0$. Let ϕ be this $1_{\{v\}}$, and let $u = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in B(F)$ such that $x \notin \ker \psi$. Then

$$\varrho \cdot (u\phi - \phi) = \varrho \cdot (1_{\{u^{-1}v\}} - 1_{\{v\}}) = \varrho(v)(1_{\{u^{-1}v\}} - 1_{\{v\}}) \in C_c^0(\tilde{\mathcal{V}}, R)$$

by lemma 2.11, so $0 \neq u\phi - \phi \in \text{Im } \tilde{\delta}_{\alpha,\nu}$, but $\lambda(u\phi - \phi) = \psi(x)\lambda(\phi) - \lambda(\phi) \neq 0$. So $\lambda \circ \tilde{\delta}_{\alpha,\nu} \neq 0$ is indeed a Whittaker functional. By replacing λ by a scalar multiple, we can assume $\lambda \circ \tilde{\delta}_{\alpha,\nu} = (9)$.

Considering λ as an element of $\mathcal{B}^{a\nu,\nu^{-1}}(F, \mathbb{C}) \cong \text{Hom}(\mathcal{B}_{a,\nu}(F, \mathbb{C}), \mathbb{C})$, we have

$$\begin{aligned} \delta^{\alpha,\nu}(\lambda)(f) &= \langle \tilde{\delta}_{\alpha,\nu}(f), \lambda \rangle \\ &= \Lambda(\chi_\alpha \chi_\nu^{-1} f) \\ &= \int_{F^*} \chi_\alpha(x) \chi_\nu^{-1}(x) f(x) \psi(x) dx \\ &= \mu_{\alpha,\nu}(f). \end{aligned}$$

(b), (c) follow from (a) as in [Sp14]. □

2.6 SEMI-LOCAL THEORY

We can generalize many of the previous constructions to the semi-local case, considering all primes $\mathfrak{p}|p$ at once.

So let F_1, \dots, F_m be finite extensions of \mathbb{Q}_p , and for each i , let q_i be the number of elements of the residue field of F_i . We put $\underline{F} := F_1 \times \dots \times F_m$.

Let R again be a ring, and $a_i \in R, \nu_i \in R^*$ for each $i \in \{1, \dots, m\}$. Put $\underline{a} := (a_1, \dots, a_m), \underline{\nu} := (\nu_1, \dots, \nu_m)$. We define $\mathcal{B}_{\underline{a}, \underline{\nu}}(\underline{F}, R)$ as the tensor product

$$\mathcal{B}_{\underline{a}, \underline{\nu}}(\underline{F}, R) := \bigotimes_{i=1}^m \mathcal{B}_{a_i, \nu_i}(F_i, R).$$

For an R -module M , we define $\mathcal{B}^{\underline{a}, \underline{\nu}^{-1}}(\underline{F}, M) := \text{Hom}_R(\mathcal{B}_{\underline{a}, \underline{\nu}}(\underline{F}, R), M)$; let

$$\langle \cdot, \cdot \rangle : \mathcal{B}_{\underline{a}, \underline{\nu}}(\underline{F}, R) \times \mathcal{B}^{\underline{a}, \underline{\nu}^{-1}}(\underline{F}, M) \rightarrow M \tag{11}$$

denote the evaluation pairing.

We have an obvious isomorphism

$$\bigotimes_{i=1}^m C_c^0(F_i^*, R) \rightarrow C_c^0(\underline{F}^*, R), \quad \bigotimes_i f_i \mapsto \left((x_i)_{i=1, \dots, m} \mapsto \prod_{i=1}^m f_i(x_i) \right). \tag{12}$$

Now when we have $\alpha_{i,1}, \alpha_{i,2} \in R^*$ such that $a_i = \alpha_{i,1} + \alpha_{i,2}$ and $\nu_i = \alpha_{i,1}\alpha_{i,2}q_i^{-1}$, we can define the $T^1(\underline{F})$ -equivariant map

$$\delta_{\underline{a}_1, \underline{a}_2} := \delta_{\underline{a}_1, \underline{\nu}} : C_c^0(\underline{F}, R) \rightarrow \mathcal{B}_{\underline{a}, \underline{\nu}}(\underline{F}, R)$$

as the inverse of (12) composed with $\bigotimes_{i=1}^m \delta_{\alpha_{i,1}, \nu_i}$.

Again, we will often write $\mathcal{B}_{\underline{\alpha}_1, \underline{\alpha}_2}(F, R) := \mathcal{B}_{\underline{a}, \underline{\nu}^{-1}}(F, R)$ and $\mathcal{B}^{\underline{\alpha}_1, \underline{\alpha}_2}(F, M) := \mathcal{B}^{\underline{a}, \underline{\nu}^{-1}}(F, M)$.

If $H \subseteq G(F)$ is a subgroup, and M an $R[H]$ -module, we define an H -action on $\mathcal{B}^{\underline{a}, \underline{\nu}^{-1}}(F, M)$ by requiring $\langle \phi, h\lambda \rangle = h \cdot \langle h^{-1}\phi, \lambda \rangle$ for all $\phi \in \mathcal{B}_{\underline{a}, \underline{\nu}}(F, M), \lambda \in \mathcal{B}^{\underline{a}, \underline{\nu}^{-1}}(F, M), h \in H$, and get a $T^1(\underline{F}) \cap H$ -equivariant mapping

$$\delta^{\underline{\alpha}_1, \underline{\alpha}_2} : \mathcal{B}^{\underline{a}, \underline{\nu}^{-1}}(F, M) \rightarrow \text{Dist}(\underline{F}^*, M), \quad \delta^{\underline{\alpha}_1, \underline{\alpha}_2}(\lambda) := \langle \delta_{\underline{\alpha}_1, \underline{\alpha}_2}(\cdot), \lambda \rangle.$$

Finally, we have a homomorphism

$$\begin{aligned} \bigotimes_{i=1}^m \mathcal{B}^{a_i, \nu_i^{-1}}(F_i, R) &\xrightarrow{\cong} \bigotimes_{i=1}^m \text{Hom}_R(\mathcal{B}_{a_i, \nu_i^{-1}}(F_i, R), R) \\ &\rightarrow \text{Hom}(\mathcal{B}_{a_1, \nu_1}(F_1, R), \text{Hom}(\mathcal{B}_{a_2, \nu_2}(F_2, R), \text{Hom}(\dots, R))\dots) \\ &\xrightarrow{\cong} \mathcal{B}^{\underline{a}, \underline{\nu}^{-1}}(F, R). \end{aligned} \tag{13}$$

where the second map is given by $\otimes_i f_i \mapsto (x_1 \mapsto (x_2 \mapsto (\dots \mapsto \prod_i f_i(x_i))\dots))$, and the last map by iterating the adjunction formula of the tensor product.

3 COHOMOLOGY CLASSES AND GLOBAL MEASURES

3.1 DEFINITIONS

From now on, let F denote a number field, with ring of integers \mathcal{O}_F . For each finite prime v , let $U_v := \mathcal{O}_v^*$. Let $\mathbb{A} = \mathbb{A}_F$ denote the ring of adèles of F , and $\mathbb{I} = \mathbb{I}_F$ the group of ideles of F . For a finite subset S of the set of places of F , we denote by $\mathbb{A}^S := \{x \in \mathbb{A}_F \mid x_v = 0 \ \forall v \in S\}$ the S -adèles and by \mathbb{I}^S the S -ideles, and put $F_S := \prod_{v \in S} F_v$, $U_S := \prod_{v \in S} U_v$, $U^S := \prod_{v \notin S} U_v$ (if S contains all infinite places of F), and similarly for other global groups.

For ℓ a prime number or ∞ , we write S_ℓ for the set of places of F above ℓ , and abbreviate the above notations to $\mathbb{A}^\ell := \mathbb{A}^{S_\ell}$, $\mathbb{A}^{p,\infty} := \mathbb{A}^{S_p \cup S_\infty}$, and similarly write \mathbb{I}^p , \mathbb{I}^∞ , F_p , F_∞ , U^∞ , U_p , $U^{p,\infty}$, \mathbb{I}_∞ etc.

Let F have r real embeddings and s pairs of complex embeddings. Set $d := r + s - 1$. Let $\{\sigma_0, \dots, \sigma_{r-1}, \sigma_r, \dots, \sigma_d\}$ be a set of representatives of these embeddings (i.e. for $i \geq r$, choose one from each pair of complex embeddings), and denote by $\infty_0, \dots, \infty_d$ the corresponding archimedean primes of F . We let $S_\infty^0 := \{\infty_1, \dots, \infty_d\} \subseteq S_\infty$.

For each place v , let dx_v denote the associated self-dual Haar measure on F_v , and $dx := \prod_v dx_v$ the associated Haar measure on \mathbb{A}_F . We define Haar measures $d^\times x_v$ on F_v^* by $d^\times x_v := c_v \frac{dx_v}{|x_v|_v}$, where $c_v = (1 - \frac{1}{q_v})^{-1}$ for v finite, $c_v = 1$ for $v|\infty$. For $v|\infty$ complex, we use the decomposition $\mathbb{C}^* = \mathbb{R}_+^* \times S^1$ (with $S^1 = \{x \in \mathbb{C}^*; |x| = 1\}$) to write $d^\times x_v = d^\times r_v d\vartheta_v$ for variables r_v, ϑ_v with $r_v \in \mathbb{R}_+^*$, $\vartheta_v \in S^1$.

Let $S_1 \subseteq S_p$ be a set of primes of F lying above p , $S_2 := S_p - S_1$. Let R be a topological Hausdorff ring.

DEFINITION 3.1. We define the module of continuous functions

$$\mathcal{C}(S_1, R) := C(F_{S_1} \times F_{S_2}^* \times \mathbb{I}^{p,\infty} / U^{p,\infty}, R);$$

and let $\mathcal{C}_c(S_1, R)$ be the submodule of all compactly supported $f \in \mathcal{C}(S_1, R)$. We write $\mathcal{C}^0(S_1, R)$, $\mathcal{C}_c^0(S_1, R)$ for the submodules of locally constant maps (or of continuous maps where R is assumed to have the discrete topology). We further define

$$\mathcal{C}_c^b(S_1, R) := \mathcal{C}_c(\emptyset, R) + \mathcal{C}_c^b(S_1, R) \subseteq \mathcal{C}_c^b(S_1, R)$$

to be the module of continuous compactly supported maps that are “constant near $(0_{\mathfrak{p}}, x^{\mathfrak{p}})$ ” for each $\mathfrak{p} \in S_1$.

DEFINITION 3.2. For an R -module M , let $\mathcal{D}_f(S_1, M)$ denote the R -module of maps

$$\phi : \mathfrak{C}\mathfrak{a}(F_{S_1} \times F_{S_2}^*) \times \mathbb{I}_F^{p,\infty} \rightarrow M$$

that are $U^{p,\infty}$ -invariant and such that $\phi(\cdot, x^{p,\infty})$ is a distribution for each $x^{p,\infty} \in \mathbb{I}_F^{p,\infty}$.

Since $\mathbb{I}_F^{p,\infty}/U^{p,\infty}$ is a discrete topological group, $\mathcal{D}_f(S_1, M)$ naturally identifies with the space of M -valued distributions on $F_{S_1} \times F_{S_2}^* \times \mathbb{I}_F^{p,\infty}/U^{p,\infty}$. So there exists a canonical R -bilinear map

$$\mathcal{D}_f(S_1, M) \times \mathcal{C}_c^0(S_1, R) \rightarrow M, \quad (\phi, f) \mapsto \int f \, d\phi, \tag{14}$$

which is easily seen to induce an isomorphism $\mathcal{D}_f(S_1, M) \cong \text{Hom}_R(\mathcal{C}_c^0(S_1, R), M)$.

For a subgroup $E \subseteq F^*$ and an $R[E]$ -module M , we let E operate on $\mathcal{D}_f(S_1, M)$ and $\mathcal{C}_c^0(S_1, R)$ by $(a\phi)(U, x^{p,\infty}) := a\phi(a^{-1}U, a^{-1}x^{p,\infty})$ and $(af)(x^\infty) := f(a^{-1}x^\infty)$ for $a \in E$, $U \in \mathfrak{C}\mathfrak{o}(F_{S_1} \times F_{S_2}^*)$, $x \in \mathbb{I}_F$; thus we have $\int (af) \, d(a\phi) = a \int f \, d\phi$ for all a, f, ϕ .

When $M = V$ is a finite-dimensional vector space over a p -adic field, we write $\mathcal{D}_f^b(S_1, V)$ for the subset of $\phi \in \mathcal{D}_f(S_1, V)$ such that ϕ is even a measure on $F_{S_1} \times F_{S_2} \times \mathbb{I}_F^{p,\infty}/U^{p,\infty}$.

DEFINITION 3.3. For a \mathbb{C} -vector space V , define $\mathcal{D}(S_1, V)$ to be the set of all maps $\phi : \mathfrak{C}\mathfrak{o}(F_{S_1} \times F_{S_2}^*) \times \mathbb{I}^p \rightarrow V$ such that:

- (i) ϕ is invariant under F^\times and $U^{p,\infty}$.
- (ii) For $x^p \in \mathbb{I}^p$, $\phi(\cdot, x^p)$ is a distribution of $F_{S_1} \times F_{S_2}$.
- (iii) For all $U \in \mathfrak{C}\mathfrak{o}(F_{S_1} \times F_{S_2}^*)$, the map $\phi_U : \mathbb{I} = F_p^\times \times \mathbb{I}^p \rightarrow V, (x_p, x^p) \mapsto \phi(x_p U, x^p)$ is smooth, and rapidly decreasing as $|x| \rightarrow \infty$ and $|x| \rightarrow 0$.

We will need a variant of this last set: Let $\mathcal{D}'(S_1, V)$ be the set of all maps $\phi \in \mathcal{D}(S_1, V)$ that are " $(S^1)^s$ -invariant", i.e. such that for all complex primes ∞_j of F and all $\zeta \in S^1 = \{x \in \mathbb{C}^*; |x| = 1\}$, we have

$$\phi(U, x^{p,\infty_j}, \zeta x_{\infty_j}) = \phi(U, x^{p,\infty_j}, x_{\infty_j}) \text{ for all } x^p = (x^{p,\infty_j}, x_{\infty_j}) \in \mathbb{I}^p.$$

There is an obvious surjective map

$$\mathcal{D}(S_1, V) \rightarrow \mathcal{D}'(S_1, V), \quad \phi \mapsto \left((U, x) \mapsto \int_{(S^1)^s} \phi(U, x) d\vartheta_r \cdots d\vartheta_{r+s-1} \right)$$

given by integrating over $(S^1)^s \subseteq (\mathbb{C}^*)^s \hookrightarrow \mathbb{I}_\infty$.

Let F_+^* denote the set of all $x \in F^*$ that are totally positive, i.e. positive with respect to every real embedding of F . (For F totally imaginary, we have $F^* = F_+^*$.) Let $F^{*'} \subseteq F_+^*$ be a maximal torsion-free subgroup of F_+^* . If F has at least one real embedding, we obviously have $F^{*'} = F_+^*$; for totally imaginary F , $F^{*'}$ is a subgroup of finite index of F^* with $F/F^{*'} \cong \mu_F$, the roots of unity of F .

We set

$$E' := F^{*'} \cap O_F^\times \subseteq O_F^\times,$$

so E' is a torsion-free \mathbb{Z} -module of rank d . E' operates freely and discretely on the space

$$\mathbb{R}_0^{d+1} := \left\{ (x_0, \dots, x_d) \in \mathbb{R}^{d+1} \mid \sum_{i=0}^d x_i = 0 \right\}$$

via the embedding

$$\begin{aligned} E' &\hookrightarrow \mathbb{R}_0^{d+1} \\ a &\mapsto (\log |\sigma_i(a)|)_{i \in S_\infty} \end{aligned}$$

(cf. proof of Dirichlet's unit theorem, e.g. in [Neu92], Ch. 1), and the quotient \mathbb{R}_0^{d+1}/E' is compact. We choose the orientation on \mathbb{R}_0^{d+1} induced by the natural orientation on \mathbb{R}^d via the isomorphism $\mathbb{R}^d \cong \mathbb{R}_0^{d+1}$, $(x_1, \dots, x_d) \mapsto (-\sum_{i=1}^d x_i, x_1, \dots, x_d)$. So \mathbb{R}_0^{d+1}/E' becomes an oriented compact d -dimensional manifold.

Let \mathcal{G}_p be the Galois group of the maximal abelian extension of F which is unramified outside p and ∞ ; for a \mathbb{C} -vector space V , let $\text{Dist}(\mathcal{G}_p, V)$ be the set of V -valued distributions of \mathcal{G}_p . Denote by $\varrho : \mathbb{I}_F/F^* \rightarrow \mathcal{G}_p$ the projection given by global reciprocity.

3.2 GLOBAL MEASURES

Now let $V = \mathbb{C}$, equipped with the trivial $F^{*'}$ -action. We want to construct a commutative diagram

$$\begin{array}{ccc} \mathcal{D}(S_1, \mathbb{C}) & \xrightarrow{\phi \mapsto \kappa_\phi} & H^d(F^{*'}, \mathcal{D}_f(S_1, \mathbb{C})) \\ & \searrow \phi \mapsto \mu_\phi & \swarrow \kappa \mapsto \mu_\kappa = \kappa \cap \partial(\cdot) \\ & & \text{Dist}(\mathcal{G}_p, \mathbb{C}) \end{array} \tag{15}$$

First, let R be any topological Hausdorff ring. Let $\overline{E'}$ denote the closure of E' in U_p . The projection map $\text{pr} : \mathbb{I}^\infty/U^{p,\infty} \rightarrow \mathbb{I}^\infty/(\overline{E'} \times U^{p,\infty})$ induces an isomorphism

$$\text{pr}^* : C_c(\mathbb{I}^\infty/(\overline{E'} \times U^{p,\infty}), R) \rightarrow H^0(E', C_c(\mathbb{I}^\infty/U^{p,\infty}, R)),$$

and the reciprocity map induces a surjective map $\overline{\varrho} : \mathbb{I}^\infty/(\overline{E'} \times U^{p,\infty}) \rightarrow \mathcal{G}_p$. Now we can define a map

$$\begin{aligned} \varrho^\# : H_0(F^{*'}/E', C_c(\mathbb{I}^\infty/(\overline{E'} \times U^{p,\infty}), R)) &\rightarrow C(\mathcal{G}_p, R), \\ [f] &\mapsto \left(\overline{\varrho}(x) \mapsto \sum_{\zeta \in F^{*'}/E'} f(\zeta x) \text{ for } x \in \mathbb{I}^\infty/(\overline{E'} \times U^{p,\infty}) \right). \end{aligned}$$

This is an isomorphism, with inverse map $f \mapsto [(f \circ \overline{\varrho}) \cdot 1_{\mathcal{F}}]$, where $1_{\mathcal{F}}$ is the characteristic function of a fundamental domain \mathcal{F} of the action of $F^{*'}/E'$ on $\mathbb{I}^\infty/U^\infty$.

We get a composite map

$$\begin{aligned} C(\mathcal{G}_p, R) &\xrightarrow{(\varrho^\sharp)^{-1}} H_0(F^{*'} / E', C_c(\mathbb{I}^\infty / (\overline{E'} \times U^{p,\infty}), R)) \\ &\xrightarrow{\text{pr}^*} H_0(F^{*'} / E', H^0(E', C_c(\mathbb{I}^\infty / U^{p,\infty}, R))) \\ &\longrightarrow H_0(F^{*'} / E', H^0(E', C_c(S_1, R))), \end{aligned} \tag{16}$$

where the last arrow is induced by the “extension by zero” from $C_c(\mathbb{I}^\infty / U^{p,\infty}, R)$ to $C_c(S_1, R)$.

Now let $\eta \in H_d(E', \mathbb{Z}) \cong \mathbb{Z}$ be the generator that corresponds to the given orientation of \mathbb{R}_0^{d+1} . This gives us, for every R -module A , a homomorphism

$$H_0(F^{*'} / E', H^0(E', A)) \xrightarrow{\cap \eta} H_0(F^{*'} / E', H_d(E', A))$$

Composing this with the edge morphism

$$H_0(F^{*'} / E', H_d(E', A)) \rightarrow H_d(F^{*'}, A) \tag{17}$$

(and setting $A = C_c(S_1, R)$) gives a map

$$H_0(F^{*'} / E', H^0(E', C_c(S_1, R))) \rightarrow H_d(F^{*'}, C_c(S_1, R)) \tag{18}$$

We define

$$\partial : C(\mathcal{G}_p, R) \rightarrow H_d(F^{*'}, C_c(S_1, R))$$

as the composition of (16) with this map.

Now, letting M be an R -module equipped with the trivial $F^{*'}$ -action, the bilinear form (14)

$$\begin{aligned} \mathcal{D}_f(S_1, M) \times C_c(S_1, R) &\rightarrow M \\ (\phi, f) &\mapsto \int f \, d\phi \end{aligned}$$

induces a cap product

$$\cap : H^d(F^{*'}, \mathcal{D}_f(S_1, M)) \times H_d(F^{*'}, C_c(S_1, R)) \rightarrow H_0(F^{*'}, M) = M. \tag{19}$$

Thus for each $\kappa \in H^d(F^{*'}, \mathcal{D}_f(S_1, M))$, we get a distribution μ_κ on \mathcal{G}_p by defining

$$\int_{\mathcal{G}_p} f(\gamma) \mu_\kappa(d\gamma) := \kappa \cap \partial(f) \tag{20}$$

for all continuous maps $f : \mathcal{G}_p \rightarrow R$.

Now let $M = V$ be a finite-dimensional vector space over a p -adic field K , and let $\kappa \in H^d(F^{*'}, \mathcal{D}_f^b(S_1, V))$. We identify κ with its image in $H^d(F^{*'}, \mathcal{D}_f(S_1, V))$; then it is easily seen that μ_κ is also a measure, i.e. we have a map

$$H^d(F^{*'}, \mathcal{D}_f^b(S_1, V)) \rightarrow \text{Dist}^b(\mathcal{G}_p, V), \quad \kappa \mapsto \mu_\kappa. \tag{21}$$

Let $L|F$ be a \mathbb{Z}_p -extension of F . Since it is unramified outside p , it gives rise to a continuous homomorphism $\mathcal{G}_p \rightarrow \text{Gal}(L|F)$ via $\sigma \mapsto \sigma|_L$. Fixing an isomorphism $\text{Gal}(L|F) \cong p^{\varepsilon_p} \mathbb{Z}_p$ (where $\varepsilon_p = 2$ for $p = 2$, $\varepsilon_p = 1$ for p odd), we obtain a surjective homomorphism $\ell : \mathcal{G}_p \rightarrow p^{\varepsilon_p} \mathbb{Z}_p$. (Note that $p^{\varepsilon_p} \mathbb{Z}_p$ is the space of definition of the p -adic exponential function \exp_p .)

Example 3.4. Let L be the cyclotomic \mathbb{Z}_p -extension of F . Then we can take $\ell = \log_p \circ \mathcal{N}$, where $\mathcal{N} : \mathcal{G}_p \rightarrow \mathbb{Z}_p^*$ is the p -adic cyclotomic character, defined by requiring $\gamma \zeta = \zeta^{\mathcal{N}(\gamma)}$ for all $\gamma \in \mathcal{G}_p$ and all p -power roots of unity ζ . It is well-known (cf. [Wa82], par. 5) that $\log_p(\mathbb{Z}_p^*) = p^{\varepsilon_p} \mathbb{Z}_p$.

It is well-known that F has t independent \mathbb{Z}_p -extensions, where $s + 1 \leq t \leq [F : \mathbb{Q}]$; the Leopoldt conjecture implies $t = s + 1$. μ_κ defines a t -variable p -adic L-function as follows:

DEFINITION 3.5. Let K be a p -adic field, V a finite-dimensional K -vector space, $\kappa \in H^d(F^{*'}, \mathcal{D}_f^b(S_1, V))$. Let $\ell_1, \dots, \ell_t : \mathcal{G}_p \rightarrow p^{\varepsilon_p} \mathbb{Z}_p$ be continuous homomorphisms. The p -adic L-function of κ is given by

$$L_p(\underline{s}, \kappa) := L_p(s_1, \dots, s_t, \kappa) := \int_{\mathcal{G}_p} \left(\prod_{i=1}^t \exp_p(s_i \ell_i(\gamma)) \right) \mu_\kappa(d\gamma)$$

for all $s_1, \dots, s_t \in \mathbb{Z}_p$.

Remark 3.6. Let $\Sigma := \{\pm 1\}^r$, where r is the number of real embeddings of F . The group isomorphism $\mathbb{Z}/2\mathbb{Z} \cong \{\pm 1\}$, $\varepsilon \mapsto (-1)^\varepsilon$, induces a pairing

$$\langle \cdot, \cdot \rangle : \Sigma \rightarrow \{\pm 1\}, \quad \langle ((-1)^{\varepsilon_i})_i, ((-1)^{\varepsilon'_i})_i \rangle := (-1)^{\sum_i \varepsilon_i \varepsilon'_i}.$$

For a field k of characteristic zero, a $k[\Sigma]$ -module V and $\underline{\mu} = (\mu_0, \dots, \mu_{r-1}) \in \Sigma$, we put $V_{\underline{\mu}} := \{v \in V \mid \langle \underline{\mu}, \underline{\nu} \rangle v = \underline{\nu} v \ \forall \underline{\nu} \in \Sigma\}$, so that we have $V = \bigoplus_{\underline{\mu} \in \Sigma} V_{\underline{\mu}}$. We write $v_{\underline{\mu}}$ for the projection of $v \in V$ to $V_{\underline{\mu}}$, and $v_+ := v_{(1, \dots, 1)}$.

For $r > 0$, we identify Σ with $F^*/F^{*'}$ via the isomorphism $\Sigma \cong \prod_{i=0}^{r-1} \mathbb{R}^*/\mathbb{R}_+^* \cong F^*/F^{*'} = F^*/F_+^*$. Then for each F^* -module M , Σ acts on $H^d(F^{*'}, \mathcal{D}_f^b(S_1, M))$ and on $H^d(F^{*'}, \mathcal{D}_f^b(S_1, M))$. For $r = 0$, we let the trivial group Σ act on these groups as well for ease of notation. The exact sequence $\Sigma \cong \prod_{i=0}^{r-1} \mathbb{R}^*/\mathbb{R}_+^* = \mathbb{I}_\infty/\mathbb{I}_\infty^0 \rightarrow \mathcal{G}_p \rightarrow \mathcal{G}_p^+ \rightarrow 0$ of class field theory (where \mathbb{I}_∞^0 is the maximal connected subgroup of \mathbb{I}_∞) yields an action of Σ on \mathcal{G}_p . We easily check that (21) is Σ -equivariant, and that the maps $\gamma \mapsto \exp_p(s \ell_i(\gamma))$ factor over $\mathcal{G}_p \rightarrow \mathcal{G}_p^+$ (since \mathbb{Z}_p -extensions are unramified at ∞). Therefore we have $L_p(\underline{s}, \kappa) = L_p(\underline{s}, \kappa_+)$.

For $\phi \in \mathcal{D}(S_1, V)$ and $f \in C^0(\mathbb{I}/F^*, \mathbb{C})$, let

$$\int_{\mathbb{I}/F^*} f(x) \phi(d^\times x_p, x^p) d^\times x^p := [U_p : U] \int_{\mathbb{I}/F^*} f(x) \phi_U(x) d^\times x,$$

where we choose an open set $U \subseteq U_p$ such that $f(x_p u, x^p) = f(x_p, x^p)$ for all $(x_p, x^p) \in \mathbb{I}$ and $u \in U$; such a U exists by lemma 3.7 below. Since this integral is additive in f , there exists a unique V -valued distribution μ_ϕ on \mathcal{G}_p such that

$$\int_{\mathcal{G}_p} f \, d\mu_\phi = \int_{\mathbb{I}/F^*} f(\varrho(x)) \phi(d^\times x_p, x^p) \, d^\times x^p \tag{22}$$

for all functions $f \in C^0(\mathcal{G}_p, V)$.

LEMMA 3.7. *Let $F : \mathbb{I}/F^* \rightarrow X$ be a locally constant map to a set X . Then there exists an open subgroup $U \subseteq \mathbb{I}$ such that f factors over \mathbb{I}/F^*U .*

Proof. $\mathbb{I}_\infty = \prod_{v|\infty} F_v$ is connected, thus f factors over $\bar{f} : \mathbb{I}/F^*\mathbb{I}_\infty \rightarrow X$. Since $\mathbb{I}/F^*\mathbb{I}_\infty$ is profinite, \bar{f} further factors over a subgroup $U' \subseteq \mathbb{I}^\infty$ of finite index, which is open. \square

Let $U_\infty^0 := \prod_{v \in S_\infty^0} \mathbb{R}_+^*$; the isomorphisms $U_\infty^0 \cong \mathbb{R}^d$, $(r_v)_v \mapsto (\log r_v)_v$, and $\mathbb{R}^d \cong \mathbb{R}_0^{d+1}$ give it the structure of a d -dimensional oriented manifold (with the natural orientation). It has the d -form $d^\times r_1 \cdot \dots \cdot d^\times r_d$, where (by slight abuse of notation) we choose $d^\times r_i$ on F_{∞_i} corresponding to the Haar measure $d^\times x_i$ resp. $d^\times r_i$ on $\mathbb{R}_+^* \subseteq F_{\infty_i}^*$. E' operates on U_∞^0 via $a \mapsto (|\sigma_i(a)|)_{i \in S_\infty^0}$, so the isomorphism $U_\infty^0 \cong \mathbb{R}_0^{d+1}$ is E' -equivariant.

For $\phi \in \mathcal{D}'(S_1, V)$, set

$$\begin{aligned} \int_0^\infty \phi \, d^\times r_0 : \mathfrak{C}\mathfrak{O}(F_{S_1} \times F_{S_2}^*) \times \mathbb{I}^{p, \infty_0} &\rightarrow \mathbb{C} \\ (U, x^{p, \infty_0}) &\mapsto \int_0^\infty \phi(U, r_0, x^{p, \infty_0}) \, d^\times r_0, \end{aligned}$$

where we let $r_0 \in F_{\infty_0}$ run through the positive real line \mathbb{R}_+^* in F_{∞_0} . Composing this with the projection $\mathcal{D}(S_1, V) \rightarrow \mathcal{D}'(S_1, V)$ gives us a map

$$\begin{aligned} \mathcal{D}(S_1, V) &\rightarrow H^0(F^{*'}, \mathcal{D}_f(S_1, C^\infty(U_\infty^0, V))), \\ \phi &\mapsto \int_{(S^1)^s} \left(\int_0^\infty \phi \, d^\times r_0 \right) d\vartheta_r \, d\vartheta_{r+1} \dots d\vartheta_{r+s-1} \end{aligned} \tag{23}$$

(where $C^\infty(U_\infty^0, V)$ denotes the space of smooth V -valued functions on U_∞^0), since one easily checks that $\int_0^\infty \phi \, d^\times r_0$ is F^{*' -invariant.

Define the complex $C^\bullet := \mathcal{D}_f(S_1, \Omega^\bullet(U_\infty^0, V))$. By the Poincaré lemma, this is a resolution of $\mathcal{D}_f(S_1, V)$. We now define the map $\phi \mapsto \kappa_\phi$ as the composition of (23) with the composition

$$H^0(F^{*'}, \mathcal{D}_f(S_1, C^\infty(U_\infty^0, V))) \rightarrow H^0(F^{*'}, C^d) \rightarrow H^d(F^{*'}, \mathcal{D}_f(S_1, V)), \tag{24}$$

where the first map is induced by

$$C^\infty(U_\infty^0, V) \rightarrow \Omega^d(U_\infty^0, V), \quad f \mapsto f(r_1, \dots, r_d) d^\times r_1 \cdot \dots \cdot d^\times r_d, \tag{25}$$

and the second is an edge morphism in the spectral sequence

$$H^q(F^{*'}, C^p) \Rightarrow H^{p+q}(F^{*'}, \mathcal{D}_f(S_1, V)). \tag{26}$$

Specializing to $V = \mathbb{C}$, we now have:

PROPOSITION 3.8. *The diagram (15) commutes, i.e., for each $\phi \in \mathcal{D}(S_1, \mathbb{C})$, we have*

$$\mu_\phi = \mu_{\kappa_\phi}.$$

Proof. Analogously to [Sp14], proof of prop. 4.21, we define a pairing

$$\langle \cdot, \cdot \rangle: \mathcal{D}(S_1, \mathbb{C}) \times C^0(\mathcal{G}_p, \mathbb{C}) \rightarrow \mathbb{C}$$

as the composite of (23) \times (16) with

$$\begin{aligned} &H^0(F^{*'}, \mathcal{D}_f(S_1, C^\infty(U_\infty^0, \mathbb{C}))) \times H_0(F^{*'} / E', H^0(E', \mathcal{C}_c^0(S_1, \mathbb{C}))) \\ &\quad \cap \rightarrow H_0(F^{*'} / E', H^0(E', C^\infty(U_\infty^0, \mathbb{C}))) \rightarrow H_0(F^{*'} / E', \mathbb{C}) \cong \mathbb{C}, \end{aligned} \tag{27}$$

where \cap is the cap product induced by (14), and the second map is induced by

$$H^0(E', C^\infty(U_\infty^0, \mathbb{C})) \rightarrow \mathbb{C}, \quad f \mapsto \int_{U_\infty^0 / E'} f(r_1, \dots, r_d) d^\times r_1 \dots d^\times r_d. \tag{28}$$

Then we can show that

$$\kappa_\phi \cap \partial(f) = \langle \phi, f \rangle = \int_{\mathcal{G}_p} f(\gamma) \mu_\phi(d\gamma) \quad \text{for all } f \in C^0(\mathcal{G}_p, \mathbb{C}),$$

by copying the proof for the totally real case (replacing F_+^* by $F^{*'}, E_+$ by E'), using the fact that for a d -form on the d -dimensional oriented manifold $M := \mathbb{R}_0^{d+1} / E' \cong U_\infty^0 / E'$, integration over M corresponds to taking the cap product with the fundamental class η of M under the canonical isomorphism $H_{dR}^d(M) \cong H_{sing}^d(M) = H^d(E', \mathbb{C})$. □

3.3 EXCEPTIONAL ZEROS

Now let $\ell_1, \dots, \ell_t : \mathcal{G}_p \rightarrow \mathbb{Z}_p$ be continuous homomorphisms. Let again $S_1 = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\} \subseteq S_p$ be a set of primes above p , of cardinality $n := \#S_1$.

PROPOSITION 3.9. *For each $\underline{x} = (x_1, \dots, x_t) \in \mathbb{N}_0^t$ set $|\underline{x}| := \sum_{i=1}^t x_i$. Then*

$$\partial\left(\prod_{i=1}^t \ell_i^{x_i}\right) = 0 \quad \text{for all } \underline{x} \text{ with } |\underline{x}| \leq n - 1.$$

Proof. We can readily generalize the proof of Spieß' result for the p -adic cyclotomic character ($\ell = \log_p \circ \mathcal{N}$) in the totally real case ([Sp14], Prop. 4.6(a)),

Lemmas 4.1 and 4.7) to show that $\partial(\ell^x) = 0$ for all $0 \leq x \leq n - 1$, using the facts that we can write $F^{*'} = E' \times \mathcal{T}$ for some subgroup $\mathcal{T} \subseteq F^{*'}$ (since $F^{*'} / E' = F^* / \mathcal{O}_F^\times$ is a free \mathbb{Z} -module), and that for each homomorphism $\ell : \mathcal{G}_p \rightarrow \mathbb{Z}_p$, the composition

$$\tilde{\ell} : \mathbb{I}^\infty \xrightarrow{\ell} \mathcal{G}_p \xrightarrow{\ell} \mathbb{Z}_p \hookrightarrow \mathbb{Q}_p.$$

is zero on $\mathbb{I}^{\infty \cdot p}$ (since the pro- q -part of \mathcal{G}_p is finite for every prime $q \neq p$ and \mathbb{Q}_p is torsion-free).

Now for a ring $R \supseteq \mathbb{Q}$, each monomial $\prod_{i=1}^t X_i^{n_i} \in R[X_1, \dots, X_t]$ of degree $n = \sum_i n_i$ can be written as a linear combination of n -th powers $(X_i + r_{i,j} X_j)^n$. Therefore each product $\prod_{i=1}^t \ell_i^{x_i}$ of degree $x = |x|$ is a linear combination of x -th powers of the homomorphisms $\ell_{i,j} := \ell_i + r_{i,j} \ell_j : \mathcal{G}_p \rightarrow \mathbb{Z}_p$. This proves the proposition. \square

DEFINITION 3.10. A t -variable p -adic analytic function $f(\underline{s}) = f(s_1, \dots, s_t)$ ($s_i \in \mathbb{Z}_p$) has vanishing order $\geq n$ at the point $\underline{0} = (0, \dots, 0)$ if all its partial derivatives of total order $\leq n - 1$ vanish, i.e. if

$$\frac{\partial^k}{(\partial \underline{s})^{\underline{k}}} f(\underline{0}) := \frac{\partial^k}{\partial s_1^{k_1} \dots \partial s_t^{k_t}} f(\underline{0}) = 0$$

for all $\underline{k} = (k_1, \dots, k_t) \in \mathbb{N}_0^t$ with $k := |\underline{k}| \leq n - 1$. We write $\text{ord}_{\underline{s}=\underline{0}} f(\underline{s}) \geq n$ in this case.

THEOREM 3.11. Let $n := \#(S_1)$, $\kappa \in H^d(F^{*'}, \mathcal{D}_f^b(S_1, V))$, V a finite-dimensional vector space over a p -adic field. Then $L_p(\underline{s}, \kappa)$ is a locally analytic function, and we have

$$\text{ord}_{\underline{s}=\underline{0}} L_p(\underline{s}, \kappa) \geq n.$$

Proof. We have

$$\frac{\partial^k}{(\partial \underline{s})^{\underline{k}}} L_p(\underline{0}, \kappa) = \int_{\mathcal{G}_p} \left(\prod_{i=1}^t \ell_i(\gamma)^{k_i} \right) \mu_\kappa(d\gamma) = \kappa \cap \partial \left(\prod_{i=1}^t \ell_i(\gamma)^{k_i} \right)$$

for all $\underline{k} = (k_1, \dots, k_t) \in \mathbb{N}_0^t$. Thus the theorem follows from proposition 3.9. \square

3.4 INTEGRAL COHOMOLOGY CLASSES

DEFINITION 3.12. A nonzero cohomology class $\kappa \in H^d(F^{*'}, \mathcal{D}_f(S_1, \mathbb{C}))$ is called integral if κ lies in the image of

$$H^d(F^{*'}, \mathcal{D}_f(S_1, R)) \otimes_R \mathbb{C} \rightarrow H^d(F^{*'}, \mathcal{D}_f(S_1, \mathbb{C}))$$

for some Dedekind ring $R \subseteq \overline{\mathbb{O}}$. If, in addition, there exists a torsion-free R -submodule $M \subseteq H^d(F^{*'}, \mathcal{D}_f(S_1, R))$ of rank ≤ 1 (i.e. M can be embedded into R) such that κ lies in the image of $M \otimes_R \mathbb{C} \rightarrow H^d(F^{*'}, \mathcal{D}_f(S_1, \mathbb{C}))$, then κ is integral of rank ≤ 1 .

For κ as in def. 3.12 and $R \subseteq \mathbb{C}$, we let $L_{\kappa,R}$ be the image of

$$H_d(F^{*'}, \mathcal{C}_c^0(S_1, R)) \rightarrow H_0(F^{*'}, \mathbb{C}) = \mathbb{C}, \quad x \mapsto \kappa \cap x.$$

PROPOSITION 3.13. *Let $\kappa \in H^d(F^{*'}, \mathcal{D}_f(S_1, \mathbb{C}))$ be integral. Then*

- (a) μ_κ is a p -adic measure.
- (b) *There exists a Dedekind ring $R \subseteq \overline{\mathbb{O}}$ such that $L_{\kappa,R}$ is a finitely generated R -module (resp. a torsion-free R -module of rank ≤ 1 , if κ is integral of rank ≤ 1).*

For each such R , the map $H^d(F^{'}, \mathcal{D}_f(S_1, L_{\kappa,R})) \otimes \overline{\mathbb{Q}} \rightarrow \mathcal{H}^d(F^{*'}, \mathcal{D}_f(S_1, \mathbb{C}))$ is injective and κ lies in its image.*

Proof. The proofs of the corresponding results for totally real F ([Sp14], prop. 4.17 and cor. 4.18) also work in the general case. □

Remark 3.14. Let κ be integral with Dedekind ring R as above. By (b) of the proposition, we can view κ as an element of $H^d(F^{*'}, \mathcal{D}_f(S_1, L_{\kappa,R})) \otimes \overline{\mathbb{Q}}$. Put $V_\kappa := L_{\kappa,R} \otimes_R \mathbb{C}_p$; let $\overline{\kappa}$ be the image of κ under the composition

$$\begin{aligned} H^d(F^{*'}, \mathcal{D}_f(S_1, L_{\kappa,R})) \otimes_R \overline{\mathbb{Q}} &\rightarrow H^d(F^{*'}, \mathcal{D}_f(S_1, L_{\kappa,R})) \otimes_R \mathbb{C}_p \\ &\rightarrow H^d(F^{*'}, \mathcal{D}_f^b(S_1, V_\kappa)), \end{aligned}$$

where the second map is induced by $\mathcal{D}_f(S_1, L_{\kappa,R}) \otimes_R \mathbb{C}_p \rightarrow \mathcal{D}_f^b(S_1, V_\kappa)$. By [Sp14], lemma 4.15, $\overline{\kappa}$ does not depend on the choice of R .

Since μ_κ is a p -adic measure, $\mu_{\overline{\kappa}}$ allows integration of all continuous functions $f \in C(\mathcal{G}_p, \mathbb{C}_p)$, and by abuse of notation, we write $L_p(\underline{s}, \kappa) := L_p(\underline{s}, \overline{\kappa})$ (cf. remark 3.6). So $L_p(\underline{s}, \kappa)$ has values in the finite-dimensional \mathbb{C}_p -vector space V_κ .

4 p -ADIC L-FUNCTIONS OF AUTOMORPHIC FORMS

We keep the notations from chapter 3; so F is again a number field with r real embeddings and s pairs of complex embeddings.

For an ideal $0 \neq \mathfrak{m} \subseteq \mathcal{O}_F$, we let $K_0(\mathfrak{m})_v \subseteq G(\mathcal{O}_{F_v})$ be the subgroup of matrices congruent to an upper triangular matrix modulo \mathfrak{m} , and we set $K_0(\mathfrak{m}) := \prod_{v \nmid \infty} K_0(\mathfrak{m})_v$, $K_0(\mathfrak{m})^S := \prod_{v \nmid \infty, v \notin S} K_0(\mathfrak{m})_v$ for a finite set of primes S . For each $\mathfrak{p} | p$, let $q_{\mathfrak{p}} = N(\mathfrak{p})$ denote the number of elements of the residue class field of $F_{\mathfrak{p}}$.

We denote by $|\cdot|_{\mathbb{C}}$ the square of the usual absolute value on \mathbb{C} , i.e. $|z|_{\mathbb{C}} = z\overline{z}$ for all $z \in \mathbb{C}$, and write $|\cdot|_{\mathbb{R}}$ for the usual absolute value on \mathbb{R} in context. We write $|\alpha| := |\alpha|_{\mathbb{C}}^{\frac{1}{2}}$ for the archimedian absolute value when α is given as a complex number in the context; whereas in the context of the p -adic characters, $|\cdot|$ denotes the p -adic absolute value, unless otherwise noted.

DEFINITION 4.1. Let $\mathfrak{A}_0(G, \underline{2}, \chi_Z)$ denote the set of all *cuspidal automorphic representations* $\pi = \otimes_v \pi_v$ of $G(\mathbb{A}_F)$ with central character χ_Z such that $\pi_v \cong$

$\sigma(| \cdot |_{F_v}^{1/2}, | \cdot |_{F_v}^{-1/2})$ at all archimedean primes v . Here we follow the notation of [JL70]; so $\sigma(| \cdot |_{F_v}^{1/2}, | \cdot |_{F_v}^{-1/2})$ is the discrete series of weight 2, $\mathcal{D}(2)$, if v is real, and is isomorphic to the principal series representation $\pi(\mu_1, \mu_2)$ with $\mu_1(z) = z^{1/2}\bar{z}^{-1/2}$, $\mu_2(z) = z^{-1/2}\bar{z}^{1/2}$ if v is complex (cf. section 4.5 below).

We will only consider automorphic representations that are *p-ordinary*, i.e. $\pi_{\mathfrak{p}}$ is ordinary (in the sense of chapter 2) for every $\mathfrak{p}|p$.

Therefore, for each $\mathfrak{p}|p$ we fix two non-zero elements $\alpha_{\mathfrak{p},1}, \alpha_{\mathfrak{p},2} \in \overline{\mathcal{O}} \subseteq \mathbb{C}$ such that $\pi_{\alpha_{\mathfrak{p},1}, \alpha_{\mathfrak{p},2}}$ is an ordinary, unitary representation. By the classification of unitary representations (see e.g. [Ge75], Thm. 4.27), a spherical representation $\pi_{\alpha_{\mathfrak{p},1}, \alpha_{\mathfrak{p},2}} = \pi(\chi_1, \chi_2)$ is unitary if and only if either χ_1, χ_2 are both unitary characters (i.e. $|\alpha_{\mathfrak{p},1}| = |\alpha_{\mathfrak{p},2}| = \sqrt{q_{\mathfrak{p}}}$), or $\chi_{1,2} = \chi_0 \cdot | \cdot |^{\pm s}$ with χ_0 unitary and $-\frac{1}{2} < s < \frac{1}{2}$. A special representation $\pi_{\alpha_{\mathfrak{p},1}, \alpha_{\mathfrak{p},2}} = \pi(\chi_1, \chi_2)$ is unitary if and only if the central character $\chi_1\chi_2$ is unitary. In all three cases, we have thus $\max\{|\alpha_{\mathfrak{p},1}|, |\alpha_{\mathfrak{p},2}|\} \geq \sqrt{q_{\mathfrak{p}}}$. Without loss of generality, we will assume the $\alpha_{\mathfrak{p},i}$ to be ordered such that $|\alpha_{\mathfrak{p},1}| \leq |\alpha_{\mathfrak{p},2}|$ for all $\mathfrak{p}|p$.

As in chapter 2, we define $a_{\mathfrak{p}} := \alpha_{\mathfrak{p},1} + \alpha_{\mathfrak{p},2}$, $\nu_{\mathfrak{p}} := \alpha_{\mathfrak{p},1}\alpha_{\mathfrak{p},2}/q_{\mathfrak{p}}$.

Let $\underline{\alpha}_i := (\alpha_{\mathfrak{p},i}, \mathfrak{p}|p)$, for $i = 1, 2$. We denote by $\mathfrak{A}_0(G, \underline{2}, \chi_Z, \underline{\alpha}_1, \underline{\alpha}_2)$ the subset of all $\pi \in \mathfrak{A}_0(G, \underline{2}, \chi_Z)$ such that $\pi_{\mathfrak{p}} = \pi_{\alpha_{\mathfrak{p},1}, \alpha_{\mathfrak{p},2}}$ for all $\mathfrak{p}|p$.

For later use we note that $\pi^\infty = \otimes_{v \nmid \infty} \pi_v$ is known to be defined over a finite extension of \mathbb{Q} , the smallest such field being the *field of definition* of π (cf. [Sp14]).

4.1 UPPER HALF-SPACE

For $k \in \{\mathbb{R}, \mathbb{C}\}$, let $\mathcal{H}_m := \mathcal{H}_k := k \times \mathbb{R}_+^*$ be the upper half-space of dimension $m := [k : \mathbb{R}] + 1$. Each \mathcal{H}_m is a differentiable manifold of dimension m . If we write $x = (u, t) \in \mathcal{H}_m$ with $t \in \mathbb{R}_+^*$, u in \mathbb{R} or \mathbb{C} , respectively, it has a Riemannian metric $ds^2 = \frac{dt^2 + du d\bar{u}}{t}$, which induces a hyperbolic geometry on \mathcal{H}_m , i.e. the geodesic lines on \mathcal{H}_m are given by “vertical” lines $\{u\} \times \mathbb{R}_+^*$ and half-circles with center in the line or plane $t = 0$. $\mathcal{H}_{\mathbb{R}}$ is naturally isomorphic to the complex upper half-plane $\{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$.

We have the decompositions $\text{GL}_2(\mathbb{C})^+ = B'_C \cdot Z(\mathbb{C}) \cdot K_C$ and $\text{GL}_2(\mathbb{R})^+ = B'_R \cdot Z(\mathbb{R}) \cdot K_R$, where $B'_k \subseteq \text{GL}_2(k)$ is the subgroup of matrices $\begin{pmatrix} \mathbb{R}_+^* & k \\ 0 & 1 \end{pmatrix}$ for $k = \mathbb{R}, \mathbb{C}$, Z is the center, and $K_{\mathbb{R}} = \text{SO}(2)$, $K_{\mathbb{C}} = \text{SU}(2)$ (cf. [By98], Cor. 43). Identifying B'_k with \mathcal{H}_k via $\begin{pmatrix} t & z \\ 0 & 1 \end{pmatrix} \mapsto (z, t)$ gives natural projections

$$\pi_{\mathbb{R}} : \text{GL}_2(\mathbb{R})^+ \twoheadrightarrow \text{GL}_2(\mathbb{R})^+ / Z(\mathbb{R}) \text{SO}(2) \cong \mathcal{H}_{\mathbb{R}},$$

$$\pi_{\mathbb{C}} : \text{GL}_2(\mathbb{C}) \twoheadrightarrow \text{GL}_2(\mathbb{C}) / Z(\mathbb{C}) K_{\mathbb{C}} \cong \mathcal{H}_{\mathbb{C}}$$

and corresponding left $\text{GL}_2(k)$ -actions on cosets.

A differential form ω on \mathcal{H}_m is called *left-invariant* if it is invariant under the pullback L_g^* of left multiplication $L_g : x \mapsto gx$ on \mathcal{H}_m , for all $g \in G$.

Following [By98], eqs. (4.20), (4.24), we choose the following basis of left-invariant differential 1-forms on \mathcal{H}_3 :

$$\beta_0 := -\frac{dz}{t}, \quad \beta_1 := \frac{dt}{t}, \quad \beta_2 := \frac{d\bar{z}}{t},$$

and on \mathcal{H}_2 (writing $z = x + iy \in \mathcal{H}_2 \subseteq \mathbb{C}$):

$$\beta_1 := \frac{dz}{y}, \quad \beta_2 := -\frac{d\bar{z}}{y}.$$

We note that a form $f_1\beta_1 + f_2\beta_2$ is harmonic on \mathcal{H}_2 if and only if f_1/y and f_2/y are holomorphic functions in z ([By98], lemma 60).

The Jacobian $J(g, (0, 1))$ of left multiplication by g in $(0, 1) \in \mathcal{H}_m$ with respect to the basis $(\beta_i)_i$ gives rise to a representation

$$\varrho = \varrho_k : Z(k) \cdot K_k \rightarrow \mathrm{SL}_m(\mathbb{C})$$

with $\varrho|_{Z(k)}$ trivial, which on K_k is explicitly given by

$$\varrho_{\mathbb{C}}(h) = \begin{pmatrix} u^2 & 2uv & v^2 \\ -u\bar{v} & u\bar{u} - v\bar{v} & v\bar{u} \\ \bar{v}^2 & -2\bar{u}\bar{v} & \bar{u}^2 \end{pmatrix} \quad \text{for } h = \begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix} \in \mathrm{SU}(2),$$

resp.

$$\varrho_{\mathbb{R}} \begin{pmatrix} \cos(\vartheta) & \sin(\vartheta) \\ -\sin(\vartheta) & \cos(\vartheta) \end{pmatrix} = \begin{pmatrix} e^{2i\vartheta} & 0 \\ 0 & e^{-2i\vartheta} \end{pmatrix}$$

([By98], (4.27), (4.21)). In the real case, we will only consider harmonic forms on \mathcal{H}_2 that are multiples of β_1 , thus we sometimes identify $\varrho_{\mathbb{R}}$ with its restriction $\varrho_{\mathbb{R}}^{(1)}$ to the first basis vector β_1 ,

$$\varrho_{\mathbb{R}}^{(1)} : \mathrm{SO}(2) \rightarrow S^1 \subseteq \mathbb{C}^*, \quad \kappa_{\vartheta} = \begin{pmatrix} \cos(\vartheta) & \sin(\vartheta) \\ -\sin(\vartheta) & \cos(\vartheta) \end{pmatrix} \mapsto e^{2i\vartheta}.$$

For each i , let ω_i be the left-invariant differential 1-form on $\mathrm{GL}_2(k)$ which coincides with the pullback $(\pi_{\mathbb{C}})^*\beta_i$ at the identity. Write $\underline{\omega}$ (resp. $\underline{\beta}$) for the column vector of the ω_i (resp. β_i). Then we have the following lemma from [By98]:

LEMMA 4.2. *For each i , the differential ω_i on G induces β_i on \mathcal{H}_m , by restriction to the subgroup $B'_k \cong \mathcal{H}_m$. For a function $\phi : G \rightarrow \mathbb{C}^m$, the form $\phi \cdot \underline{\omega}$ (with \mathbb{C}^m considered as a row vector, so \cdot is the scalar product of vectors) induces $f \cdot \underline{\beta}$, where $f : \mathcal{H}_m \rightarrow \mathbb{C}^m$ is given by*

$$f(z, t) := \phi \left(\begin{pmatrix} t & z \\ 0 & 1 \end{pmatrix} \right).$$

(See [By98], Lemma 57.)

To consider the infinite primes of F all at once, we define

$$\mathcal{H}_\infty := \prod_{i=0}^d \mathcal{H}_{m_i} = \prod_{i=0}^{r-1} \mathcal{H}_2 \times \prod_{i=r}^d \mathcal{H}_3$$

(where $m_i = 2$ if σ_i is a real embedding, and $m_i = 3$ if σ_i is complex), and let $\mathcal{H}_\infty^0 := \prod_{i=1}^d \mathcal{H}_{m_i}$ be the product with the zeroth factor removed. (The choice of the 0-th factor is for convenience; we could also choose any other infinite place, whether real or complex.)

For each embedding σ_i , the elements of $\mathbb{P}^1(F)$ are cusps of \mathcal{H}_{m_i} : for a given complex embedding $F \hookrightarrow \mathbb{C}$, we can identify F with $F \times \{0\} \hookrightarrow \mathbb{C} \times \mathbb{R}_{\geq 0}$ and define the "extended upper half-space" as $\overline{\mathcal{H}}_3 := \mathcal{H}_3 \cup F \cup \{\infty\} \subseteq \mathbb{C} \times \mathbb{R}_{\geq 0} \cup \{\infty\}$; similarly for a given real embedding $F \hookrightarrow \mathbb{R}$, we get the extended upper half-plane $\overline{\mathcal{H}}_2 := \mathcal{H}_2 \cup F \cup \{\infty\}$. A basis of neighbourhoods of the cusp ∞ is given by the sets $\{(u, t) \in \mathcal{H}_m | t > N\}$, $N \gg 0$, and of $x \in F$ by the open half-balls in \mathcal{H}_m with center $(x, 0)$.

Let $G(F)^+ \subseteq G(F)$ denote the subgroup of matrices with totally positive determinant. It acts on \mathcal{H}_∞^0 by composing the embedding

$$G(F)^+ \hookrightarrow \prod_{v|\infty, v \neq v_0} G(F_v)^+, \quad g \mapsto (\sigma_1(g), \dots, \sigma_d(g)),$$

with the actions of $G(\mathbb{C})^+ = G(\mathbb{C})$ on \mathcal{H}_3 and $G(\mathbb{R})^+$ on \mathcal{H}_2 as defined above, and on $\Omega_{\text{harm}}^d(\mathcal{H}_\infty^0)$ by the inverse of the corresponding pullback, $\gamma \cdot \underline{\omega} := (\gamma^{-1})^* \underline{\omega}$. Both are left actions.

For each complex v , we write the codomain of ϱ_{F_v} as

$$\varrho_{F_v} : Z(F_v) \cdot K_{F_v} \rightarrow \text{SL}_3(\mathbb{C}) =: \text{SL}(V_v),$$

for a three-dimensional \mathbb{C} -vector space V_v . We denote the harmonic forms on $\text{GL}_2(F_v)$, \mathcal{H}_{F_v} defined above by $\underline{\omega}_v, \underline{\beta}_v$ etc.

Let $V = \bigotimes_{v \in S_{\mathbb{C}}} V_v \cong (\mathbb{C}^3)^{\otimes s}$, $Z_\infty = \prod_{v|\infty} Z(F_v)$, $K_\infty = \prod_{v|\infty} K_{F_v}$. Denoting by $S_{\mathbb{C}}$ (resp. $S_{\mathbb{R}}$) the set of complex (resp. real) archimedean primes of F , we can merge the representations ϱ_{F_v} for each $v|\infty$ into a representation

$$\varrho = \varrho_\infty := \bigotimes_{v \in S_{\mathbb{C}}} \varrho_{\mathbb{C}} \otimes \bigotimes_{v \in S_{\mathbb{R}}} \varrho_{\mathbb{R}}^{(1)} : Z_\infty \cdot K_\infty \rightarrow \text{SL}(V),$$

and define V -valued vectors of differential forms

$$\underline{\omega} := \bigotimes_{v \in S_{\mathbb{C}}} \underline{\omega}_v \otimes \bigotimes_{v \in S_{\mathbb{R}}} \underline{\omega}_v^1, \quad \underline{\beta} := \bigotimes_{v \in S_{\mathbb{C}}} \underline{\beta}_v \otimes \bigotimes_{v \in S_{\mathbb{R}}} (\underline{\beta}_v)_1$$

on $\text{GL}_2(F_\infty)$ and \mathcal{H}_∞ , respectively.

4.2 AUTOMORPHIC FORMS

Let $\chi_Z : \mathbb{A}_F^*/F^* \rightarrow \mathbb{C}^*$ be a Hecke character that is trivial at the archimedean places. We also denote by χ_Z the corresponding character on $Z(\mathbb{A}_F)$ under the isomorphism $\mathbb{A}_F^* \rightarrow Z(\mathbb{A}_F)$, $a \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$.

DEFINITION 4.3. An automorphic cusp form of parallel weight $\underline{2}$ with central character χ_Z is a map $\phi : G(\mathbb{A}_F) \rightarrow V$ such that

- (i) $\phi(z\gamma g) = \chi_Z(z)\phi(g)$ for all $g \in G(\mathbb{A})$, $z \in Z(\mathbb{A})$, $\gamma \in G(F)$.
- (ii) $\phi(gk_\infty) = \phi(g)\varrho(k_\infty)$ for all $k_\infty \in K_\infty$, $g \in G(\mathbb{A})$ (considering V as a row vector).
- (iii) ϕ has “moderate growth“ on $B'_\mathbb{A} := \left\{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \in G(\mathbb{A}) \right\}$, i.e. $\exists C, \lambda \forall A \in B'_\mathbb{A} : \|\phi(A)\| \leq C \cdot \sup(|y|^\lambda, |y|^{-\lambda})$ (for any fixed norm $\|\cdot\|$ on V);
and $\phi|_{G(\mathbb{A}_\infty)} \cdot \underline{\omega}$ is the pullback of a harmonic form $\omega_\phi = f_\phi \cdot \underline{\beta}$ on \mathcal{H}_∞ .
- (iv) There exists a compact open subgroup $K' \subseteq G(\mathbb{A}^\infty)$ such that $\phi(gk) = \phi(g)$ for all $g \in G(\mathbb{A})$ and $k \in K'$.
- (v) For all $g \in G(\mathbb{A}_F)$,

$$\int_{\mathbb{A}_F/F} \phi \left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) dx = 0. \quad (\text{“Cuspidality”})$$

We denote by $\mathcal{A}_0(G, \text{harm}, \underline{2}, \chi_Z)$ the space of all such maps ϕ .

For each $g^\infty \in \mathbb{A}_F^\infty$, let $\omega_\phi(g^\infty)$ be the restriction of $\phi(g^\infty, \cdot) \cdot \underline{\omega}$ from $G(\mathbb{A}_F^\infty)$ to \mathcal{H}_∞ ; it is a $(d+1)$ -form on \mathcal{H}_∞ .

We want to integrate $\omega_\phi(g^\infty)$ between two cusps of the space \mathcal{H}_{m_0} . (We will identify each $x \in \mathbb{P}^1(F)$ with its corresponding cusp in $\overline{\mathcal{H}_{m_0}}$ in the following.) The geodesic between the cusps $x \in F$ and ∞ in $\overline{\mathcal{H}_{m_0}}$ is the line $\{x\} \times \mathbb{R}_+^* \subseteq \mathcal{H}_{m_0}$ and the integral of ω_ϕ along it is finite since ϕ is uniformly rapidly decreasing:

THEOREM 4.4. (Gelfand, Piatetski-Shapiro) An automorphic cusp form ϕ is rapidly decreasing modulo the center on a fundamental domain \mathcal{F} of $\text{GL}_2(F) \backslash \text{GL}_2(\mathbb{A}_F)$; i.e. there exists an integer r such that for all $N \in \mathbb{N}$ there exists a $C > 0$ such that

$$\phi(zg) \leq C|z|^r \|g\|^{-N}$$

for all $z \in Z(\mathbb{A}_F)$, $g \in \mathcal{F} \cap \text{SL}_2(\mathbb{A}_F)$. Here $\|g\| := \max\{|g_{i,j}|, |(g^{-1})_{i,j}|\}_{i,j \in \{1,2\}}$.

(See [CKM04], Thm. 2.2; or [Kur78], (6) for quadratic imaginary F .)
 In fact, the integral of $\omega_\phi(g^\infty)$ along $\{x\} \times \mathbb{R}_+^* \subseteq \mathcal{H}_{m_0}$ equals the integral of $\phi(g^\infty, \cdot) \cdot \underline{\omega}$ along a path $g_t \in \mathrm{GL}_2(F_{\infty_0})$, $t \in \mathbb{R}_+^*$, where we can choose

$$g_t = \frac{1}{\sqrt{t}} \begin{pmatrix} t & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{t}} & \frac{x}{\sqrt{t}} \\ 0 & \sqrt{t} \end{pmatrix},$$

and thus have $\|g_t\| = \sqrt{t}$ for all $t \gg 0$, $\|g_t\| = C\frac{1}{\sqrt{t}}$ for $t \ll 1$, so the integral $\int_x^\infty \omega_\phi(g^\infty) \in \Omega_{\mathrm{harm}}^d(\mathcal{H}_\infty^0)$ is well-defined by the theorem.
 For any two cusps $a, b \in \mathbb{P}^1(F)$, we now define

$$\int_a^b \omega_\phi(g^\infty) := \int_a^\infty \omega_\phi(g^\infty) - \int_b^\infty \omega_\phi(g^\infty) \in \Omega_{\mathrm{harm}}^d(\mathcal{H}_\infty^0).$$

Since ϕ is *uniformly* rapidly decreasing ($\|g_t\|$ does not depend on x , for $t \gg 0$), this integral along the path $(a, 0) \rightarrow (a, \infty) = (b, \infty) \rightarrow (b, 0)$ in $\overline{\mathcal{H}}_{m_0}$ is the same as the limit (for $t \rightarrow \infty$) of the integral along $(a, 0) \rightarrow (a, t) \rightarrow (b, t) \rightarrow (b, 0)$; and since ω_ϕ is harmonic (and thus integration is path-independent within \mathcal{H}_{m_0}) the latter is in fact independent of t , so equality holds for each $t > 0$, or along any path from $(a, 0)$ to $(b, 0)$ in \mathcal{H}_{m_0} . Thus $\int_a^b \omega_\phi(g^\infty)$ equals the integral of $\omega_\phi(g^\infty)$ along the geodesic from a to b , and we have

$$\int_a^b \omega_\phi(g^\infty) + \int_b^c \omega_\phi(g^\infty) = \int_a^c \omega_\phi(g^\infty)$$

for any three cusps $a, b, c \in \mathbb{P}^1(F)$. Let $\mathrm{Div}(\mathbb{P}^1(F))$ denote the free abelian group of divisors of $\mathbb{P}^1(F)$, and let $\mathcal{M} := \mathrm{Div}_0(\mathbb{P}^1(F))$ be the subgroup of divisors of degree 0.

We can extend the definition of the integral linearly to get a homomorphism

$$\mathcal{M} \rightarrow \Omega_{\mathrm{harm}}^d(\mathcal{H}_\infty^0), \quad m \mapsto \int_m \omega_\phi(g^\infty),$$

and easily check that

$$\gamma^* \left(\int_{\gamma m} \omega_\phi(\gamma g) \right) = \int_m \omega_\phi(g). \tag{29}$$

for all $\gamma \in G(F)^+$, $g \in G(\mathbb{A}^\infty)$, $m \in \mathcal{M}$.

Now let \mathfrak{m} be an ideal of F prime to p , let χ_Z be a Hecke character of conductor dividing \mathfrak{m} , and $\underline{\alpha}_1, \underline{\alpha}_2$ as above.

DEFINITION 4.5. We define $S_2(G, \mathfrak{m}, \underline{\alpha}_1, \underline{\alpha}_2)$ to be the \mathbb{C} -vector space of all maps

$$\Phi : G(\mathbb{A}^p) \rightarrow \mathcal{B}^{\underline{\alpha}_1, \underline{\alpha}_2}(F_p, V) = \mathrm{Hom}(\mathcal{B}_{\underline{\alpha}_1, \underline{\alpha}_2}(F_p, \mathbb{C}), V)$$

such that:

- (a) ϕ is “almost” $K_0(\mathfrak{m})$ -invariant (in the notation of [Ge75]), i.e. $\phi(gk) = \phi(g)$ for all $g \in G(\mathbb{A}^p)$ and $k \in \prod_{v \nmid \mathfrak{m}p} G(\mathcal{O}_v)$, and $\phi(gk) = \chi_Z(a)\phi(g)$ for all $v|\mathfrak{m}$, $k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(\mathfrak{m})_v$ and $g \in G(\mathbb{A}^p)$.

- (b) For each $\psi \in \mathcal{B}_{\underline{\alpha}_1, \underline{\alpha}_2}(F_p, \mathbb{C})$, the map

$$\langle \Phi, \psi \rangle : G(\mathbb{A}) = G(F_p) \times G(\mathbb{A}^p) \rightarrow V, (g_p, g^p) \mapsto \Phi(g^p)(g_p\psi)$$

lies in $\mathcal{A}_0(G, \text{harm}, \underline{2}, \chi_Z)$.

Note that (a) implies that ϕ is K' -invariant for some open subgroup $K' \subseteq K_0(\mathfrak{m})^p$ of finite index ([By98]/[We71]).

4.3 COHOMOLOGY OF $\text{GL}_2(F)$

Let M be a left $G(F)$ -module and N an $R[H]$ -module, for a ring R and a subgroup $H \subseteq G(F)$. Let $S \subseteq S_p$ be a set of primes of F dividing p ; as above, let $\chi = \chi_Z$ be a Hecke character of conductor \mathfrak{m} prime to p .

DEFINITION 4.6. For a compact open subgroup $K \subseteq K_0(\mathfrak{m})^S \subseteq G(\mathbb{A}^{S, \infty})$, we denote by $\mathcal{A}_f(K, S, M; N)$ the R -module of all maps $\Phi : G(\mathbb{A}^{S, \infty}) \times M \rightarrow N$ such that

1. $\Phi(gk, m) = \Phi(g, m)$ for all $g \in G(\mathbb{A}^{S, \infty})$, $m \in M$, $k \in \prod_{v \nmid \mathfrak{m}p} G(\mathcal{O}_v)$;
2. $\Phi(gk) = \chi_Z(a)\Phi(g)$ for all $v|\mathfrak{m}$, $k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(\mathfrak{m})_v$ and $g \in G(\mathbb{A}^{S, \infty})$, $m \in M$.

We denote by $\mathcal{A}_f(S, M; N)$ the union of the $\mathcal{A}_f(K, S, M; N)$ over all compact open subgroups K .

$\mathcal{A}_f(S, M; N)$ is a left $G(\mathbb{A}^{S, \infty})$ -module via $(\gamma \cdot \Phi)(g, m) := \Phi(\gamma^{-1}g, m)$ and has a left H -operation given by $(\gamma \cdot \Phi)(g, m) := \gamma\Phi(\gamma^{-1}g, \gamma^{-1}m)$, commuting with the $G(\mathbb{A}^{S, \infty})$ -operation.

In contrast to our previous notation, we consider two subsets $S_1 \subseteq S_2 \subseteq S_p$ in this section. We put $(\underline{\alpha}_1, \underline{\alpha}_2)_{S_1} := \{(\alpha_{\mathfrak{p},1}, \alpha_{\mathfrak{p},2}) | \mathfrak{p} \in S_1\}$, we set

$$\mathcal{A}_f((\underline{\alpha}_1, \underline{\alpha}_2)_{S_1}, S_2, M; N) = \mathcal{A}_f(S_2, M; \mathcal{B}^{(\underline{\alpha}_1, \underline{\alpha}_2)_{S_1}}(F_{S_1}, N));$$

we write $\mathcal{A}_f(\mathfrak{m}, (\underline{\alpha}_1, \underline{\alpha}_2)_{S_1}, S_2, M; N) := \mathcal{A}_f(K_0(\mathfrak{m}), (\underline{\alpha}_1, \underline{\alpha}_2)_{S_1}, S_2, M; N)$. If $S_1 = S_2$, we will usually drop S_2 from all these notations.

We have a natural identification of $\mathcal{A}_f(\mathfrak{m}, (\underline{\alpha}_1, \underline{\alpha}_2)_S, M; N)$ with the space of maps $G(\mathbb{A}^{S, \infty}) \times M \times \mathcal{B}^{(\underline{\alpha}_1, \underline{\alpha}_2)_S}(F_S, R) \rightarrow N$ that are “almost” K -invariant.

Let $S_0 \subseteq S_1 \subseteq S_2 \subseteq S_p$ be subsets. The pairing (11) induces a pairing

$$\mathcal{A}_f((\underline{\alpha}_1, \underline{\alpha}_2)_{S_1}, S_2, M; N) \times \mathcal{B}^{(\underline{\alpha}_1, \underline{\alpha}_2)_{S_0}}(F_{S_0}, R) \rightarrow \mathcal{A}_f((\underline{\alpha}_1, \underline{\alpha}_2)_{S_0}, S_2, M; N)$$

which, when restricting to *K*-invariant elements, induces an isomorphism

$$\mathcal{A}_f(K, (\underline{\alpha}_1, \underline{\alpha}_2)_{S_1}, S_2, M; N) \cong \mathcal{B}^{(\underline{\alpha}_1, \underline{\alpha}_2)_{S_1 - S_0}}(F_{S_1 - S_0}, \mathcal{A}_f(\underline{\alpha}_1, \underline{\alpha}_2)_{S_0}, S_2, M; N).$$

Putting $S_0 := S_1 - \{\mathfrak{p}\}$ for a prime $\mathfrak{p} \in S_1$, we specifically get an isomorphism

$$\mathcal{A}_f(K, (\underline{\alpha}_1, \underline{\alpha}_2)_{S_1}, S_2, M; N) \cong \mathcal{B}^{\alpha_{\mathfrak{p},1}, \alpha_{\mathfrak{p},2}}(F_{\mathfrak{p}}, \mathcal{A}_f(\underline{\alpha}_1, \underline{\alpha}_2)_{S_0}, S_2, M; N).$$

Lemmas 2.9 and 2.10 now immediately imply the following:

LEMMA 4.7. *Let $S \subseteq S_p$, $\mathfrak{p} \in S$, $S_0 := S - \{\mathfrak{p}\}$. Let $K \subseteq G(\mathbb{A}^{S,\infty})$ be a compact open subgroup.*

(a) *If $\pi_{\alpha_{\mathfrak{p},1}, \alpha_{\mathfrak{p},2}}$ is spherical, we have exact sequences*

$$0 \rightarrow \mathcal{A}_f(K, (\underline{\alpha}_1, \underline{\alpha}_2)_S, M; N) \rightarrow Z \xrightarrow{N - \nu_{\mathfrak{p}}} Z \rightarrow 0$$

and

$$0 \rightarrow Z \rightarrow \mathcal{A}_f(K_0, (\underline{\alpha}_1, \underline{\alpha}_2)_{S_0}, M; N) \xrightarrow{T - a_{\mathfrak{p}}} \mathcal{A}_f(K_0, (\underline{\alpha}_1, \underline{\alpha}_2)_{S_0}, M; N) \rightarrow 0$$

for a $G(\mathbb{A}^{S_0,\infty})$ -module Z and a compact open subgroup $K_0 = K \times K_{\mathfrak{p}}$ of $G(\mathbb{A}^{S_0,\infty})$.

(b) *If $\pi_{\alpha_{\mathfrak{p},1}, \alpha_{\mathfrak{p},2}}$ is special (with central character $\chi_{\mathfrak{p}}$), we have exact sequences*

$$0 \rightarrow \mathcal{A}_f(K, (\underline{\alpha}_1, \underline{\alpha}_2)_S, M; N) \rightarrow Z' \rightarrow Z \rightarrow 0$$

and

$$\begin{aligned} 0 \rightarrow Z \rightarrow \mathcal{A}_f(K_0, (\underline{\alpha}_1, \underline{\alpha}_2)_{S_0}, M; N)^2 \rightarrow \mathcal{A}_f(K_0, (\underline{\alpha}_1, \underline{\alpha}_2)_{S_0}, M; N)^2 \rightarrow 0, \\ 0 \rightarrow Z' \rightarrow \mathcal{A}_f(K'_0, (\underline{\alpha}_1, \underline{\alpha}_2)_{S_0}, M; N)^2 \rightarrow \mathcal{A}_f(K'_0, (\underline{\alpha}_1, \underline{\alpha}_2)_{S_0}, M; N)^2 \rightarrow 0, \end{aligned}$$

with $Z^{(\prime)} := \mathcal{A}_f(K_0^{(\prime)}, (\underline{\alpha}_1, \underline{\alpha}_2)_{S_0}, S, M; N(\chi_{\mathfrak{p}}))$, where $K_0^{(\prime)} = K \times K_{\mathfrak{p}}^{(\prime)}$ are compact open subgroups of $G(\mathbb{A}^{S_0,\infty})$.

PROPOSITION 4.8. *Let $S \subseteq S_p$ and let K be a compact open subgroup of $G(\mathbb{A}^{S,\infty})$.*

(a) *For each flat R -module N (with trivial $G(F)$ -action), the canonical map*

$$H^q(G(F)^+, \mathcal{A}_f(K, (\underline{\alpha}_1, \underline{\alpha}_2)_S, M; R)) \otimes_R N \rightarrow H^q(G(F)^+, \mathcal{A}_f(K, (\underline{\alpha}_1, \underline{\alpha}_2)_S, M; N))$$

is an isomorphism for each $q \geq 0$.

(b) *If R is finitely generated as a \mathbb{Z} -module, $H^q(G(F)^+, \mathcal{A}_f(K, (\underline{\alpha}_1, \underline{\alpha}_2)_S, M; R))$ is finitely generated over R .*

Proof. We can copy the proof of [Sp14], Prop. 5.6, using lemma 4.7 instead of [Sp14], lemma 5.4 to reduce to the case $S = \emptyset$. □

We define

$$H_*^q(G(F)^+, \mathcal{A}_f((\underline{\alpha}_1, \underline{\alpha}_2)_S, M; R)) := \varinjlim H^q(G(F)^+, \mathcal{A}_f(K, (\underline{\alpha}_1, \underline{\alpha}_2)_S, M; R))$$

where the limit runs over all compact open subgroups $K \subseteq G(\mathbb{A}^{S, \infty})$; and similarly define $H_*^q(B(F)^+, \mathcal{A}_f((\underline{\alpha}_1, \underline{\alpha}_2)_S, \mathcal{M}; R))$. The proposition immediately implies

COROLLARY 4.9. *Let $R \rightarrow R'$ be a flat ring homomorphism. Then the canonical map*

$$H_*^q(G(F)^+, \mathcal{A}_f((\underline{\alpha}_1, \underline{\alpha}_2)_S, \mathcal{M}; R)) \otimes_R R' \rightarrow H_*^q(G(F)^+, \mathcal{A}_f((\underline{\alpha}_1, \underline{\alpha}_2)_S, \mathcal{M}; R'))$$

is an isomorphism, for all $q \geq 0$.

If $R = k$ is a field of characteristic zero, $H_*^q(G(F)^+, \mathcal{A}_f((\underline{\alpha}_1, \underline{\alpha}_2)_S, M; R))$ is a smooth $G(\mathbb{A}^{S, \infty})$ -module, and we have

$$H_*^q(G(F)^+, \mathcal{A}_f((\underline{\alpha}_1, \underline{\alpha}_2)_S, M; k)^K = H^q(G(F)^+, \mathcal{A}_f(K, (\underline{\alpha}_1, \underline{\alpha}_2)_S, M; k)).$$

We identify $G(F)/G(F)^+$ with the group $\Sigma = \{\pm 1\}^r$ via the isomorphism

$$G(F)/G(F)^+ \xrightarrow{\det} F^*/F_+^* \cong \Sigma$$

(with all groups being trivial for $r = 0$). Then Σ acts on $H_*^q(G(F)^+, \mathcal{A}_f((\underline{\alpha}_1, \underline{\alpha}_2)_S, M; k))$ and $H^q(G(F)^+, \mathcal{A}_f(K, (\underline{\alpha}_1, \underline{\alpha}_2)_S, M; k))$ by conjugation. For $\pi \in \mathfrak{A}_0(G, \underline{2})$ and $\underline{\mu} \in \Sigma$, we write $H_*^q(G(F)^+, \cdot)_{\pi, \underline{\mu}} := \text{Hom}_{G(\mathbb{A}^{S, \infty})}(\pi^S, H_*^q(G(F)^+, \cdot))_{\underline{\mu}}$.

PROPOSITION 4.10. *Let $\pi \in \mathfrak{A}_0(G, \underline{2}, \chi_Z, \underline{\alpha}_1, \underline{\alpha}_2)$, $S \subseteq S_p$. Let k be a field which contains the field of definition of π . Then for every $\underline{\mu} \in \Sigma$, we have*

$$H_*^q(G(F)^+, \mathcal{A}_f((\underline{\alpha}_1, \underline{\alpha}_2)_S, \mathcal{M}; k)_{\pi, \underline{\mu}} = \begin{cases} k, & \text{if } q = d; \\ 0, & \text{if } q \in \{0, \dots, d-1\} \end{cases} \quad (30)$$

Proof. The case $S = \emptyset$ is proved analogously to [Sp14], prop. 5.8, using the results of Harder [Ha87]. For $S = S_0 \cup \{\mathfrak{p}\}$ and $\pi_{\mathfrak{p}}$ spherical, lemma 4.7(a) and the statement for S_0 give an isomorphism

$$H_*^q(G(F)^+, \mathcal{A}_f((\underline{\alpha}_1, \underline{\alpha}_2)_{S_0}, \mathcal{M}; k)_{\pi, \underline{\mu}} \cong H_*^q(G(F)^+, \mathcal{A}_f((\underline{\alpha}_1, \underline{\alpha}_2)_S, \mathcal{M}; k)_{\pi, \underline{\mu}}$$

since the Hecke operators $T_{\mathfrak{p}}$ and $N_{\mathfrak{p}}$ act on the left-hand side by multiplication with $a_{\mathfrak{p}}$ and $\nu_{\mathfrak{p}}$, respectively. If $\pi_{\mathfrak{p}}$ is special, we can similarly deduce the statement for S from that for S_0 , using the first exact sequence of lemma 4.7(b), since the results of [Ha87] also hold when twisting k by a (central) character. \square

4.4 EICHLER-SHIMURA MAP

From now on, let $S_1 \subseteq S_p$ be the set of places such that π_p is the Steinberg representation (i.e. $\alpha_{p,1} = \nu_p = 1, \alpha_{p,2} = q$).

Given a subgroup $K_0(\mathfrak{m})^p \subseteq G(\mathbb{A}^{p,\infty})$ as above, there is a map

$$I_0 : S_2(G, \mathfrak{m}, \underline{\alpha}_1, \underline{\alpha}_2) \rightarrow H^0(G(F)^+, \mathcal{A}_f(\mathfrak{m}, \underline{\alpha}_1, \underline{\alpha}_2, \mathcal{M}; \Omega_{\text{harm}}^d(\mathcal{H}_\infty^0)))$$

given by

$$I_0(\Phi) : (\psi, (g, m)) \mapsto \int_m \omega_{\langle \Phi, \psi \rangle}(1_p, g),$$

for $\psi \in \mathcal{B}_{\underline{\alpha}_1, \underline{\alpha}_2}(F_p, \mathbb{C}), g \in G(\mathbb{A}^{p,\infty}), m \in \mathcal{M}$, where 1_p denotes the unity element in $G(F_p)$.

This is well-defined since both sides are “almost” $K_0(\mathfrak{m})$ -invariant, and the $G(F)^+$ -invariance of $I_0(\Phi)$ follows from a straightforward calculation, using (29).

From the complex

$$\mathcal{A}_f(m, \underline{\alpha}_1, \underline{\alpha}_2, \mathcal{M}; \mathbb{C}) \rightarrow C^\bullet := \mathcal{A}_f(\mathfrak{m}, \underline{\alpha}_1, \underline{\alpha}_2, \mathcal{M}; \Omega_{\text{harm}}^\bullet(\mathcal{H}_\infty^0))$$

we get a map

$$S_2(G, \mathfrak{m}, \underline{\alpha}_1, \underline{\alpha}_2) \rightarrow H^d(G(F)^+, \mathcal{A}_f(\mathfrak{m}, \underline{\alpha}_1, \underline{\alpha}_2, \mathcal{M}; \mathbb{C})) \tag{31}$$

by composing I_0 with an edge morphism of the spectral sequence

$$H^q(G(F)^+, C^p) \implies H^{p+q}(G(F)^+, C^\bullet).$$

Using the map $\delta^{\underline{\alpha}_1, \underline{\alpha}_2} : \mathcal{B}^{\underline{\alpha}_1, \underline{\alpha}_2}(F, V) \rightarrow \text{Dist}(F_p^*, V)$ from section 2.6, we next define a map

$$\Delta_{\mathbb{V}}^{\underline{\alpha}_1, \underline{\alpha}_2} : S_2(G, \mathfrak{m}, \underline{\alpha}_1, \underline{\alpha}_2) \rightarrow \mathcal{D}(S_1, V) \tag{32}$$

by

$$\Delta_{\mathbb{V}}^{\underline{\alpha}_1, \underline{\alpha}_2}(\Phi)(U, x^p) = \delta^{\underline{\alpha}_1, \underline{\alpha}_2} \left(\Phi \begin{pmatrix} x^p & 0 \\ 0 & 1 \end{pmatrix} \right) (U)$$

for $U \in \mathfrak{C}\mathfrak{o}(F_{S_1} \times F_{S_2}), x^p \in \mathbb{I}^p$, and we denote by $\Delta^{\underline{\alpha}_1, \underline{\alpha}_2} : S_2(G, \mathfrak{m}, \underline{\alpha}_1, \underline{\alpha}_2) \rightarrow \mathcal{D}(S_1, \mathbb{C})$ its $(1, \dots, 1)$ th coordinate function (i.e. corresponding to the harmonic forms $\otimes_{v|\infty}(\omega_v)_1, \otimes_{v|\infty}(\beta_v)_1$ in section 4.1):

$$\Delta^{\underline{\alpha}_1, \underline{\alpha}_2}(\Phi)(U, x^p) = \delta^{\underline{\alpha}_1, \underline{\alpha}_2} \left(\Phi \begin{pmatrix} x^p & 0 \\ 0 & 1 \end{pmatrix} \right)_{(1, \dots, 1)} (U).$$

Since for each complex prime $v, S^1 \cong \text{SU}(2) \cap T(\mathbb{C})$ operates on Φ via $\varrho_v, \Delta^{\underline{\alpha}_1, \underline{\alpha}_2}$ is easily seen to be S^1 -invariant, i.e. it lies in $\mathcal{D}'(S_1, \mathbb{C})$.

We also have a natural (i.e. commuting with the complex maps of each C^\bullet) family of maps

$$\mathcal{A}_f(\mathfrak{m}, \underline{\alpha}_1, \underline{\alpha}_2, \mathcal{M}, \Omega_{\text{harm}}^i(\mathcal{H}_\infty^0)) \rightarrow \mathcal{D}_f(S_1, \Omega^i(U_\infty^0, \mathbb{C})) \tag{33}$$

for all $i \geq 0$, and

$$\mathcal{A}_f(\mathfrak{m}, \underline{\alpha}_1, \underline{\alpha}_2, \mathcal{M}, \mathbb{C}) \rightarrow \mathcal{D}_f(S_1, \mathbb{C}) \tag{34}$$

(the $i = -1$ -th term in the complexes), by mapping $\Phi \in \mathcal{A}_f(\mathfrak{m}, \underline{\alpha}_1, \underline{\alpha}_2, \mathcal{M}, \cdot)$ first to

$$(U, x^{p, \infty}) \mapsto \Phi \left(\begin{pmatrix} x^{p, \infty} & 0 \\ 0 & 1 \end{pmatrix}, \infty - 0 \right) (\delta_{\underline{\alpha}_1, \underline{\alpha}_2}(1_U)) \in \Omega_{\text{harm}}^i(\mathcal{H}_{\infty}^0) \text{ resp. } \in \mathbb{C},$$

and then for $i \geq 0$ restricting the differential forms to $\Omega^i(U_{\infty}^0)$ via

$$U_{\infty}^0 = \prod_{v \in S_{\infty}^0} \mathbb{R}_+^* \hookrightarrow \prod_{v \in S_{\infty}^0} \mathcal{H}_v = \mathcal{H}_{\infty}^0.$$

One easily checks that (33) and (34) are compatible with the homomorphism of “acting groups” $F^{*'} \hookrightarrow G(F)^+$, $x \mapsto \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$, so we get induced maps in cohomology

$$H^0(G(F)^+, \mathcal{A}_f(\mathfrak{m}, \underline{\alpha}_1, \underline{\alpha}_2, \mathcal{M}, \Omega_{\text{harm}}^d(\mathcal{H}_{\infty}^0))) \rightarrow H^0(F^{*'}, \mathcal{D}_f(S_1, \Omega^d(U_{\infty}^0, \mathbb{C}))) \tag{35}$$

and

$$H^d(G(F)^+, \mathcal{A}_f(\mathfrak{m}, \underline{\alpha}_1, \underline{\alpha}_2, \mathcal{M}, \mathbb{C})) \rightarrow H^d(F^{*'}, \mathcal{D}_f(S_1, \mathbb{C})), \tag{36}$$

which are linked by edge morphisms of the respective spectral sequences to give a commutative diagram (given in the proof below).

PROPOSITION 4.11. *We have a commutative diagram:*

$$\begin{array}{ccc} S_2(G, \mathfrak{m}, \underline{\alpha}_1, \underline{\alpha}_2) & \xrightarrow{(31)} & H^d(G(F)^+, \mathcal{A}_f(\mathfrak{m}, \underline{\alpha}_1, \underline{\alpha}_2, \mathcal{M}, \mathbb{C})) \\ \downarrow \Delta^{\underline{\alpha}_1, \underline{\alpha}_2} & & \downarrow (36) \\ \mathcal{D}'(S_1, \mathbb{C}) & \xrightarrow{\phi \mapsto \kappa_{\phi}} & H^d(F^{*'}, \mathcal{D}_f(S_1, \mathbb{C})) \end{array}$$

Proof. The given diagram factorizes as

$$\begin{array}{ccccc} S_2(G, \mathfrak{m}, \underline{\alpha}_1, \underline{\alpha}_2) & \xrightarrow{I_0} & H^0(G(F)^+, \mathcal{A}_f(\Omega_{\text{harm}}^d(\mathcal{H}_{\infty}^0))) & \longrightarrow & H^d(G(F)^+, \mathcal{A}_f(\mathbb{C})) \\ \downarrow \Delta^{\underline{\alpha}_1, \underline{\alpha}_2} & & \downarrow (35) & & \downarrow (36) \\ \mathcal{D}'(S_1, \mathbb{C}) & \longrightarrow & H^0(F^{*'}, \mathcal{D}_f(S_1, \Omega^d(U_{\infty}^0, \mathbb{C}))) & \longrightarrow & H^d(F^{*'}, \mathcal{D}_f(S_1, \mathbb{C})) \end{array}$$

(where we write $\mathcal{A}_f(\cdot)$ instead of $\mathcal{A}_f(\mathfrak{m}, \underline{\alpha}_1, \underline{\alpha}_2, \mathcal{M}, \cdot)$ for brevity). The right-hand square is the naturally commutative square mentioned above; the commutativity of the left-hand square can easily be checked by hand. \square

4.5 WHITTAKER MODEL

We now consider an automorphic representation $\pi = \otimes_v \pi_v \in \mathfrak{A}_0(G, \underline{2}, \chi_Z, \underline{\alpha}_1, \underline{\alpha}_2)$. Denote by $\mathfrak{c}(\pi) := \prod_{v \text{ finite}} \mathfrak{c}(\pi_v)$ the conductor of π .

Let $\chi : \mathbb{I}^\infty \rightarrow \mathbb{C}^*$ be a unitary character of the finite ideles; for each finite place v , set $\chi_v = \chi|_{F_v^*}$. For each prime v of F , let \mathcal{W}_v denote the Whittaker model of π_v . For each finite and each real prime, we choose $W_v \in \mathcal{W}_v$ such that the local L-factor equals the local zeta function at $g = 1$, i.e. such that

$$L(s, \pi_v \otimes \chi_v) = \int_{F_v^*} W_v \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \chi_v(x) |x|^{s-\frac{1}{2}} d^\times x \tag{37}$$

for any unramified quasi-character $\chi_v : F_v^* \rightarrow \mathbb{C}^*$ and $\text{Re}(s) \gg 0$.

This is possible by [Ge75], Thm. 6.12 (ii); and by loc.cit., Prop. 6.17, W_v can be chosen such that $\text{SO}(2)$ operates on W_v via ϱ_v for real archimedean v , and is “almost” $K_0(\mathfrak{c}(\pi_v))$ -invariant for finite v .

For complex primes v of F , we can also choose a W_v satisfying (37) and which behaves well with respect to the $\text{SU}(2)$ -action ϱ_v , as follows:

By [Kur77], there exists a function

$$\underline{W}_v = (W_v^0, W_v^1, W_v^2) : G(F_v) \rightarrow \mathbb{C}^3$$

such that $W_v^i \in \mathcal{W}_v$ for all i , and such that $\text{SU}(2)$ operates by the right via ϱ_v on \underline{W}_v ; i.e. for all $g \in G(F_v)$ and $h \in \text{SU}(2)$, we have

$$\underline{W}_v(gh) = \underline{W}_v(g)\varrho_{\mathbb{C}}(h).$$

Note that W_v^1 is thus invariant under right multiplication by a diagonal matrix $\begin{pmatrix} u & 0 \\ 0 & \bar{u} \end{pmatrix}$ with $u \in S^1 \subseteq \mathbb{C}$. Since π_v has trivial central character for archimedean v by our assumption, a function in \mathcal{W}_v is also invariant under $Z(F_v)$. Thus we have

$$W_v^1 \left(g \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \right) = W_v^1(g) \quad \text{for all } g \in G(F_v), u \in S^1.$$

W_v^1 can be described explicitly in terms of a certain Bessel function, as follows. The modified Bessel differential equation of order $\alpha \in \mathbb{C}$ is

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - (x^2 + \alpha^2)y = 0.$$

Its solution space (on $\{\text{Re } z > 0\}$) is two-dimensional; we are only interested in the second standard solution K_v , which is characterised by the asymptotics

$$K_v(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z}$$

(cf. [We71]). By [Kur77],³ we have $W_v^1 \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} = \frac{2}{\pi} x^2 K_0(4\pi x)$.

(W_v^0 and W_v^2 can also be described in terms of Bessel functions; they are linearly dependent and scalar multiples of $x^2 K_1(4\pi x)$.)

By [JL70], Ch. 1, Thm. 6.2(vi), $\sigma(| \cdot |_{\mathbb{C}}^{1/2}, | \cdot |_{\mathbb{C}}^{-1/2}) \cong \pi(\mu_1, \mu_2)$ with

$$\mu_1(z) = z^{1/2} \bar{z}^{-1/2} = |z|_{\mathbb{C}}^{-1/2} z, \quad \mu_2(z) = z^{-1/2} \bar{z}^{1/2} = |z|_{\mathbb{C}}^{-1/2} \bar{z};$$

and the L-series of the representation is the product of the L-factors of these two characters:

$$\begin{aligned} L_v(s, \pi_v) = L(s, \mu_1)L(s, \mu_2) &= 2(2\pi)^{-(s+\frac{1}{2})}\Gamma(s+\frac{1}{2}) \cdot 2(2\pi)^{-(s+\frac{1}{2})}\Gamma(s+\frac{1}{2}) \\ &= 4(2\pi)^{-(2s+1)}\Gamma(s+\frac{1}{2})^2. \end{aligned}$$

On the other hand, letting $d^\times x = \frac{dx}{|x|_{\mathbb{C}}} = \frac{dr}{r} d\vartheta$ (for $x = re^{i\vartheta}$), we have for $\operatorname{Re}(s) > -\frac{1}{2}$:

$$\begin{aligned} \int_{\mathbb{C}^*} W_v^1 \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} |x|_{\mathbb{C}}^{s-\frac{1}{2}} d^\times x &= \int_{S^1} \int_{\mathbb{R}_+} W_v^1 \begin{pmatrix} re^{i\vartheta} & 0 \\ 0 & 1 \end{pmatrix} |x|_{\mathbb{C}}^{s-\frac{1}{2}} \frac{dr}{r} d\vartheta \\ &= 4 \int_0^\infty x^2 K_0(4\pi x) x^{2s-1} \frac{dx}{x} \\ &\text{(invariance under } \operatorname{SU}(2) \cdot Z(F_v) \text{ gives a constant integral w.r.t. } \vartheta) \\ &= 4(4\pi)^{-2s+1} \int_0^\infty K_0(x) x^{2s} dx \\ &= 4(4\pi)^{-2s+1} 2^{2s-1} \Gamma(s+\frac{1}{2})^2 \\ &= 4(2\pi)^{-2s+1} \Gamma(s+\frac{1}{2})^2 \end{aligned}$$

by ([DLMF] 10.43.19). Thus we have

$$\int_{\mathbb{C}^*} W_v^1 \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} |x|_{\mathbb{C}}^{s-\frac{1}{2}} d^\times x = (2\pi)^2 L_v(s, \pi_v)$$

for all $\operatorname{Re}(s) > -\frac{1}{2}$. We set $W_v := (2\pi)^{-2} W_v^1$; thus (37) holds also for complex primes.

Now that we have defined W_v for all primes v , we put $W^P(g) := \prod_{v \nmid p} W_v(g_v)$ for all $g = (g_v)_v \in G(\mathbb{A}^P)$. We will also need the vector-valued function $\underline{W}^P : G(\mathbb{A}_F) \rightarrow V$ given by

$$\underline{W}^P(g) := \prod_{v \nmid p \text{ finite or } v \text{ real}} W_v(g_v) \cdot \bigotimes_{v \text{ complex}} (2\pi)^{-2} \underline{W}_v(g_v).$$

³Note that [Kur77] uses a slightly different definition of the K_v , which is $\frac{2}{\pi}$ times our K_v .

4.6 p -ADIC MEASURES OF AUTOMORPHIC FORMS

Now return to our $\pi \in \mathfrak{A}_0(G, \underline{2}, \chi_Z, \underline{\alpha}_1, \underline{\alpha}_2)$. We fix an additive character $\psi : \mathbb{A} \rightarrow \mathbb{C}^*$ which is trivial on F , and let ψ_v denote the restriction of ψ to $F_v \hookrightarrow \mathbb{A}$, for all primes v . We further require that $\ker(\psi_{\mathfrak{p}}) \supseteq \mathcal{O}_{\mathfrak{p}}$ and $\mathfrak{p}^{-1} \not\subseteq \ker \psi_{\mathfrak{p}}$ for all $\mathfrak{p}|p$, so that we can apply the results of chapter 2.

As in chapter 2, let $\mu_{\pi_{\mathfrak{p}}} := \mu_{\alpha_{\mathfrak{p},1}/\nu_{\mathfrak{p}}} = \mu_{q_{\mathfrak{p}}/\alpha_{\mathfrak{p},2}}$ denote the distribution $\chi_{q_{\mathfrak{p}}/\alpha_{\mathfrak{p},2}}(x)\psi_{\mathfrak{p}}(x)dx$ on $F_{\mathfrak{p}}$, and let $\mu_{\pi_p} := \prod_{\mathfrak{p}|p} \mu_{\pi_{\mathfrak{p}}}$ be the product distribution on $F_p := \prod_{\mathfrak{p}|p} F_{\mathfrak{p}}$.

Define $\phi = \phi_{\pi} : \mathfrak{C}\mathfrak{o}(F_{S_1} \times F_{S_2}^*) \times \mathbb{I}^p \rightarrow \mathbb{C}$ by

$$\phi(U, x^p) := \sum_{\zeta \in F^*} \mu_{\pi_p}(\zeta U) W^p \begin{pmatrix} \zeta x^p & 0 \\ 0 & 1 \end{pmatrix}.$$

By proposition 2.13(a), we have for each $U \in \mathfrak{C}\mathfrak{o}(F_{S_1} \times F_{S_2}^*)$:

$$\begin{aligned} \phi_U(x) := \phi(x_p U, x^p) &= \sum_{\zeta \in F^*} \mu_{\pi_p}(\zeta x_p U) W^p \begin{pmatrix} \zeta x^p & 0 \\ 0 & 1 \end{pmatrix} \\ &= \sum_{\zeta \in F^*} W \begin{pmatrix} \zeta x & 0 \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

where $W(g) := W_U(g_p)W^p(g^p)$ lies in the global Whittaker model $\mathcal{W} = \mathcal{W}(\pi)$ for all $g = (g_p, g^p) \in G(\mathbb{A})$, putting $W_U := W_{1_U}$; so ϕ is well-defined and lies in $\mathcal{D}(S_1, \mathbb{C})$ (since W is smooth and rapidly decreasing; distribution property, F^* - and $U^{p,\infty}$ -invariance being clear by the definitions of ϕ and W^p).

Let $\mu_{\pi} := \mu_{\phi_{\pi}}$ be the distribution on \mathcal{G}_p corresponding to ϕ_{π} , as defined in (22), and let $\kappa_{\pi} := \kappa_{\phi_{\pi}} \in H^d(F^{*'}, \mathcal{D}_f(S_1, \mathbb{C}))$ be the cohomology class defined by (23) and (24).

THEOREM 4.12. *Let $\pi \in \mathfrak{A}_0(G, \underline{2}, \chi_Z, \underline{\alpha}_1, \underline{\alpha}_2)$; we assume the $\alpha_{\mathfrak{p},i}$ to be ordered such that $|\alpha_{\mathfrak{p},1}| \leq |\alpha_{\mathfrak{p},2}|$ for all $\mathfrak{p}|p$. (So $\chi_{\mathfrak{p},1} = |\cdot| \chi_{\mathfrak{p},2}$ for all special $\pi_{\mathfrak{p}}$.)*

(a) *Let $\chi : \mathcal{G}_p \rightarrow \mathbb{C}^*$ be a character of finite order with conductor $\mathfrak{f}(\chi)$. Then we have the interpolation property*

$$\int_{\mathcal{G}_p} \chi(\gamma) \mu_{\pi}(d\gamma) = \tau(\chi) \prod_{\mathfrak{p} \in S_p} e(\pi_{\mathfrak{p}}, \chi_{\mathfrak{p}}) \cdot L(\frac{1}{2}, \pi \otimes \chi),$$

where

$$e(\pi_{\mathfrak{p}}, \chi_{\mathfrak{p}}) = \begin{cases} \frac{(1 - \alpha_{\mathfrak{p},1} x_{\mathfrak{p}} q_{\mathfrak{p}}^{-1})(1 - \alpha_{\mathfrak{p},2} x_{\mathfrak{p}}^{-1} q_{\mathfrak{p}}^{-1})(1 - \alpha_{\mathfrak{p},2} x_{\mathfrak{p}} q_{\mathfrak{p}}^{-1})}{(1 - x_{\mathfrak{p}} \alpha_{\mathfrak{p},2}^{-1})}, & \text{ord}_{\mathfrak{p}}(\mathfrak{f}(\chi)) = 0 \\ & \text{and } \pi \text{ spherical,} \\ \frac{(1 - \alpha_{\mathfrak{p},1} x_{\mathfrak{p}} q_{\mathfrak{p}}^{-1})(1 - \alpha_{\mathfrak{p},2} x_{\mathfrak{p}}^{-1} q_{\mathfrak{p}}^{-1})}{(1 - x_{\mathfrak{p}} \alpha_{\mathfrak{p},2}^{-1})}, & \text{ord}_{\mathfrak{p}}(\mathfrak{f}(\chi)) = 0 \\ & \text{and } \pi \text{ special,} \\ (\alpha_{\mathfrak{p},2}/q_{\mathfrak{p}})^{\text{ord}_{\mathfrak{p}}(\mathfrak{f}(\chi))}, & \text{ord}_{\mathfrak{p}}(\mathfrak{f}(\chi)) > 0 \end{cases}$$

and $x_{\mathfrak{p}} := \chi_{\mathfrak{p}}(\varpi_{\mathfrak{p}})$.

(b) κ_{π} is integral (cf. definition 3.12). For $\underline{\mu} \in \Sigma$, let $\kappa_{\pi, \underline{\mu}}$ be the projection of κ_{π} to $H^d(F^{*'}, \mathcal{D}_f(S_1, \mathbb{C}))_{\pi, \underline{\mu}}$. Then $\kappa_{\pi, \underline{\mu}}$ is integral of rank ≤ 1 .

Proof. (a) We consider χ as a character on \mathbb{F}_F/F^* , and choose a subgroup $V = \prod_{\mathfrak{p}|p} V_{\mathfrak{p}} \subseteq U_p$ such that $\chi_{\mathfrak{p}}|_V = 1$.

Since π is unitary, we have $|\alpha_{\mathfrak{p}, 2}| \geq \sqrt{q_{\mathfrak{p}}} > 1 = |\chi_{\mathfrak{p}}(\varpi_{\mathfrak{p}})|$ for all \mathfrak{p} , thus $e(\pi_{\mathfrak{p}}, \chi_{\mathfrak{p}}|\cdot|_{\mathfrak{p}}^s)$ is non-singular for all $s \geq 0$, and we will be able to apply proposition 2.4 locally below.

We have

$$\int_{\mathcal{G}_p} \chi(\gamma) \mu_{\pi}(d\gamma) = [U_p : V] \int_{\mathbb{F}_F/F^*} \chi(x) \phi_V(x) d^{\times} x,$$

and therefore we have to show that the equality

$$[U_p : V] \int_{\mathbb{F}_F/F^*} \chi(x) |x|^s \phi_V(x) d^{\times} x = N(\mathfrak{f}(\chi))^s \tau(\chi) \prod_{\mathfrak{p}|p} e(\pi_{\mathfrak{p}}, \chi_{\mathfrak{p}}|\cdot|_{\mathfrak{p}}^s) \cdot L(s + \frac{1}{2}, \pi \otimes \chi)$$

holds for $s = 0$. Since both the left-hand side and $L(s + \frac{1}{2}, \pi \otimes \chi)$ are holomorphic in s (cf. [Ge75], Thm. 6.18), it suffices to show this for $\operatorname{Re}(s) \gg 0$. But for such s , we have

$$\begin{aligned} [U_p : V] \int_{\mathbb{F}_F/F^*} \chi(x) |x|^s \phi_V(x) d^{\times} x &= \int_{\mathbb{F}_F} \chi(x) |x|^s W \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} d^{\times} x \\ &= [U_p : V] \int_{F_p^*} \chi_p(x) |x|^s W_V \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} d^{\times} x \cdot \int_{\mathbb{F}_F^p} \chi^p(y) |y|^s W^p \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} d^{\times} y \\ &= \prod_{\mathfrak{p}|p} \int_{F_{\mathfrak{p}}^*} \chi_{\mathfrak{p}}(x) |x|_{\mathfrak{p}}^s \mu_{\pi_{\mathfrak{p}}}(dx) \cdot L_{S_p}(s + \frac{1}{2}, \pi \otimes \chi) \\ &= \prod_{\mathfrak{p}|p} (e(\pi_{\mathfrak{p}}, \chi_{\mathfrak{p}}|\cdot|_{\mathfrak{p}}^s) \tau(\chi_{\mathfrak{p}}|\cdot|_{\mathfrak{p}}^s)) \cdot L(s + \frac{1}{2}, \pi \otimes \chi) \\ &= N(\mathfrak{f}(\chi))^s \tau(\chi) \prod_{\mathfrak{p}|p} e(\pi_{\mathfrak{p}}, \chi_{\mathfrak{p}}|\cdot|_{\mathfrak{p}}^s) \cdot L(s + \frac{1}{2}, \pi \otimes \chi) \end{aligned}$$

by propositions 2.13, 2.4 and equation (37).

(b) Let $\lambda_{\underline{\alpha}_1, \underline{\alpha}_2} \in \mathcal{B}^{\underline{\alpha}_1, \underline{\alpha}_2}(F_p, \mathbb{C})$ be the image of $\otimes_{v|p} \lambda_{a_v, v_v}$ under the map (13). For each $\psi \in \mathcal{B}^{\underline{\alpha}_1, \underline{\alpha}_2}(F_p, \mathbb{C})$, define

$$\begin{aligned} \langle \Phi_{\pi}, \psi \rangle(g^p, g_p) &:= \sum_{\zeta \in F^*} \lambda_{\underline{\alpha}_1, \underline{\alpha}_2} \left(\begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix} g_p \cdot \psi \right) \underline{W}^p \left(\begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix} g^p \right) \\ &=: \sum_{\zeta \in F^*} \underline{W}_{\psi} \left(\begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix} g \right) \end{aligned}$$

for a V -valued function \underline{W}_ψ whose every coordinate function is in $\mathcal{W}(\pi)$. This defines a map $\Phi_\pi : G(\mathbb{A}^p) \rightarrow \mathcal{B}^{\alpha_1, \alpha_2}(F_p, V)$. In fact, Φ_π lies in $S_2(G, \mathfrak{m}, \underline{\alpha}_1, \underline{\alpha}_2)$, where \mathfrak{m} is the prime-to- p part of $\mathfrak{f}(\pi)$: Condition (a) of definition 4.5 follows from the fact that the W_v are almost $K_0(\mathfrak{c}(\pi_v))$ -invariant, for $v \nmid p, \infty$. For condition (b), we check that $\langle \Phi_\pi, \psi \rangle$ satisfies the conditions (i)-(v) in the definition of $\mathcal{A}_0(G, \text{harm}, \underline{2}, \chi)$: Each coordinate function of $\langle \Phi_\pi, \psi \rangle$ lies in (the underlying space of) π by [Bu98], Thm. 3.5.5, thus $\langle \Phi, \psi \rangle$ fulfills (i) and (v), and has moderate growth. (ii) and (iv) follow from the choice of the W_v and \underline{W}_v . Now since $\pi_v \cong \sigma(| \cdot |_v^{1/2}, | \cdot |_v^{-1/2})$ for $v | \infty$, $\langle \Phi, \psi \rangle|_{B_{F_v}'} \cdot \underline{\beta}_v = C \sum_{\zeta \in F^*} \underline{W}_v \begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix} \cdot \underline{\beta}_v$ is harmonic for each archimedean place v of F : for real v , it is well-known that $f(z)/y$ is holomorphic for $f \in \mathcal{D}(2)$, and thus $f \cdot (\beta_v)_1$ is harmonic; for complex v , harmonicity follows from the other conditions, see e.g. [Kur78], p. 546 or [We71].

An easy calculation shows that

$$\lambda_{\underline{\alpha}_1, \underline{\alpha}_2} \left(\begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix} \delta_{\underline{\alpha}_1, \underline{\alpha}_2}(1_U) \right) = \int_{\zeta U} \prod_{\mathfrak{p} | p} \chi_{\alpha_{\mathfrak{p}, 2}}(-x) \psi_{\mathfrak{p}}(-x) dx = \mu_{\pi_p}(\zeta U)$$

for all $\zeta \in F^*$, and therefore we have

$$\begin{aligned} \Delta^{\alpha_1, \alpha_2}(\Phi_\pi)(U, x^p) &= \sum_{\zeta \in F^*} \lambda_{\underline{\alpha}_1, \underline{\alpha}_2} \left(\begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix} \delta_{\underline{\alpha}_1, \underline{\alpha}_2}(1_U) \right) W^p \begin{pmatrix} \zeta x^p & 0 \\ 0 & 1 \end{pmatrix} \\ &= \sum_{\zeta \in F^*} \mu_{\pi_p}(\zeta U) W^p \begin{pmatrix} \zeta x^p & 0 \\ 0 & 1 \end{pmatrix} = \phi_\pi(U, x^p). \end{aligned}$$

Let R be the integral closure of $\mathbb{Z}[a_{\mathfrak{p}}, \nu_{\mathfrak{p}}; \mathfrak{p} | p]$ in its field of fractions; thus R is a Dedekind ring $\subseteq \overline{\mathcal{O}}$ for which $\mathcal{B}_{\underline{\alpha}_1, \underline{\alpha}_2}(F, R)$ is defined. Since \mathbb{C} is a flat R -module,

$$H^d(G(F)^+, \mathcal{A}_f(\mathfrak{m}, \underline{\alpha}_1, \underline{\alpha}_2, \mathcal{M}, R)) \otimes \mathbb{C} \rightarrow H^d(G(F)^+, \mathcal{A}_f(\mathfrak{m}, \underline{\alpha}_1, \underline{\alpha}_2, \mathcal{M}, \mathbb{C}))$$

is an isomorphism by proposition 4.8. The map (36) can be described as the "R-valued" map

$$H^d(G(F)^+, \mathcal{A}_f(\mathfrak{m}, \underline{\alpha}_1, \underline{\alpha}_2, \mathcal{M}, R)) \rightarrow H^d(F^{*'}, \mathcal{D}_f(R))$$

tensored with \mathbb{C} . By proposition 4.11, κ_π lies in its image, and thus in $H^d(F^{*'}, \mathcal{D}_f(R)) \otimes \mathbb{C}$; i.e. it is integral.

Similarly, it follows from propositions 4.8 and 4.10 that $\kappa_{\pi, \underline{\mu}}$ is integral of rank ≤ 1 . □

COROLLARY 4.13. μ_π is a p -adic measure.

Proof. By proposition 3.8, $\mu_\pi = \mu_{\phi_\pi} = \mu_{\kappa_\pi}$. Since κ_π is integral, μ_{κ_π} is a p -adic measure by corollary 3.13. □

4.7 VANISHING ORDER OF THE p -ADIC L-FUNCTION

Let L_1, \dots, L_t be independent \mathbb{Z}_p -extensions of F , and let $\ell_1, \dots, \ell_t : \mathcal{G}_p \rightarrow p^{\varepsilon_p} \mathbb{Z}_p$ be the homomorphisms corresponding to them (as in section 3.2). Then we have the p -adic L -function

$$L_p(\underline{s}, \pi) := L_p(\underline{s}, \kappa_\pi) := L_p(s_1, \dots, s_t, \kappa_{\pi,+}) := \int_{\mathcal{G}_p} \prod_{i=1}^t \exp_p(s_i \ell_i(\gamma)) \mu_\pi(d\gamma)$$

of definition 3.5, with $s_1, \dots, s_t \in \mathbb{Z}_p$. $L_p(\underline{s}, \pi)$ is a locally analytic function with values in the one-dimensional \mathbb{C}_p -vector space $V_{\kappa_{\pi,+}} = L_{\kappa, \overline{\sigma}, +} \otimes_{\overline{\sigma}} \mathbb{C}_p$. By theorem 3.11, we have

THEOREM 4.14. *$L_p(\underline{s}, \pi)$ is a locally analytic (t -variable) function, and all partial derivatives of order $\leq n := \#(S_1)$ vanish; i.e. we have*

$$\text{ord}_{\underline{s}=\underline{0}} L_p(\underline{s}, \pi) \geq n.$$

Now let E be a modular elliptic curve over F , corresponding to an automorphic representation π ; by this we mean that the local L-factors of the Hasse-Weil L-function $L(E, s)$ and of the automorphic L-function $L(s - \frac{1}{2}, \pi)$ coincide at all places v of F . From the definition of the respective L-factors (cf. [Si86] for the Hasse-Weil L-function, [Ge75] for the automorphic L-function) we know that π has trivial central character. Moreover, for $\mathfrak{p}|p$, $\pi_{\mathfrak{p}}$ is a principal series representation iff E has good reduction at \mathfrak{p} , and in this case $\pi_{\mathfrak{p}}$ is ordinary iff E is ordinary (i.e. not supersingular) at \mathfrak{p} ; $\pi_{\mathfrak{p}}$ is a special (resp. Steinberg) representation iff E has multiplicative (resp. split multiplicative) reduction at \mathfrak{p} . For $v|\infty$, π_v is “of weight 2” as assumed before.

We say that E is p -ordinary if it has good ordinary or multiplicative reduction at all places $\mathfrak{p}|p$ of F . So E is p -ordinary iff π is ordinary at all $\mathfrak{p}|p$. In this case, we define the p -adic L-function of E by $L_p(E, \underline{s}) := L_p(\underline{s}, \pi)$.

For each $i \in \{1, \dots, t\}$ and each prime $\mathfrak{p}|p$ of F , we write $\ell_{\mathfrak{p},i}$ for the restriction of ℓ_i to $F_{\mathfrak{p}} \hookrightarrow \mathbb{I} \rightarrow \mathcal{G}_p$. Let $q_{\mathfrak{p}}$ be the Tate period of $E|F_{\mathfrak{p}}$ and $\text{ord}_{\mathfrak{p}}$ the normalized valuation on $F_{\mathfrak{p}}^*$. Defining the L -invariants of $E|F_{\mathfrak{p}}$ with respect to L_i by

$$\mathcal{L}_{\mathfrak{p},i}(E) := \ell_{\mathfrak{p},i}(q_{\mathfrak{p}}) / \text{ord}_{\mathfrak{p}}(q_{\mathfrak{p}}),$$

we can generalize Hida’s exceptional zero conjecture to general number fields:

CONJECTURE 4.15. *Let S_1 be the set of $\mathfrak{p}|p$ at which E has split multiplicative reduction, $n := \#S_1$, $S_2 := S_p \setminus S_1$. Then*

$$\text{ord}_{\underline{s}=\underline{0}} L_p(E, \underline{s}) \geq n, \tag{38}$$

and we have

$$\frac{\partial^n}{\partial s_i^n} L_p(E, \underline{s})|_{\underline{s}=\underline{0}} = n! \prod_{\mathfrak{p} \in S_1} \mathcal{L}_{\mathfrak{p},i}(E) \prod_{\mathfrak{p} \in S_2} e(\pi_{\mathfrak{p}}, 1) \cdot L(E, 1), \tag{39}$$

for all $i = 1, \dots, t$, where $e(\pi_{\mathfrak{p}}, 1) = (1 - \alpha_{\mathfrak{p},1}^{-1})^2$ if E has good ordinary reduction at \mathfrak{p} , and $e(\pi_{\mathfrak{p}}, 1) = 2$ if E has non-split multiplicative reduction at \mathfrak{p} .

Note that the conjecture (when considered for all sets of independent \mathbb{Z}_p -extensions of F) also determines the “mixed” partial derivatives $\frac{\partial^k}{\partial \underline{s}^k} L_p(E, \underline{0})$ of order n , since they can be written as \mathbb{Q} -linear combinations of n -th “pure” partial derivatives $\frac{\partial^n}{\partial s_i^n} L_p(E, \underline{0})$ with respect to other choices of independent \mathbb{Z}_p -extensions of F (cf. the proof of proposition 3.9).

Theorem 4.14 immediately implies the first part (38) of the conjecture:

COROLLARY 4.16. *Let E be a p -ordinary modular elliptic curve over F . Let n be the number of places $\mathfrak{p}|p$ at which E has split multiplicative reduction. Then we have*

$$\text{ord}_{\underline{s}=\underline{0}} L_p(E, \underline{s}) \geq n.$$

In future work, we hope to also establish formula (39) for a class of non-totally-real number fields.

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