Rigidity and Frobenius Structure

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Abstract. We show that an irreducible ordinary differential equation on the projective line has a Frobenius structure for a power of some prime p if it is rigid in the sense of Katz and satisfies some other reasonable (and necessary) conditions relative to the prime p.

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Introduction

The purpose of this note is show that an irreducible rigid differential equation on an open subset of $\mathbb{P}^1$ with regular singularities and rational exponents has, with reasonable local assumptions relative a prime p, a Frobenius structure relative to some power of p.

Katz showed that any irreducible rigid local system on an open subset $\mathbb{P}^1$ can be built up by repeated tensor product and convolution operations of a suitable sort from local systems of rank one [8]. One therefore expects that if the corresponding regular singular differential equation is defined, say, over $\mathbb{Q}$ and has rational exponents, it should have a Frobenius structure for almost all p. What we show in this paper, in effect, is that if the differential equation has rational exponents, defines an overconvergent isocrystal for some prime p and satisfies a few other reasonable local conditions, it will have a Frobenius structure for that particular p. It is well known that the existence of a Frobenius structure implies that the equation comes from a convergent isocrystals, and overconvergence then follows from the other assumptions. We remark that when the equation is irreducible, this Frobenius structure is unique up to a scalar multiple, as was shown by Dwork [7].

Katz used the theory of algebraic D-modules in [8]; Berthelot's theory of arithmetic D-modules is not *a priori* applicable here since it relies heavily on the existence of a Frobenius structure (it is not known how to define "holonomic" without one). On the other hand, once an overconvergent isocrystal is known to have a Frobenius structure, its direct image by specialization is to be a holonomic $\mathcal{D}^\dagger$-module to which the methods of [8] could be applied. The present approach is elementary in that it uses only the cohomological criterion for rigidity, together with a p-adic analogue (theorem 1 below) in terms of rigid cohomology. The main point is that if a regular singular differental equation on an open subset of $\mathbb{P}^1$ is rigid and irreducible, and defines an overconvergent isocrystal, then that isocrystal is p-adically rigid (theorem 2). The existence of a Frobenius structure follows from this, assuming rational exponents and other suitable conditions (theorem 3).

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1 Classical and p-adic Rigidity

Let U be a nonempty Zariski open subset of $\mathbb{P}^1_{\mathbb{C}}$, with analytification U^{an}. We recall that a local system V on U^{an} is *rigid* if any other local system on U^{an} with the same local monodromy as V is isomorphic to V. Denote by $j : U^{an} \to \mathbb{P}^1$ the natural inclusion, and set $S = \mathbb{P}^1 \setminus U$. Katz shows that an irreducible V is rigid if and only if $H^1(\mathbb{P}^1, j_*End(V)) = 0$, or equivalently if $\chi(\mathbb{P}^1, j_*End(V)) = 2$. That this condition is sufficient is relatively easy, and we will see that it can be extends to the case of p-adic differential equations.

A p-adic analogue of the rigidity condition can be formulated for the category of overconvergent isocrystals on an open subset $\mathbb{P}^1$ over a p-adic base. We will assume that the reader is familiar with this theory, but it will be useful to recall a few basic constructions.

Fix a complete discrete valuation ring $\mathcal{V}$ of mixed characteristic p, with fraction field K and residue field k. Let $\mathbb{P}^1_{\mathcal{V}}$ be the projective line considered as a formal $\mathcal{V}$-scheme, $U \subset \mathbb{P}^1$ a nonempty formal affine subscheme with closed fiber U_k. The complement $S = \mathbb{P}^1 \setminus U_k$ is then a finite set of points. As usual, $U^{an} \subset \mathbb{P}^1_K$ will denote the corresponding affinoid space; it is the same as the tube $]U[=]U_k[$ (c.f. [2]). Recall that in this setting, an overconvergent isocrystal on U can be identified with a locally free module with an overconvergent connection (M, ∇) over the dagger-algebra

$$A^\dagger = \varinjlim_{W} \Gamma(W, \mathcal{O}_W) \tag{1.1}$$

where W runs over the directed system of strict neighborhoods of U^{an}, i.e. a rigid-analytic open neighborhoods W of U^{an} such that $\{W, \mathbb{P}^1 \setminus U^{an}\}$ is an admissible cover of $\mathbb{P}^1$. We will usually abbreviate (M, ∇) by M.

If s is a point of S and W is a strict neighborhood of U^{an}, the open set $W \cap]s[$

is isomorphic to a rigid-analytic annulus, and we denote by $\mathcal{R}_s$ the direct limit

$$\mathcal{R}_s = \varinjlim_{W} \Gamma(W \cap]s[, \mathcal{O}_W) \tag{1.2}$$

of the function algebras of these annuli; this is the *Robba ring* at s. If x is a local parameter of $\mathbb{P}^1_{\mathcal{V}}$ at s (i.e. reduces to a local parameter of $\mathbb{P}^1_k$ at s) then $\mathcal{R}_s \simeq \mathcal{R}$ where $\mathcal{R}$ is the "standard" Robba ring, i.e. the ring of formal Laurent series in x converging in some annulus $r < |x| <$. If s is a point of S, the natural inclusions $W \cap]s[\hookrightarrow W$ induce injective ring homomorphisms $\Gamma(W, \mathcal{O}_W) \hookrightarrow \Gamma(W \cap]s[, \mathcal{O}_W)$, whence a continuous ring homomorphism $A^\dagger \hookrightarrow \mathcal{R}_s$ for all $s \in S$. If (M, ∇) is an overconvergent isocrystal on U, we set

$$M_s = \varinjlim_{W} \Gamma(W \cap]s[, M)$$

which, since M is a a coherent $\mathcal{O}_W$-module, is a $\mathcal{R}_s$-module of finite presentation. The connection on the $\mathcal{R}_s$-module M_s induced by ∇ will be denoted ∇_s, and finally the pair (M_s, ∇_s) will be also denoted by M_s; it is an "overconvergent isocrystal on $\mathcal{R}_s$" that represents the mondromy of M about s.

We therefore make the following definition. An overconvergent isocrystal M on U is *p-adically rigid* if it has the following property: if N is another overconvergent isocrystal on U such that $M_s \simeq N_s$ for all $s \in S$, then $M \simeq N$. As in the classical case we do not make a definition in the case of curves of higher genus, or varieties of higher dimension (although for curves of higher genus, the definition of "weakly rigid" extends in an obvious way).

To formulate a cohomological condition for the p-adic rigidity of an overconvergent isocrystal (M, ∇), we recall that for a complex local system V on an open $U \subseteq \mathbb{P}^1_{\mathbb{C}}$, $H^1(\mathbb{P}^1_{\mathbb{C}}, j_*V)$ is the same as the parabolic cohomology $H^1_p(U, V)$, i.e. the image of the forget supports map $H^1_c(U, V) \to H^1(U, V)$. In fact the long exact sequences arising from the exact triangles

$$\begin{gathered} j_!V \to j_*V \to \bigoplus_{s \in S} (j_*V)_s \xrightarrow{+1} \\ j_*V \to Rj_*V \to \bigoplus_{s \in S} (R^1 j_*V)_s[-1] \xrightarrow{+1} \end{gathered} \tag{1.3}$$

reduce to exact sequences

$$\begin{gathered} 0 \to H^0(U, V) \to \bigoplus_{s \in S} (j_*V)_s \to H^1_c(U, j_*V) \to H^1(\mathbb{P}^1, j_*V) \to 0 \\ 0 \to H^1(\mathbb{P}^1, j_*V) \to H^1(U, V) \to \bigoplus_{s \in S} (R^1 j_*V)_s \to H^2(\mathbb{P}^1, j_*V) \to 0 \end{gathered} \tag{1.4}$$

and an isomorphism

$$H^2_c(U, V) \simeq H^2(\mathbb{P}^1, j_*V). \tag{1.5}$$

The assertion follows from this, given that $H^1_c(U,V) \to H^1(U,V)$ is induced by the composite $j_!V \to j_*V \to Rj_*V$. From the definitions and 1.5 we get equalities

$$\begin{aligned}\chi(\mathbb{P}^1, j_*V) &= \dim H^0(U,V) - \dim H^1(\mathbb{P}^1, j_*V) + \dim H^2_c(U,V)\\ &= \chi_c(U,V) + \sum_{s\in S} \dim V_s.\end{aligned} \tag{1.6}$$

The p-adic analogue is straightforward, using rigid cohomology (see [2] for the general definition, and [6] for the case of an affine curve). The first fact we need is the existence of a six-term exact sequence

$$\begin{aligned}0 \to H^0(U,M) \to \bigoplus_{s\in S} H^0_{DR}(M_s) \to H^1_c(U,M) \to\\ \xrightarrow{\partial} H^1(U,M) \to \bigoplus_{s\in S} H^1_{DR}(M_s) \to H^2_c(U,M) \to 0\end{aligned} \tag{1.7}$$

for any overconvergent isocrystal M on U [6, 9.5.2]. In 1.7 the "local cohomology" $H^i_{DR}(M_s)$ is just the ordinary de Rham cohomology of $M_s = (M_s, \nabla_s)$. We then define the parabolic cohomology $H^1_p(U,M)$ by

$$H^1_p(U,M) = \mathrm{Im}(\partial : H^1_c(U,M) \to H^1(U,M)). \tag{1.8}$$

From this we see that 1.7 is the p-adic analogue of the result of gluing together the exact sequences 1.4 at the term $H^1(\mathbb{P}^1, j_*V)$.
When $H^1_p(U,M)$ has finite dimension, we can define the "parabolic" Euler characteristic of M by analogy with the first part of 1.6

$$\chi_p(M) = \dim H^0(U,M) - \dim H^1_p(U,V) + \dim H^2_c(U,M) \tag{1.9}$$

and from 1.7 and 1.9 we get the equality

$$\chi_p(U,M) = \chi_c(M) + \sum_{s\in S} \dim H^0_{DR}(M_s) \tag{1.10}$$

analogous to second part of 1.6.
The space $H^1_p(U,M)$ will of course have finite dimension if either of $H^1(U,M)$ or $H^1_c(U,M)$. The finite-dimensionality of these latter spaces depends on the behavior of M at the points of S. The next proposition extends slightly the main result of [6]:

1 Proposition *Let M be an overconvergent isocrystal on U. If $H^1_{DR}(M_s)$ has finite dimension for every $s \in S$, the K-vector spaces $H^1(U,M)$, $H^1_c(U,M)$ and $H^1_p(U,M)$ have finite dimension and there are canonical duality isomorphisms*

$$\begin{aligned}H^i(U,M)^\vee &\simeq H^{2-i}_c(U,M^\vee)\\ H^1_p(U,M)^\vee &\simeq H^1_p(U,M^\vee)\end{aligned} \tag{1.11}$$

for $0 \le i \le 2$.

Proof. By theorem 9.5 of [6] it suffices to show that for all $s \in U_k$ the K-linear map $\nabla_s : M_s \to M_s \otimes \Omega^1$ is a strict morphism of topological vector spaces. Since M_s and $M_s \otimes \Omega^1$ are LF-spaces this follows from the next lemma. ∎

In fact this is standard but I do not know a convenient reference:

1 Lemma *Suppose $u : V \to W$ is a continuous map of LF-spaces such that* $\mathrm{Coker}(u)$ *has finite dimension. Then u is strict.*

Proof. There is a subspace $H \subset W$ of finite dimension that is an algebraic supplement to $u(V)$. Since H is separated, its topology is the unique separated topology of a finite-dimensional vector space, and $H \oplus V$ is an LF-space. The natural map $f : H \oplus V \to W$ is surjective and therefore open by the open mapping theorem [10, Prop. 8.8]. Suppose now $A \subset V$ is open; then $H \oplus A \subset H \oplus V$ is open and consequently $f(H \oplus A) = H + u(A)$ is open. Since $u(V) \cap (H + u(A)) = u(A)$, $u(A)$ is open in $u(V)$. ∎

The condition that $\dim(H^1_{DR}(M_s)) < \infty$ in proposition 1 is a consequence of the "NL property" of Christol and Mebkhout. The definition is rather involved and we refer the reader to [4] and the references therein. The one consequence of this condition we need is the following: if as before M is an overconvergent isocrystal of rank d on U and satisfies condition NL at every point of S, then

$$\chi_c(U, M) = d\chi_c(U) - \sum_{s \in S} \mathrm{Irr}(M_s) \tag{1.12}$$

where $\mathrm{Irr}(M_s)$ is the irregularity of the isocrystal M_s, defined in [4]. In particular, $\chi_c(U, M)$ only depends on U, the rank of M and the irregularities $\mathrm{Irr}(M_s)$.
We can now state:

1 Theorem *Suppose M is an irreducible overconvergent isocrystal on $U \subset \mathbb{P}^1$ such that $End(M)$ satisfies condition NL at every point of S. If $\chi_p(End(M)) = 2$ then M is p-adically rigid.*

Proof. The argument is the same as in [8]. Suppose that N is an overconvergent isocrystal such that $M_s \simeq N_s$ for all $s \in S$; in particular M and N have the same rank. Since $Hom(M, N)_s \simeq End(M)_s$ for all $s \in S$, $Hom(M, N)$ satisfies condition NL at every s. Then it follows from $\chi_p(End(M)) = 2$ and the index formula 1.12 that $\chi_p(Hom(M, N)) = 2$, and therefore

$$\dim H^0(\mathbb{P}^1, Hom(M, N)) + \dim H^2_c(\mathbb{P}^1, Hom(M, N)) \geq 2.$$

On the other hand $Hom(M, N)$ and $Hom(N, M)$ are dual, so the duality 1.11 yields

$$\dim H^0(\mathbb{P}^1, Hom(M, N)) + \dim H^0(\mathbb{P}^1, Hom(N, M)) \geq 2.$$

and we conclude that one of $Hom(M, N)$, $Hom(N, M)$ is nonzero. Since M and N have the same rank and M is irreducible, we conclude that $M \simeq N$. ∎

We note that since $End(M)$ is canonically self-dual, the irreducibility of M implies that either $\chi_p(M) = 2$ or $\chi_p(M) \leq 0$, so that $\chi_p(M) = 2$ in this case is equivalent to $H^1_p(U, End(M)) = 0$. As in the classical case we can think of $\dim H^1_p(U, End(M))$ as the number of "accessory parameters" of M (see [8], p. 5). I do not know if there is a converse to theorem 1, as is the case over $\mathbb{C}$; it would be of interest to settle this question.

2 Comparison Theorems

Suppose M is a module with a connection with regular singularites on, say, an open subset U of $\mathbb{P}^1_{\mathbb{Q}}$, and denote by V the corresponding local system on $U^{an}_{\mathbb{C}}$. The aim of this section is to show, under a few (necessary) assumptions, that if V is rigid, the p-adic completion of M is p-adically rigid (one condition, obviously, is that this p-adic completion defines an overconvergent isocrystal). We need not, however, restrict ourselves to the case where M is defined over $\mathbb{Q}$, or over a number field. In fact, the condition that V be rigid is essentially an algebraic condition on M:

2 Lemma *Suppose M is a module with a connection with regular singularities on some open subset of $\mathbb{P}^1_K$, where K is a field of characteristic zero embeddable into $\mathbb{C}$. If the local system $(M \otimes_{K,\iota} \mathbb{C})^{an}$ is rigid for one choice of embedding $\iota : K \to \mathbb{C}$, it is rigid for any other choice.*

Proof. By Katz's criterion, it suffices to show that $\chi_p((M \otimes_{K,\iota} \mathbb{C})^{an}) = 2$ if and only if $\chi_p(M) = 2$ (with the latter defined, say by algebraic D-module theory), but this is just a special case of the Riemann-Hilbert correspondence. ■

If K is any field of characteristic zero and M is a module with regular connection on $\mathbb{P}^1_K$, we can say that M is *rigid* if there is an absolutely finitely generated subfield $K_0 \subset K$ over which M has a model M_0, and an embedding $\iota : K_0 \to \mathbb{C}$ such that $\iota(M_0)^{an}$ is a rigid local system; this is evidently independent of the choice of model, and, by the lemma, of ι. We remark that a model over an absolutely finitely generated subfield always exists.

Now in fact one could give a purely algebraic definition of rigidity, analogous to the definition for local systems, and with this definition one could prove that $\chi_p(M) = 2$ implies that M is rigid. The converse, however, would not be available without the above comparison lemma, since it requires a transcendental argument.

Suppose now $\mathcal{V}$ is a complete discrete valuation ring of mixed characteristic p, with fraction field K and residue field k. Let $S \subset \mathbb{P}^1_{\mathcal{V}}$ be a closed subscheme that finite, flat and integral over $\mathcal{V}$ and set $U = \mathbb{P}^1 \setminus S$. The U and S that appeared in the last section are now $\hat{U}$ (the p-adic completion) and S_k. As before, U^{an} is the affinoid space associated to $\hat{U}$. Finally we denote by U_K, S_K the fibers of U and S over K. Note that S_K can be identified with a finite subset of the tube $]S_k[$, and in fact every point of S_K is contained in exactly one disk $]s[$ with $s \in S_k$.

Suppose now that (M, ∇) (as before, usually referred to as M) is a coherent $\mathcal{O}_U$-module with (integrable) connection. We denote by M_K the corresponding module with connection on U_K. If the formal horizontal sections of M_K have radius of convergence equal to 1 at every point of U_K, then M_K defines an overconvergent isocrystal on $\hat{U}$ which we denote by $M^\dagger$. We are interested in comparing various properties of M_K and $M^\dagger$, subject to a number of assumptions. The first is purely geometrical:

C1 For all $s \in S_k$, the disk $]s[$ contains exactly one point of S_K, which is a K-rational point.

Thus each disk contains at most one singular point of M_K. The remaining conditions refer specifically to M. Recall that $a \in \mathbb{Z}_p$ is p-adic Liouville if for every positive real $r < 1$, $|a - n| < r^{|n|}$ has infinitely many solutions $n \in \mathbb{Z}$.

C2 M defines an overconvergent isocrystal $M^\dagger$ on $\hat{U}$.

C3 M_K is regular singular, and the exponents of $End(M_K)$ belong to $\mathbb{Z}_p$ and are not p-adic Liouville numbers (in particular the exponents of M_K itself do not differ by p-adic Liouville numbers).

In the next theorem and further on we will need a consequence of Christol's transfer theorem [3, thm. 1], which can be stated as follows. First, if A is any $n \times n$ matrix A with entries in K, we denote by M_A the free $\mathcal{R}$-module $\mathcal{R}^n$ with connection given by

$$\nabla(u) = du + Au \otimes \frac{dx}{x} \tag{2.1}$$

where x is the parameter of $\mathcal{R}$. Recall finally that for $s \in S_k$, a local parameter of $\mathbb{P}^1_{\mathcal{V}}$ at s fixes an identification $\mathcal{R}_s = \mathcal{R}$.

3 Lemma *Suppose (M, ∇) satisfies C2-C3. If $s \in S_k$, $M_s^\dagger$ is isomorphic as an isocrystal on $\mathcal{R}$ to M_A for some $n \times n$ matrix A with entries in $\mathcal{V}$.*

In fact Christol's theorem is a purely local statement and we refer the reader to [5, thm. 3.6] for an explanation of how the lemma follows from [3, thm. 1].

2 Theorem *Suppose M satisfies conditions C1-C3. If M_K is irreducible as a module with connection, $M^\dagger$ is irreducible as an overconvergent isocrystal. If in addition M_K is rigid, $M^\dagger$ is p-adically rigid.*

Proof. The first part follows from theorem 2.5 of [5], which asserts that the differential galois group of M_K is isomorphic to the differential galois group (in the category of overconvergent isocrystals) of $M^\dagger$. Thus if M_K corresponds to an irreducible representation of its differential galois group, so does $M^\dagger$.
If M_K is rigid, then $\chi(U_K, j_* End(M_K)) = 2$, where as before $j : U_K \to \mathbb{P}^1_K$ is the inclusion (and M_K is now regarded as a local system on U_K). By theorem 1 it suffices to show that $\chi_p(End(M^\dagger)) = 2$.

By C3, $End(M^\dagger)$ satisfies condition NL at every point of S_k, and furthermore $\mathrm{Irr}_s(End(M^\dagger)) = 0$ for all $s \in S_k$. Thus

$$\chi_c(End(M^\dagger)) = d\chi_c(U) = \chi_c(End(M_K))$$

where d is the rank of $End(M_K)$. One can also deduce this equality from the comparison theorem of Baldassarri-Chiarellotto [1].

To show that $\chi_p(End(M^\dagger)) = \chi_p(End(M_K)) = 2$, it thus suffices to show that $(j_*M_K)_s$ and $M_s^\dagger$ have the same dimension for all $s \in S_k$. Suppose t is a local parameter of $\mathbb{P}^1_{\mathcal{V}}$ such that $t = 0$ defines a point of S_K in $\mathbb{P}^1_K$, and its reduction in $\mathbb{P}^1_k$. Then $(j_*M_K)_s$ and $M_s^\dagger$ are the spaces horizontal sections of the connection in respectively in the ring of formal Laurent series $K((t))$, and in the ring of elements of $\mathcal{R}_s$ convergent for $0 < |t| < 1$. Since the exponents of $End(M_K)$ are not p-adic Liouville, lemma 3 implies that $M_s^\dagger$ is isomorphic, as $\mathcal{R}_s$-module with connection, to a free $\mathcal{R}_s$-module with connection given by the matrix of 1-forms $A \otimes_K dt/t$, where A is a constant matrix. The verification that these spaces have the same dimension is then straightforward (see [5, Lemma 3.4] for the case of $M^\dagger$). ■

3 Frobenius Structure

We now apply theorem 2 to the question of Frobenius structures. To the assumptions already made we add:

C4 The exponents of M_K are rational.

It is known that if M satisfies C1-C3 and has a Frobenius structure, then C4 holds as well.

If $q = p^f$ is a power of p, we denote by $\phi : \hat{U} \to \hat{U}$ a lifting of the q^{th}-power Frobenius of U_k. If t is a global parameter on $\mathbb{P}^1_{\mathcal{V}}$, we could of course take $\phi(t) = t^q$; the theorem in this section allows more general choices. We denote by $\sigma : \mathcal{V} \to \mathcal{V}$ the restriction of ϕ.

The main theorem of this section follows from the following local-to-global principle:

4 Lemma *Suppose M_K is an irreducible module with connection satisfying C3. If M_K is rigid and $\phi^* M_s^\dagger \simeq M_s^\dagger$ for all $s \in S_k$, $M^\dagger$ has a q^{th}-power Frobenius structure.*

Proof. By theorem 2 we know that $M^\dagger$ is irreducible and p-adically rigid, and the assertion follows from the definition. ■

We denote by N the least common multiple of the denominators of the exponents of M_K at all points of S_k.

3 Theorem *Suppose M satisfies conditions C1-C4. If M_K is irreducible and rigid, then $M^\dagger$ has a q^{th}-power Frobenius structure for any $q = p^f$ such that $q \equiv 1 \pmod{N}$.*

We remarked in the introduction that the Frobenius structure is unique up to scalar multiples.

Proof. Fix an $s \in S_k$ and a local parameter x of the Robba ring $\mathcal{R}_s$. By lemma 3 there is an isomorphism $M_s \simeq M_A$ (depending on the choice of x, i.e. on the identification $\mathcal{R}_s \simeq \mathcal{R}$) for some A with rational, p-adically integral eigenvalues. In view of lemma 4 we must show that $\phi^* M_A \simeq M_A$.

We first remark that $M_A \simeq M_{qA}$ for q as above. This is elementary: we can assume A is in Jordan normal form, since the eigenvalues are rational. We reduce immediately to the case when A is a single Jordan block with eigenvalue λ; then qA is similar to a block with eigenvalue $q\lambda$, say A', and it suffices to show that $M_A \simeq M_{A'}$. Since $q \equiv 1 \pmod N$ we can write $q\lambda = \lambda + k$ with $k \in \mathbb{Z}$, and the map $\mathcal{R}^n \to \mathcal{R}^n$ given by $u \mapsto x^k u$ is the desired isomorphism.

To conclude we show that $\phi^* M_A \simeq M_{qA}$. We give two arguments, an elementary one that needs restrictions on $\mathcal{V}$ and a second, less elementary one with no restrictions.

First method. Let π be a uniformizer of $\mathcal{V}$ and let e be its absolute ramification index. For this proof we assume that $e < p - 1$ (note that this excludes $p = 2$). Before going on we recall that in general, if ∇ and ∇' are connections on $\mathcal{R}^n$ given by $n \times n$ matrices of 1-forms B and B', then an isomorphism $(\mathcal{R}^n, \nabla) \simeq (\mathcal{R}^n, \nabla')$ is a matrix $C \in \mathrm{GL}_n(\mathcal{R})$ such that

$$dC \cdot C^{-1} = CBC^{-1} - B'. \tag{3.1}$$

In particular if B, B' are conjugate by a constant matrix, that matrix also yields an isomorphism $(\mathcal{R}^n, \nabla) \simeq (\mathcal{R}^n, \nabla')$.

Now

$$\phi^* \left(A \otimes \frac{dx}{x} \right) = A \otimes \frac{d\phi(x)}{\phi(x)} = qA \otimes \frac{dx}{x} + A \otimes \frac{dh(x)}{h(x)} \tag{3.2}$$

with $h(x) = x^{-q}\phi(x)$. We need a C satisfying 3.1, where B is the right hand side of 3.2 and $B' = qA \otimes dx/x$. We will find one that commutes with B, in which case 3.1 reduces to $dC \cdot C^{-1} = A \otimes dh/h$. If we denote by $\mathcal{R}^0$ the integral Robba ring, i.e. the subring of $\mathcal{R}$ with coefficients in $\mathcal{V}$, then $h(x) \equiv 1 \pmod{\pi \mathcal{R}^0}$. We may then define $\log h(x)$ by the usual power series, and $\log h(x) \equiv 1 \pmod{\pi \mathcal{R}^0}$ as well. Since $e < p - 1$ the exponential $C(x) = \exp(A \otimes \log h(x))$ converges to an element of $\mathrm{GL}_n(\mathcal{R}^0)$. Since $C(x)$ commutes with A, the change of basis by $C(x)$ is the desired isomorphism $\phi^* M_A \simeq M_{qA}$.

Second method. Let t be a global parameter on $\mathbb{P}^1_{\mathcal{V}}$. If $s \in S_k$ corresponds to $t = a$ we set $x = t - a$, which is a local parameter at s. We denote by ϕ_x the lifting of the qth power Frobenius to $\mathcal{R}_s$ defined by $\phi_x(x) = x^q$. For this particular lifting it is evident from 3.1 that $\phi_x^* M_A \simeq M_{qA}$, so we must show that $\phi^* M_s \simeq \phi_x^* M_s$. Because of our choice of x, ϕ_x extends to a lifting of Frobenius on all of $\mathbb{P}^1_{\mathcal{V}}$, namely $\phi_x(t) = a^\sigma + (t - a)^q$. Then there is an isomorphism $\phi^* M^\dagger \simeq \phi_x^* M^\dagger$ if overconvergent isocrystals on U; this is the standard "independence of lifting" property of overconvergent isocrystals. More specifically it follows from [9, Prop. 7.1.6] with $Y = Y' = \mathbb{P}^1_k$, $X = X' = U$, $P = P' = \mathbb{P}^1_{\mathcal{V}}$, $u_1 = \phi$ and $u_2 = \phi_x$. Restricting the isomorphism $\phi^* M^\dagger \simeq \phi_x^* M^\dagger$ to the tube $]s[$ yields $\phi^* M_s \simeq \phi_x^* M_s$. ■

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