Automorphisms of a Symmetric Product of a Curve (with an Appendix by Najmuddin Fakhruddin)

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Abstract. Let X be an irreducible smooth projective curve of genus $g > 2$ defined over an algebraically closed field of characteristic different from two. We prove that the natural homomorphism from the automorphisms of X to the automorphisms of the symmetric product $\mathrm{Sym}^d(X)$ is an isomorphism if $d > 2g-2$. In an appendix, Fakhruddin proves that the isomorphism class of the symmetric product of a curve determines the isomorphism class of the curve.

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1. Introduction

Automorphisms of varieties is currently a very active topic in algebraic geometry; see [Og], [HT], [Zh] and references therein. Hurwitz's automorphisms theorem, [Hu], says that the order of the automorphism group $\mathrm{Aut}(X)$ of a compact Riemann surface X of genus $g \geq 2$ is bounded by $84(g-1)$. The group of automorphisms of the Jacobian $J(X)$ preserving the theta polarization is generated by $\mathrm{Aut}(X)$, translations and inversion [We], [La]. There is a universal constant c such that the order of the group of all automorphisms of any smooth minimal complex projective surface S of general type is bounded above by $c \cdot K_S^2$ [Xi].

Let X be a smooth projective curve of genus g, with $g > 2$, over an algebraically closed field of characteristic different from two. Take any integer $d > 2g-2$. Let $\mathrm{Sym}^d(X)$ be the d-fold symmetric product of X. Our aim here is to study the group $\mathrm{Aut}(\mathrm{Sym}^d(X))$ of automorphisms of the algebraic variety $\mathrm{Sym}^d(X)$. An automorphism f of the algebraic curve X produces an algebraic automorphism $\rho(f)$ of $\mathrm{Sym}^d(X)$ that sends any $\{x_1, \cdots, x_d\} \in \mathrm{Sym}^d(X)$ to $\{f(x_1), \cdots, f(x_d)\}$. This map

$$\rho : \mathrm{Aut}(X) \longrightarrow \mathrm{Aut}(\mathrm{Sym}^d(X)), \quad f \longmapsto \rho(f)$$

is clearly a homomorphism of groups. We prove the following:

Theorem 1.1. *The natural homomorphism*

$$\rho : \mathrm{Aut}(X) \longrightarrow \mathrm{Aut}(\mathrm{Sym}^d X)$$

is an isomorphism.

The idea of the proof of Theorem 1.1 is as follows. The homomorphism ρ is evidently injective, so we have to show that it is also surjective. The Albanese variety of $\mathrm{Sym}^d(X)$ is the Jacobian $J(X)$ of X. So an automorphism of $\mathrm{Sym}^d(X)$ induces an automorphism of $J(X)$. Using results of Fakhruddin (Appendix A) and Collino–Ran ([Co], [Ra]), we show that the induced automorphisms of $J(X)$ respects the theta divisor up to translation. Invoking the strong form of the Torelli theorem for the Jacobian mentioned above, it follows that such automorphisms are generated by automorphisms of the curve X, translations of $J(X)$, and the inversion of $J(X)$ that sends each line bundle to its dual. Using a result of Kempf we show that if an automorphism α of $J(X)$ lifts to $\mathrm{Sym}^d(X)$, then α is induced by an automorphism of X, and this finishes the proof.

It should be clarified that we need a slight generalization of the result of Kempf [Ke]; this is proved in Section 2. The proof of Theorem 1.1 is in Section 3.

In Appendix A by Fakhruddin the following is proved.

Let C_1 and C_2 be smooth projective curves of genus $g \geq 2$ over an algebraically closed field k. If $\mathrm{Sym}^d C_1 \cong \mathrm{Sym}^d C_2$ for some $d \geq 1$, then $C_1 \cong C_2$ unless $g = d = 2$.

2. Some properties of the Picard bundle

The degree of a line bundle ξ over a smooth projective variety Z is the class of the first Chern class $c_1(\xi)$ in the Néron-Severi group $\mathrm{NS}(Z)$, so the line bundles of degree zero on Z are classified by the Jacobian $J(Z)$.

As before, X is a smooth projective curve of genus g, with $g > 2$, over an algebraically closed field of characteristic different from two. For any integer d, let $P^d = \mathrm{Pic}^d(X)$ be the abelian variety that parametrizes the line bundles on X of degree d. It is a torsor for $J(X)$.

A *branding* of P^d is a Poincaré line bundle $\mathcal{Q}$ on $X \times P^d$ [Ke, p. 245]. Two brandings differ by tensoring with the pullback of a line bundle on P^d. A *normalized branding* is a branding such that $\mathcal{Q}|_{\{x\}\times P^d}$ has degree zero for one point $x \in X$ (equivalently, for all points of X). Two normalized brandings differ by tensoring with the pullback of a degree zero line bundle on P^d.

The natural projection of $X \times P^d$ to P^d will be denoted by π_{P^d}. A normalized branding $\mathcal{Q}$ induces an embedding

$$I_{\mathcal{Q}} : X \longrightarrow J(P^d) =: J \tag{2.1}$$

that sends any $x \in X$ to the point of J corresponding to the line bundle $\mathcal{Q}|_{\{x\}\times P^d}$ on P^d. If

$$\mathcal{Q}' = \mathcal{Q} \otimes \pi_{P^d}^* L_j \, ,$$

where L_j is the line bundle corresponding to a point $j \in J(P^d) = J$, then we have $I_{\mathcal{Q}'} = I_{\mathcal{Q}} + j$.

Assume that $d > 2g-2$. Since $H^1(X, L) = H^0(X, L^* \otimes K_X)^* = 0$ if L is a line bundle with degree$(L) > 2g-2$, the direct image $\pi_{P^d *}\mathcal{Q}$, where $\mathcal{Q}$ is a branding, is locally free. A *Picard bundle* $W(\mathcal{Q})$ on P^d is the vector bundle $\pi_{P^d *}\mathcal{Q}$, where $\mathcal{Q}$ is a normalized branding. From the projection formula it follows that two Picard bundles differ by tensoring with a degree zero line bundle on P^d.

There is a version of the following proposition for $d < 0$ in [Ke, Corollary 4.4] (for negative degree, the Picard bundle is defined using the first direct image).

PROPOSITION 2.1. *Let $d > 2g-2$.*

1. *$H^1(P^d, W(\mathcal{Q}))$ is non-zero (in fact, it is one-dimensional if it is non-zero) if and only if $0 \in I_{\mathcal{Q}}(X)$.*
2. *Let L_j be the line bundle on P^d corresponding to a point $j \in J$. Then $H^1(P^d, L_j \otimes W(\mathcal{Q}))$ is non-zero (in fact, it is one-dimensional if it is non-zero) if and only if $-j \in I_{\mathcal{Q}}(X)$.*

Proof. Part (1). If $0 \notin I_{\mathcal{Q}}(X)$, then $H^1(P^d, W(\mathcal{Q})) = 0$ by [Ke, p. 252, Theorem 4.3(c)]. Fix a line bundle M on X of degree one, and consider the associated Abel-Jacobi map

$$X \longrightarrow J(X), \quad x \longmapsto M^{-1} \otimes \mathcal{O}_X(x) \, .$$

Let $N_{X/J(X)}$ be the normal bundle of the image of X under this Abel-Jacobi map. If $0 \in I_{\mathcal{Q}}(X)$, then using [Ke, p. 252, Theorem 4.3(d)] it follows that $H^1(P^d, W(\mathcal{Q}))$ is canonically isomorphic to the space of sections of the skyscraper sheaf on X

$$K_X^{-1} \otimes \wedge^0 N_{X/J} \otimes M^d|_{I_{\mathcal{Q}}^{-1}(0)} = K_X^{-1} \otimes M^d|_{I_{\mathcal{Q}}^{-1}(0)},$$

where $I_{\mathcal{Q}}$ is constructed in (2.1). But the space of sections of this skyscraper sheaf is clearly one-dimensional, because $I_{\mathcal{Q}}^{-1}(0)$ consists of one point of X.

Part (2) follows from part (1) because $L_j \otimes W(\mathcal{Q}) = W(\mathcal{Q} \otimes \pi_{P^d}^* L_j)$, and $I_{W(\mathcal{Q}\otimes\pi_{P^d}^* L_j)} = I_{W(\mathcal{Q})} + j$. □

For a point $j \in P^d$, by $-j$ we denote the point of P^{-d} corresponding to the dual of the line bundle corresponding to j. Note that for $j, j' \in P^d$, we have $-j + 2j' \in P^d$.

PROPOSITION 2.2. *Assume that $g(X) > 1$ and $d > 2g-2$. Let j be a point of* $\mathrm{Pic}^0(X)$*, and let $T_j : P^d \longrightarrow P^d$ be the translation by j. Let M be a degree zero line bundle on P^d. If*

$$T_j^*(M \otimes W(\mathcal{Q})) \cong W(\mathcal{Q}) \, ,$$

then $j = 0$ and $M = \mathcal{O}_{P^d}$.

Let $i : P^d \longrightarrow P^d$ be the inversion given by $z \longmapsto -z + 2z_0$, where z_0 is a fixed point in P^d. If

$$i^*T_j^*(M \otimes W(\mathcal{Q})) \cong W(\mathcal{Q})\,,$$

then X is a hyperelliptic curve.

Proof. The first part is [Ke, Proposition 9.1] except that there it is assumed that $d < 0$; the proof of Proposition 9.1 uses [Ke, Corollary 4.4] which requires this hypothesis. However, the case $d > 2g-2$ can be proved similarly; for the convenience of the reader we give the details.
Let $y \in J$ be the point corresponding to the line bundle M. The line bundle on P^d corresponding to any $t \in J$ will be denoted by L_t. In particular, $M = L_y$. For every $t \in J$, using the hypothesis, we have

$$T_j^*(L_{t+y} \otimes W(\mathcal{Q})) = T_j^* L_t \otimes T_j^*(M \otimes W(\mathcal{Q})) = L_t \otimes W(\mathcal{Q})\,; \tag{2.2}$$

note that the fact that a degree zero line bundle on an Abelian variety is translation invariant is used above. Combining (2.2) and the fact that T_j is an isomorphism, we have

$$H^1(P^d,\, L_t \otimes W(\mathcal{Q})) \cong H^1(P^d,\, T_j^*(L_{t+y} \otimes W(\mathcal{Q}))) \cong H^1(P^d,\, L_{t+y} \otimes W(\mathcal{Q}))\,.$$

Using Proposition 2.1 it follows that $t \in -I_{\mathcal{Q}}(X)$ if and only if $t+y \in -I_{\mathcal{Q}}(X)$. Hence $I_{\mathcal{Q}}(X) = y + I_{\mathcal{Q}}(X)$. If $g(X) > 1$, this implies that $y = 0$. Therefore, we have $W(\mathcal{Q}) = T_j^*(W(\mathcal{Q}))$. Using the fact that $c_1(W(\mathcal{Q})) = \theta$, a theta divisor, it follows that θ is rationally equivalent to the translate $\theta - j$, hence $j = 0$.
The proof of the second part is similar. We have

$$\begin{aligned} i^*T_j^*(L_{y-t} \otimes W(\mathcal{Q})) &= i^*T_j^* L_{-t} \otimes i^*T_j^*(M \otimes W(\mathcal{Q})) \\ &= i^* L_{-t} \otimes W(\mathcal{Q}) = L_t \otimes W(\mathcal{Q})\,; \end{aligned} \tag{2.3}$$

the fact that $i^* L_{-t} = L_t$ is used above. Consequently,

$$H^1(P^d,\, L_t \otimes W(\mathcal{Q})) \cong H^1(P^d,\, i^*T^*(L_{y-t} \otimes W(\mathcal{Q}))) \cong H^1(P^d,\, L_{y-t} \otimes W(\mathcal{Q}))\,,$$

and using [Ke, Corollary 4.4] it follows that $t \in -I_{\mathcal{Q}}(X)$ if and only if $y - t \in -I_{\mathcal{Q}}(X)$. Hence $I_{\mathcal{Q}}(X) = -I_{\mathcal{Q}}(X) - y$. Let

$$f : X \longrightarrow X$$

be the morphism uniquely determined by the condition

$$I_{\mathcal{Q}}(x) = -I_{\mathcal{Q}}(f(x)) - y\,.$$

We note that f is well defined because $-I_{\mathcal{Q}}$ and $y + I_{\mathcal{Q}}$ are two embeddings of X in J with the same image, so they differ by an automorphism of X which is f. In other words, if we identify X with its image under $I_{\mathcal{Q}}$, then f is induced from the automorphism $T_{-y} \circ i$ of J. This automorphism $T_{-y} \circ i$ is clearly an involution. Let $\omega \in H^0(X, \Omega_X)$ be an algebraic 1-form on X. Then $f^*\omega = -\omega$, because of the isomorphism $H^0(X, \Omega_X) = H^0(J, \Omega_J)$ induced by $I_{\mathcal{Q}}$, and the fact that i^* acts as multiplication by -1 on the 1-forms on J. It now follows by Lemma 2.3 that f is a hyperelliptic involution. □

LEMMA 2.3. *Let $g > 1$. Let $f : X \longrightarrow X$ be an involution satisfying the condition that $f^*\omega = -\omega$ for every 1-form ω. Then X is hyperelliptic with f being the hyperelliptic involution.*

Proof. Consider the canonical morphism

$$F : X \longrightarrow \mathbb{P}(H^0(X, \Omega_X))$$

that sends any $x \in X$ to the hyperplane $H^0(X, \Omega_X(-x))$ in $H^0(X, \Omega_X)$. By definition,

$$H^0(X, \Omega_X(-x)) = \{\omega \in H^0(X, \Omega_X) \mid \omega(x) = 0\},$$

but the hypothesis implies that $\omega(x) = 0$ if and only if $\omega(f(x)) = f^*(\omega)(x) = 0$. Therefore, we have

$$H^0(X, \Omega_X(-x)) = H^0(X, \Omega_X(-f(x))),$$

and it follows that $F(x) = F(f(x))$, implying that the canonical morphism is not an embedding; note that f is not the identity because there are nonzero algebraic 1-forms. Therefore, X is hyperelliptic, and f is the hyperelliptic involution. □

We note that Lemma 2.3 is clearly false if the characteristic of the base field is two. Hence the proof of Proposition 2.2 needs the assumption that the base field has characteristic different from two.

3. PROOF OF THEOREM 1.1

Using the morphism $X \longrightarrow \text{Sym}^d(X)$, $y \longmapsto dy$, it follows that the homomorphism ρ in Theorem 1.1 is injective.

Fix a point $x \in X$. Let $\mathcal{L}$ be the normalized Poincaré line bundle on $X \times J(X)$, i.e., it is trivial when restricted to the slice $\{x\} \times J(X)$. Let

$$E := q_*(\mathcal{L} \otimes p^*\mathcal{O}_X(dx))$$

be the Picard bundle, where p and q are the projections from $X \times J(X)$ to X and $J(X)$ respectively. Since $d > 2g - 2$, it follows that E is a vector bundle of rank $d - g + 1$.

We will identify $\text{Sym}^d(X)$ with the projective bundle $P(E) = \mathbb{P}(E^\vee)$.

Let θ be the theta divisor of $J(X)$; in particular, we have $\theta^g = g!$. The Chern classes of E are given by $c_i(E) = \theta^i/i!$ [ACGH].

Let Z be a smooth projective variety and $z_0 \in Z$ a point. Then there is an abelian variety $\text{Alb}(Z)$ and a morphism

$$a_Z : Z \longrightarrow \text{Alb}(Z)$$

such that $a_Z(z_0) = 0$, and given any morphism $\phi : Z \longrightarrow A$, where A is an abelian variety and $\phi(z_0) = 0$, there is a unique homomorphism $h : \text{Alb}(Z) \longrightarrow A$ such that $h \circ a_Z = \phi$. The $\text{Alb}(Z)$ is called the Albanese variety for (Z, z_0) while a_Z is called the Albanese morphism.

The Albanese variety of $(P(E), dx)$, where x is the fixed point of X, is the Jacobian $J(X)$, and the Albanese morphism sends an effective divisor $\sum_{\ell=1}^d P_\ell$

of degree d to the degree zero line bundle $\mathcal{O}_X((\sum_{\ell=1}^d P_\ell) - dx)$. Given an automorphism

$$\varphi : P(E) \longrightarrow P(E)\,,$$

the universal property of the Albanese variety yields a commutative diagram

$$\begin{array}{ccc} P(E) & \xrightarrow[\cong]{\varphi} & P(E) \\ \downarrow & & \downarrow \\ J(X) & \xrightarrow[\cong]{\alpha} & J(X) \end{array} \tag{3.1}$$

and this produces an automorphism of projective bundles

$$\begin{array}{ccc} P(E) & \xrightarrow[\cong]{\psi} & P(\alpha^*E) \\ & \searrow \quad \swarrow & \\ & J(X) & \end{array}$$

Therefore, there is a line bundle L on $J(X)$ such that there is an isomorphism

$$\alpha^*E \,\cong\, E \otimes L\,. \tag{3.2}$$

There is a commutative diagram of groups

$$\begin{array}{ccc} \mathrm{Aut}(P(E)) & \xrightarrow{\lambda} & \mathrm{Aut}(J(X)) \\ \rho \uparrow & \nearrow_{\mu} & \\ \mathrm{Aut}(X) & & \end{array} \tag{3.3}$$

where λ is constructed as above using the universal property of the Albanese variety given in (3.1), and ρ is the homomorphism in Theorem 1.1. To construct μ, note that the commutativity of the diagram (3.1) implies that $\mu(f)$, $f \in \mathrm{Aut}(X)$, has to send $\mathcal{O}_X((\sum_{\ell=1}^d P_\ell) - dx)$ to $\mathcal{O}_X((\sum_{\ell=1}^d f(P_\ell)) - dx)$. A short calculation yields

$$\mu(f) \,=\, (f^{-1})^* \circ T_{dx - df^{-1}(x)}\,, \tag{3.4}$$

where T_a, $a \in J(X)$, is translation on $J(X)$ by a.
Let $\theta' \,=\, c_1(\alpha^*E) \,=\, \theta + L$. Then

$$c_i(\alpha^*E) \,=\, \alpha^* c_i(E) \,=\, \frac{\alpha^*\theta^i}{i!} \,=\, \frac{\theta'^i}{i!}\,,$$

and $\theta'^g \,=\, \alpha^*\theta^g \,=\, g!$. Now we apply Lemma A.2; here the condition $g \,>\, 2$ is used. We obtain that $\theta^i \,=\, \theta'^i$ for all $i \,>\, 1$.
We identify X with the image in $J(X)$ of the Abel-Jacobi map. In particular X is numerically equivalent to $\theta^{g-1}/(g-1)!$. We calculate the intersection (note that the condition $g > 2$ is again used, because we need $g - 1 > 1$)

$$\theta' X = \theta' \frac{\theta^{g-1}}{(g-1)!} = \theta' \frac{\theta'^{g-1}}{(g-1)!} = g\,.$$

Invoking a characterization of a Jacobian variety due to Collino and Ran, [Co], [Ra], it follows that $(J(X)\,,\theta'\,,X)$ is a Jacobian triple, i.e., θ' is a theta divisor of the Jacobian variety $J(X)$ up to translation. This means that θ and θ' differ by translation, in other words, the class of $c_1(L)$ in the Néron-Severi group $\mathrm{NS}(J(X))$ is zero. Consequently, α is an isomorphism of polarized Abelian varieties, i.e., it sends θ' to a translate of it.

The strong form of the classical Torelli theorem ([La, Théorème 1 and 2 of Appendix]) tells us that such an automorphism α is of the form

$$\alpha \,=\, F\circ\sigma\circ T_a\,, \quad \sigma\,\in\,\{1\,,\iota\}\,,$$

where $F \,=\, (f^{-1})^*$ for an automorphism f of X, while T_a is translation by an element $a\,\in\,J(X)$ and ι sends each element of $J(X)$ to its inverse. If X is hyperelliptic, then ι is induced by the hyperelliptic involution, so we may assume that σ is the identity map of X when X is hyperelliptic.

Let f be an automorphism of X with $F\,=\,(f^{-1})^*$ being the induced isomorphism on $J(X)$. Using the definition of E, it is easy to check that

$$F^*E\,\cong\,T^*_{dx'-dx}E\,,$$

where $x'\,=\,f^{-1}(x)$.

We claim that $\alpha\,=\,F\circ T_a$.

To prove this, assume that $\alpha\,\neq\,F\circ T_a$. Then X is not hyperelliptic, and $\alpha\,=\,F\circ\iota\circ T_a$. Hence

$$\alpha^*E\,=\,T^*_a\iota^*F^*E\,=\,T^*_a\iota^*T^*_{dx'-dx}E\,,$$

and using (3.2),

$$E\,\cong\,\iota^*T^*_{dx'-dx-a}(E\otimes L)\,.$$

Now from Proposition 2.2 it follows that X is hyperelliptic, and we arrive at a contradiction. This proves the claim.

Summing up, we can assume that $\alpha\,=\,F\circ T_a$. Using (3.2),

$$E\,\cong\,T^*_{dx-dx'-a}(E\otimes L)$$

From Proposition 2.2 it follows that L is the trivial line bundle, and $a\,=\,dx-dx'$. Therefore,

$$\alpha\,=\,(f^{-1})^*\circ T_{dx-df^{-1}(x)}$$

for some automorphism f of X, and hence, by (3.4),

$$\mathrm{Image}(\lambda)\,\subset\,\mathrm{Image}(\mu)\,. \tag{3.5}$$

We will now show that the morphism λ is injective.

Suppose $\alpha\,=\,\lambda(\varphi)\,=\,\mathrm{Id}_{J(X)}$. Using (3.2), the morphism φ is induced by an isomorphism between E and $E\otimes L$. We have just seen that L is trivial, the morphism φ is induced by an automorphism of E, and this automorphism has to be multiplication by a nonzero scalar, because E is stable with respect to the polarization given by the theta divisor (cf. [EL]). Therefore, the morphism φ is the identity. This proves that the morphism λ is injective.

The homomorphism μ is also injective, since it is a composition of a translation and the pullback induced by an automorphism of X.

Combining these it follows that the morphism ρ is also injective (this can also be checked directly), and hence all the homomorphisms in the diagram (3.3) are injective. This, combined with (3.5), shows that ρ is an isomorphism.

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Appendix A. Torelli's theorem for high degree symmetric products of curves

Najmuddin Fakhruddin

Let k be an algebraically closed field and C_1 and C_2 two smooth projective curves of genus $g > 1$ over k . It is a consequence of Torelli's theorem that if $\operatorname{Sym}^{g-1} C_1 \cong \operatorname{Sym}^{g-1} C_2$, then $C_1 \cong C_2$. The same holds for the d-th symmetric products, for $1 \leq d < g-1$ as a consequence of a theorem of Martens [Mar]. We shall show that with one exception the same result continues to hold for all $d \geq 1$, i.e., we have the following

Theorem A.1. *Let C_1 and C_2 be smooth projective curves of genus $g \geq 2$ over an algebraically closed field k. If $\operatorname{Sym}^d C_1 \cong \operatorname{Sym}^d C_2$ for some $d \geq 1$, then $C_1 \cong C_2$ unless $g = d = 2$.*

It is well known that there exist non-isomorphic curves of genus 2 over $\mathbf{C}$ with isomorphic Jacobians. Since the second symmetric power of a genus 2 curve is isomorphic to the blow up of the Jacobian in a point, it follows that our result is the best possible.

Proof of Theorem. Let C_1, C_2 be two curves of genus $g > 1$ with $\operatorname{Sym}^d C_1 \cong \operatorname{Sym}^d C_2$ for some $d \geq 1$. Since the Albanese variety of $\operatorname{Sym}^d C_i$, $d \geq 1$, is isomorphic to the Jacobian $J(C_i)$, it follows that $J(C_1) \cong J(C_2)$. If $d \leq g-1$, the theorem follows immediately from [Mar], since the image of $\operatorname{Sym}^d C_i$ in $J(C_i)$ (after choosing a base point) is $W_d(C_i)$. Note that in this case it suffices to have a birational isomorphism from $\operatorname{Sym}^d C_1$ to $\operatorname{Sym}^d C_2$.
Suppose $g \leq d \leq 2g-3$. Then the Albanese map from $\operatorname{Sym}^d C_i$ to $J(C_i)$ is surjective with general fiber of dimension $d-g$. Interpreting the fibers as complete linear systems of degree d on C_i, it follows by Serre duality that the subvariety of $J(C_i)$ over which the fibers are of dimension $> d-g$ is isomorphic to $W_{2g-2-d}(C_i)$. Therefore if $\operatorname{Sym}^d C_1 \cong \operatorname{Sym}^d C_2$, then $W_{2g-2-d}(C_1) \cong W_{2g-2-d}(C_2)$, so Martens' theorem implies that $C_1 \cong C_2$.

Now suppose that $d > 2g - 2$ and $g > 2$. By choosing some isomorphism we identify $J(C_1)$ and $J(C_2)$ with a fixed abelian variety A. If $\phi : \operatorname{Sym}^d C_1 \to \operatorname{Sym}^d C_2$ is our given isomorphism, from the universal property of the Albanese morphism we obtain a commutative diagram

$$\begin{array}{ccc} \operatorname{Sym}^d C_1 & \xrightarrow{\phi} & \operatorname{Sym}^d C_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ A & \xrightarrow{f} & A \end{array}$$

where the π_i's are the Albanese morphisms corresponding to some base points and f is an automorphism of A (not necessarily preserving the origin). By replacing C_2 with $f^{-1}(C_2)$ we may then assume that f is the identity.
Since $d > 2g - 2$, the maps π_i, $i = 1, 2$ make $\operatorname{Sym}^d C_i$ into projective bundles over A. By a theorem of Schwarzenberger [Sc], $\operatorname{Sym}^d C_i \cong \mathbb{P}(E_i)$, where E_i is a vector bundle on A of rank $d - g + 1$ with $c_j(E_i) = [W_{g-j}(C_i)]$, $i = 1, 2$, $0 \leq j \leq g-1$, in the group of cycles on A modulo numerical equivalence. Since ϕ is an isomorphism of projective bundles, it follows that there exists a line bundle L on A such that $E_1 \cong E_2 \otimes L$.
Let $\theta_i = [W_{g-1}(C_i)]$, so by Poincaré's formula $[W_{g-j}(C_i)] = \theta_i^j/j!$, $i = 1, 2$, $i \leq j \leq g - 1$. Lemma A.2 below implies that $\theta_i^{g-1} = \theta_2^{g-1}$ in the group of cycles modulo numerical equivalence on A. Since $\theta_i^g = g!$, this implies that $\theta_1 \cdot [C_2] = g$. By Matsusaka's criterion [Mat], it follows that $W_{g-1}(C_1)$ is a theta divisor for C_2, which by Torelli's theorem implies that $C_1 \cong C_2$.
If $d = 2g - 2$ and $g > 2$, then we can still apply the previous argument. In this case we also have that $\operatorname{Sym}^d(C_i) \cong \mathbb{P}(E_i)$, $i = 1, 2$ but E_i is a coherent sheaf which is not locally free. However on the complement of some point of A it does become locally free and the previous formula for the Chern classes remains valid.
The above argument clearly does not suffice if $g = 2$. To handle this case we shall use some properties of Picard bundles for which we refer the reader to [Mu]. Suppose that $d > 2$ and C_i, $i = 1, 2$ are two *non-isomorphic* curves of genus 2 with $\operatorname{Sym}^d C_1 \cong \operatorname{Sym}^d C_2$. Using the same argument (and notation) as the $g > 2$ case, it follows that there exist embeddings of C_i, $i = 1, 2$, in A and a line bundle L on A such that $E_1 \cong E_2 \otimes L$ and $L^{\otimes d-1} \cong \mathcal{O}(C_1 - C_2)$ (we identify C_i, $i = 1, 2$ with their images).
For $i \geq 1$, let G_i denote the i-th Picard sheaf associated to C_2, so that $\mathbb{P}(G_i) \cong \operatorname{Sym}^i(C_2)$. ($G_i$ is the sheaf denoted by F_{2-i} in [Mu] and $G_d \cong E_2$). There is an exact sequence ([Mu, p. 172]):

$$0 \to \mathcal{O}_A \to G_i \to G_{i-1} \to 0 \qquad \text{(A.1)}$$

for all $i > 1$. We will use this exact sequence and induction on i to compute the cohomology of sheaves of the form $E_1 \otimes P \cong E_2 \otimes L \otimes P$, where $P \in \operatorname{Pic}^0(A)$. Consider first the cohomology of G_1, which is the pushforward of a line bundle of degree 1 on a translate of C_2. Since we have assumed that $C_1 \ncong C_2$, it follows

that $C_1 \cdot C_2 > 2$. Since $C_1^2 = C_2^2 = 2$, $\deg(L|_{C_2}) = (C_1 - C_2) \cdot C_2/(d-1) > 0$. By Riemann-Roch it follows that $h^j(A, G_1 \otimes L \otimes P)$, $j = 1, 2$ is independent of P, except possibly for one P if $\deg(L|_{C_2}) = 1$, and $h^2(A, G_1 \otimes L \otimes P) = 0$ since G_1 is supported on a curve.
Now $C_1 \cdot C_2 > 2$ also implies that $c_1(L)^2 < 0$. By the index theorem, it follows that $h^0(A, L \otimes P) = h^2(A, L \otimes P) = 0$ and $h^1(A, L \otimes P)$ is independent of P. Therefore by tensoring the exact sequence (A.1) with $L \otimes P$ and considering the long exact sequence of cohomology, we obtain an exact sequence

$$\begin{aligned} 0 \to H^0(A, G_i \otimes L \otimes P) \to H^0(A, G_{i-1} \otimes L \otimes P) \to H^1(A, L \otimes P) \\ \to H^1(A, G_i \otimes L \otimes P) \to H^1(A, G_{i-1} \otimes L \otimes P) \to 0 \quad \text{(A.2)} \end{aligned}$$

and isomorphisms $H^2(A, G_i \otimes L \otimes P) \to H^2(A, G_{i-1} \otimes L \otimes P)$ for all $i > 1$. By induction, it follows that $H^2(A, G_i \otimes L \otimes P) = 0$ for all $i > 0$. Since the Euler characteristic of $G_i \otimes L \otimes P$ is independent of P, the above exact sequence (A.2) along with induction shows that for all $i > 0$ and $j = 0, 1, 2$, $h^j(G_i \otimes L \otimes P)$ is independent of P, except for possibly one P. In particular, this holds for $i = d$ hence $h^j(A, E_1 \otimes P)$ is independent of P except again for possibly one P. We obtain a contradiction by using the computation of the cohomology of Picard sheaves in Proposition 4.4 of [Mu]: This implies that $h^1(A, E_1 \otimes P)$ is one or zero depending on whether the point in A corresponding to P does or does not lie on a certain curve (which is a translate of C_1). □

Lemma A.2. *Let X be an algebraic variety of dimension $g \geq 3$ and let E_i, $i = 1, 2$ be vector bundles on X of rank r. Suppose $c_1(E_i) = \theta_i$, $c_j(E_i) = \theta_i^j/j!$ for $i = 1, 2$ and $j = 2, 3$ ($j = 2$ if $g = 3$), and $E_1 \cong E_2 \otimes L$ for some line bundle L on X. Then $\theta_1^j = \theta_2^j$ for all $j > 1$ ($j = 2$ if $g = 3$).*

Proof. Since $E_1 \cong E_2 \otimes L$, $c_1(E_1) = c_1(E_2) + rc_1(L)$, hence $c_1(L) = (\theta_1 - \theta_2)/r$. For a vector bundle E of rank r and a line bundle L on any variety, we have the following formula for the Chern polynomial ([Fu], page 55):

$$c_t(E \otimes L) = \sum_{k=0}^{r} t^k c_t(L)^{r-k} c_i(E).$$

Letting $E = E_1$, $E \otimes L = E_2$, and expanding out the terms of degree 2 and 3, one easily sees that $\theta_1^j = \theta_2^j$ for $j = 2$ and also for $j = 3$ if $g > 3$. (Note that this only requires knowledge of $c_j(E_i)$ for $j = 1, 2, 3$.) Since any integer $n > 1$ can be written as $n = 2a + 3b$ with $a, b \in \mathbf{N}$, the lemma follows. □

Remark A.3. Using Theorem 1.1, one sees that Theorem A.1 holds over all perfect fields k (of characteristic > 2) if $d > 2g - 2$: For projective varieties X, Y over a field let $\underline{\mathrm{Isom}}(X, Y)$ denote the scheme of isomorphisms. For any $d > 0$, there is a natural map

$$\underline{\mathrm{Isom}}(C_1, C_2) \to \underline{\mathrm{Isom}}(\mathrm{Sym}^d C_1, \mathrm{Sym}^d C_2)$$

of finite schemes over k which one sees is a bijection on geometric points by combining Theorem 1.1 and Theorem A.1. If k is perfect[1] this implies that the map on k-rational points is also a bijection.

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References

[ACGH] E. Arbarello, M. Cornalba, P. Griffiths, J. Harris, Algebraic Curves. Grundlehren der mathematischen Wissenschaften 267. Springer-Verlag 1985.

[BDH] I. Biswas, A. Dhillon and J. Hurtubise, Automorphisms of the Quot Schemes Associated to Compact Riemann Surfaces, *Int. Math. Res. Not.* (2015), 1445–1460.

[Co] A. Collino, A new proof of the Ran-Matsusaka criterion for Jacobians, *Proc. Amer. Math. Soc.* 92 (1984), 329–331.

[EL] L. Ein and R. K. Lazarsfeld, Stability and restrictions of Picard bundles, with an application to the normal bundles of elliptic curves, *Complex projective geometry (Trieste, 1989/Bergen, 1989)*, 149–156, London Math. Soc. Lecture Note Ser., 179, Cambridge Univ. Press, Cambridge, 1992.

[Fu] W. Fulton, *Intersection theory,* Springer-Verlag, Berlin, second ed., 1998.

[HT] B. Hassett and Y. Tschinkel, Extremal rays and automorphisms of holomorphic symplectic varieties, *K3 surfaces and their moduli*, 73–95, Progr. Math., 315, Birkhäuser/Springer, Cham, 2016.

[Hu] A. Hurwitz, Über algebraische Gebilde mit Eindeutigen Transformationen in sich, *Math. Ann.* 41 (1893), 403–442.

[Ke] G. Kempf, Toward the inversion of abelian integrals. I, *Ann. of Math.* 110 (1979), 243–273.

[La] K. Lauter, Geometric methods for improving the upper bounds on the number of rational points on algebraic curves over finite fields. With an appendix by J.-P. Serre, *Jour. Alg. Geom.* 10 (2001), 19–36.

[Mar] H. H. Martens, An extended Torelli theorem, *Amer. J. Math.* 87 (1965), 257–261.

[Mat] T. Matsusaka, On a characterization of a Jacobian variety, *Memo. Coll. Sci. Univ. Kyoto. Ser. A. Math.* 32 (1959), 1–19.

[Mu] S. Mukai, Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves, *Nagoya Math. J.* 81 (1981), 153–175.

[1] In fact, using the methods of [BDH] one may see that $\underline{\mathrm{Isom}}(\mathrm{Sym}^d C_1, \mathrm{Sym}^d C_2)$ is etale, so the statement actually holds over any field.

[Og] K. Oguiso, Simple abelian varieties and primitive automorphisms of null entropy of surfaces, *K3 surfaces and their moduli*, 279–296, Progr. Math., 315, Birkhäuser/Springer, Cham, 2016.

[Ra] Z. Ran, On subvarieties of abelian varieties, *Invent. Math.* 62 (1981), 459–479.

[Sc] R. L. E. Schwarzenberger, Jacobians and symmetric products, *Illinois J. Math.* 7 (1963), 257–268.

[Xi] G. Xiao, Bound of automorphisms of surfaces of general type. I, *Ann. of Math.* 139 (1994), 51–77.

[We] A. Weil, Zum beweis des Torelli satzes, *Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II* 2 (1957), 33–53.

[Zh] D.-Q. Zhang, Birational automorphism groups of projective varieties of Picard number two, *Automorphisms in birational and affine geometry*, 231–238, Springer Proc. Math. Stat., 79, Springer, Cham, 2014.

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