Leibniz and the Infinite

Eberhard Knobloch

2010 Mathematics Subject Classification: 01A45, 28-03
Keywords and Phrases: Leibniz, infinite, infinitely small, mathematical rigour, integration theory

The German universal genius Gottfried Wilhelm Leibniz was born in Leipzig on the 21st of June according to the Julian calendar (on the 1st of July according to the Gregorian calendar) 1646. From 1661 he studied at the universities of Leipzig and Jena. On February 22, 1667 he became Doctor of Laws at the university of Nürnberg-Altdorf. He declined the professorship that was offered to him at this university. For a short time he accepted a position at the court of appeal of Mainz. From 1672 to 1676 he spent four years in Paris where he invented his differential and integral calculus in autumn 1675.

From 1676 up to the end of his life he earned his living as librarian at the court of the duke, then elector, of Hannover. In 1700 he was appointed president of the newly founded Electoral Academy of Sciences of Berlin. He contributed to nearly all scientific disciplines and left the incredibly huge amount of about 200 000 sheets of paper. Less than one half of them have been published up to now.

In Paris he became one of the best mathematicians of his time within a few years. He was especially interested in the infinite. But what did he mean by this notion? His comments on Galileo's Discorsi give the answer. Therein Galileo had demonstrated that there is a one-to-one correspondence between the set of the natural numbers and the set of the square numbers. Hence in his eyes the Euclidean axiom "The whole is greater than a part" was invalidated in the sense that it could not be applied there: infinite sets cannot be compared with each other with regard to their size. Leibniz contradicted him. For him it was impossible that this axiom failed. This only seemed to be the case because Galileo had presupposed the existence of actually infinite sets. For him the universal validity of rules was more important than the existence of objects, in this case of actually infinite numbers or actually infinite sets. Hence Leibniz did not admit actual infinity in mathematics. "Infinite" meant "larger than any given quantity". He used the mode of possibility in order to characterize the mathematical infinite: it is always possible to find a quantity that is larger than any given quantity.

Figure 1: Portrait of Leibniz by A. Scheit, 1703 (By courtesy of the Gottfried Wilhelm Leibniz Library, Hannover)

By using the mode of possibility he consciously imitated ancient models like Aristotle, Archimedes, and Euclid. Aristotle had defined the notion of quantity in his Metaphysics: quantity is what can be divided into parts being in it. Something (a division) can be done in this case. If a division of a certain object is not possible, the object cannot be a quantity. In the 17th and 18th centuries mathematics was the science of quantities. Hence it could not handle non-quantities. Hence Leibniz avoided non-quantities in mathematics by all means.

Indivisibles were non-quantities by definition: they cannot be divided. Yet they occurred even in the title of Bonaventura Cavalieri's main work Geometry developed by a new method by means of the indivisibles of continua. Cavalieri's indivisibles were points of a line, straight lines of a plane, planes of a solid. Leibniz amply used this notion, for example in the title of the first publication of his integral calculus Analysis of indivisibles and infinites that appeared in 1686. But according to his mathematical convictions he had to look for a suitable, new interpretation of the notion.

From 1673 he tried different possibilities like smallest, unassignable magnitude, smaller than any assignable quantity. He rightly rejected all of them because there are no smallest quantities and because a quantity that is smaller than any assignable quantity is equal to zero or nothing. In spring 1673 he finally stated that indivisibles have to be defined as infinitely small quantities or the ratio of which to a perceivable quantity is infinite. Thus he had shifted the problem. Now he had to answer the question: What does it mean to be infinitely small? Still in 1673 he gave an excellent answer: infinitely small means smaller than any given quantity. He again used the mode of possibility and introduced a consistent notion. Its if-then structure – if somebody proposes a quantity, then there will be a smaller quantity – rightly reminds the modern reader of Weierstraß's ϵ-δ-language. Leibniz's language can be translated into Weierstraß's language.

Leibniz used this well-defined notion throughout the longest mathematical treatise he ever wrote, in his Arithmetical quadrature of the circle, of the ellipse, and of the hyperbola. Unfortunately it remained unpublished during his lifetime though he wrote it already in the years 1675/76. Only in 1993 did the first printed version appear in Göttingen.

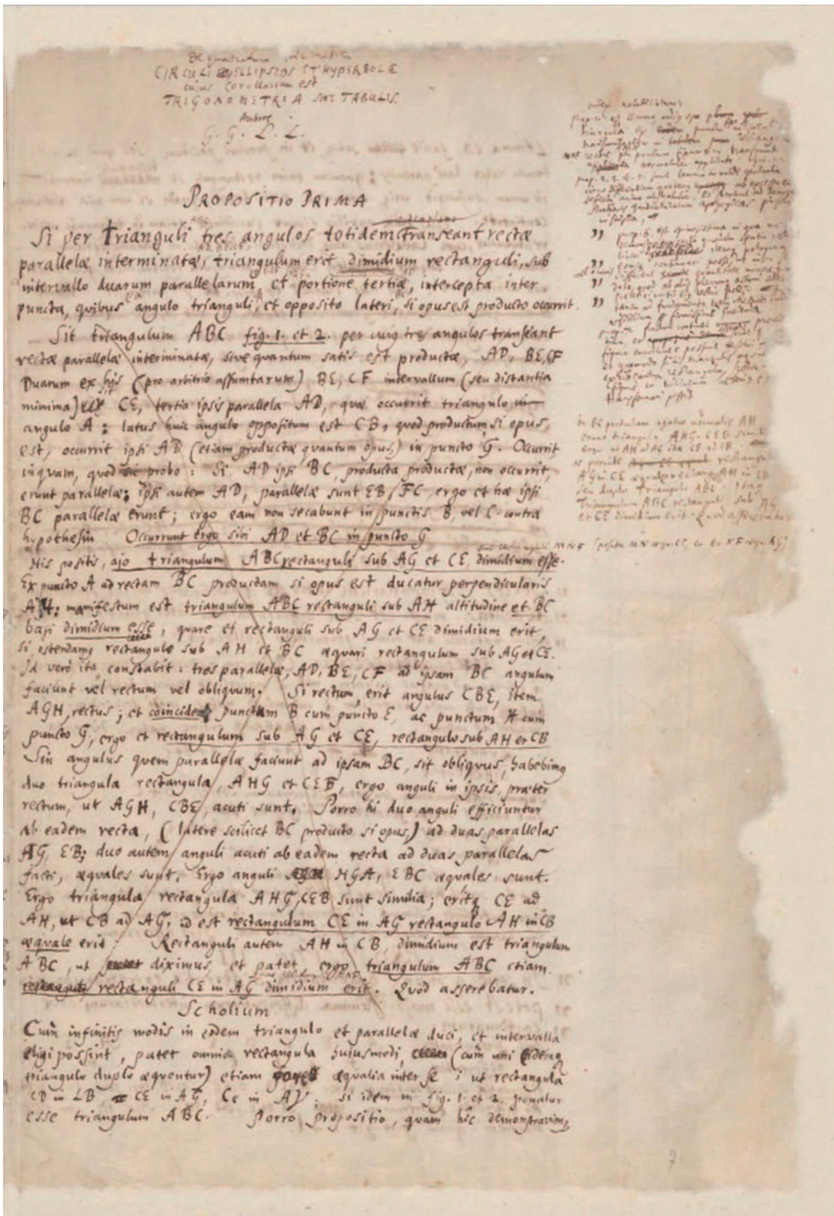
For this reason Leibniz has been falsely accused of neglecting mathematical rigour again and again up to the present day. His Arithmetical quadrature contains the counterdemonstration of that false criticism. Therein theorem 6 gives a completely rigorous foundation of infinitesimal geometry by means of Riemannian sums. Leibniz foresaw its deterrent effect saying:

> The reading of this proposition can be omitted if somebody does not want supreme rigour in demonstrating proposition 7. And it will be better that it be disregarded at the beginning and that it be read only after the whole subject has been understood, in order that its excessive exactness does not discourage the mind from the other, far more agreeable, things by making it become weary prematurely. For it achieves only this: that two spaces of which one passes into the other if we progress infinitely, approach each other with a difference that is smaller than any arbitrary assigned difference, even if the number of steps remains finite. This is usually taken for granted, even by those who claim to give rigorous demonstrations.

Leibniz referred to the ancients like Archimedes who was still the model of mathematical rigour. After the demonstration Leibniz stated: "Hence the method of indivisibles which finds the areas of spaces by means of sums of lines can be regarded as proven." He explicitly formulated the fundamental idea of the differential calculus, that is, the linearization of curves:

> The readers will notice what a large field of discovery is opened up once they have well understood only this: that every curvilinear figure is nothing but a polygon with infinitely many infinitely small sides.

When he published his differential calculus for the first time in 1684 he repeated this crucial idea. From that publication he had to justify his invention. In 1701 he rightly explained:



CIRCULI ELLIPSEOS ET HYPERBOLE
cujus Corollarium est
TRIGONOMETRIA SINE TABULIS
Autore
G. G. L. L.

PROPOSITIO PRIMA

Si per trianguli tres angulos totidem transeant rectæ parallelæ interminatæ, triangulum erit dimidium rectanguli, sub intervallo duarum parallelarum, et portione tertiæ, intercepta inter puncta, quibus angulo trianguli, et opposito lateri, si opus est producto occurrit.

Figure 2: First page of Leibniz's treatise Arithmetical quadrature of the circle etc. (By courtesy of the Gottfried Wilhelm Leibniz Library, Hannover. Shelf mark LH XXXV 2,1 folio 7r)

> Because instead of the infinite and the infinitely small one takes quantities that are as large or as small as it is necessary so that the error is smaller than the given error so that one differs from the style of Archimedes only by the expressions which are more direct in our method and more suitable for the art of invention.

The story convincingly demonstrates the correctness of his saying: "Those who know me only by my publications don't know me."

References

[1] E. Knobloch, Leibniz's rigorous foundation of infinitesimal geometry by means of Riemannian sums, Synthese 133 (2002), 59–73.

[2] [2] E. Knobloch, Galileo and German thinkers: Leibniz, in: L. Pepe (ed.), Galileo e la scuola galileiana nelle Università del Seicento, Cooperativa Libraria Universitaria Bologna, Bologna 2011, pp. 127–139.

Eberhard Knobloch
Berlin-Brandenburg Academy
of Sciences and Humanities
Technische Universität Berlin
H 72
Straße des 17. Juni 135
10623 Berlin
eberhard.knobloch@tu-berlin.de