The Princess and Infinite-Dimensional Optimization

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Abstract. Traces of infinite-dimensional optimization can be sourced to ancient Greek mathematics. According to a legend the knowledge about the solution of such kind of problems helped on the foundation of Carthage, and today's new subfields of infinite-dimensional optimization such as optimal control, shape or topology optimization are indispensable in propelling present and future technological developments.

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The wish for optimization seems to be deeply grounded in mankind. How often somebody says proudly: "Now I have optimized it *again*!" [for example, the author's spouse or colleagues from engineering departments, etc. The author will not comment here on the word "*again*".] Hence there must be traces of optimization deep in human history.

Like most mathematicians, the author likes to trace the roots of his own research area and to search for his scientific ancestors and funny stories around them. Therefore, this article tries to answer the question "What is the first infinite-dimensional constrained optimization problem?". But the reader may be warned. The answer may come from a subjective viewpoint and may be affected by the "optimization of the attractiveness" of the stories behind these questions.

For the non-experts, in infinite-dimensional optimization we want to find optimal solutions of problems where the optimization variables are elements of infinite-dimensional spaces or even more complicated objects such as functions, curves, sets, shapes, topologies etc. The search for extremal points of real-valued functions of real variables known from school is not meant. At a first glance, this may indicate that we cannot go back farther than to the invention of calculus by Leibniz and Newton at the end of the 17th century. However, this is not true as we will see.

1 Renaissance in mathematics

Johann Bernoulli's price question (acutissimis qui toto Orbe florent mathematicis, for the most astucious mathematicians of the entire globe)[1] may come into mind first: "What is the curve of quickest descent between two given fixed points in a vertical plane?" (1696) and Newton's problem: "What is the shape of a body of minimum resistance?" (1687).[2] The first problem was created by Johann Bernoulli to tease his older brother Jacob from whom he knew that he was working on those kind of problems and Johann hoped that his brother and teacher would not be able to find an answer. He erred; see, e.g., Goldstine (1980).

Both problems are typical infinite-dimensional problems. Their analytical solution is even today not possible without a solid knowledge of calculus, a few years earlier invented by Leibniz (1684),[3] resp. Newton (1736).[4]

Johann Bernoulli's problem reads as follows, cp. Fig. 1:

$$\inf_{y \in Y_{\rm ad}} \frac{1}{\sqrt{2\,g}} \int_0^{x_B} \frac{\sqrt{1+(y'(x))^2}}{\sqrt{-y(x)}}\,\mathrm{d}x\,,$$

where the set of admissible functions $Y_{\rm ad}$ is defined by

$$Y_{\rm ad} := \big\{ y \text{ is continuous on } [0,x_B] \text{ and continuously differentiable on } (0,x_B) \\ \text{with prescribed boundary conditions } y(0)=0 \text{ and } y(x_B)=y_B \big\}\,.$$

Here g denotes the Earth's gravitational acceleration.

Sir Isaac Newton's problem reads as follows: the total resistance of particles that hit the body (nose of the aircraft or its airfoils; see Fig. 9) exactly once and transfer their momentum to the body, is the sum over the body of these transfers of momentum:

$$\inf_{y \in Y_{\rm ad}} \int_\Omega \frac{\mathrm{d}x}{1+\|\nabla y(x)\|_2^2}\,,$$

with

$$Y_{\rm ad} := \{ y\colon \Omega \to [0,M] \subset \mathbb{R} :\ \Omega \subset \mathbb{R}^2 \text{ bounded and } y \text{ concave} \}\,.$$

[1]Bernoulli, Johann, Problema novum ad cujus solutionem Mathematici invitantur, *Acta Eruditorum*, pp. 269, 1696; see also *Johannis Bernoulli Basileensis Opera Omnia*, Bousquet, Lausanne and Geneva, Switzerland, Joh. Op. XXX (pars), t. I, p. 161, 1742.

[2]Newton, Isaac: *Philosophiae Naturalis Principia Mathematica*, submitted 1686 to the Royal Society, published 1687, 2nd ed. 1713, 3rd ed. 1726, commented 3rd ed. by the Franciscans Thomas Le Seur and François Jacquier using Leibniz' calculus (!), 1739–1742.

[3]Leibniz, Gottfried Wilhelm: *Nova methodus pro maximis et minimis, itemque tangentibus, quae nec fractas nec irrationales quantitates moratur, et singulare pro illis calculi genus*, Acta Eruditorum, 1984.

[4]Newton, Isaac: *The method of Fluxions and Infinite Series with its Application to the Geometry of Curve-lines*, 1736. Newton's work was already existent and ready for press in 1671 (in Latin). The English translation, however, appeared not until 1736 after Newton's death. This has contributed to the priority quarrel between Newton and Leibniz; see, e.g., Wußing (2009), p. 471ff, and the references cited therein.

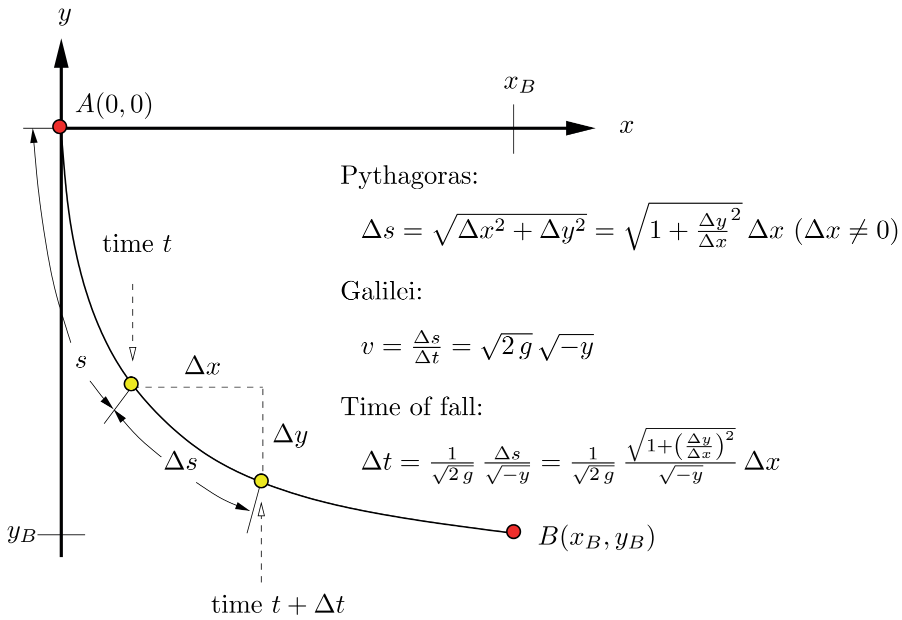

Figure 1: Bernoulli's Brachistochrone Problem: Pythagoras' theorem, Galilei's law of free fall and a summation over all infinitesimal time intervals Δt yields the minimization problem. For its solution, Johann Bernoulli applied the idea of discretization and associated the curve of quickest descent with a ray of light through layers of different media and the fall velocity with the speed of light. By Fermat's principle of least time, resp. Snell's law of refraction, Johann Bernoulli derived the differential equation $y(x)\,\left(1 + (y'(x))^2\right) = -2\,r$, $r > 0$, as necessary condition, the solutions of which were known to be cycloids: $x(\theta) = r\,(\theta - \sin\theta)$, $y(\theta) = -r\,(1 - \cos\theta)$, $0 \leq \theta \leq \theta_B$, with r and θ_B defined by the terminal conditions $x(\theta_B) = x_B$ and $y(\theta_B) = y_B$.

Newton: *I reckon that this proposition will be not without application in the building of ships.*[2]

This old problem is still inspiring current research; see, e.g., Buttazzo et. al. (1993) and Lachand-Robert and Peletier (2001).

In his famous reply[5] to the problem of his younger brother Johann, Jacob Bernoulli posed the following even more difficult problem: "What is the shape of the planar closed curve, resp. of the associated bounded set surrounded by

[5]Bernoulli, Jacob, Solutio Problematum Fraternorum, una cum Propositione reciproca aliorum, *Acta Eruditorum*, pp. 211–217, 1697; see also *Jacobi Bernoulli Basileensis Opera*, Cramer & Philibert, Geneva, Switzerland, Jac. Op. LXXV, pp. 768–778, 1744.

this curve that contains the maximum area while its perimeter is restricted?",

$$\inf_{\gamma\in\Gamma_{\mathrm{ad}}} \int_a^b \left(x\,\frac{\mathrm{d}y}{\mathrm{d}t} - y\,\frac{\mathrm{d}x}{\mathrm{d}t} \right)\,\mathrm{d}t\,,$$

where the set Γ_{ad} of admissible curves is given by

$$\Gamma_{\mathrm{ad}} := \left\{ \gamma\colon [a,b] \ni t \mapsto \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \in \mathbb{R}^2 : \int_a^b \sqrt{\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t = L > 0 \right\}.$$

Or, more generally, in modern mathematical language

$$\sup_{\Omega\in\mathcal{O}_{\mathrm{ad}}} \int_\Omega \mathrm{d}x$$

with the set $\mathcal{O}_{\mathrm{ad}}$ of all admissible sets given by

$$\mathcal{O}_{\mathrm{ad}} := \{\Omega \subset \mathbb{R}^n : \ \Omega \text{ bounded}\,,\ n \in \mathbb{N}\,,\ \text{and} \ \int_{\partial\Omega} \mathrm{d}s = L > 0\}\,.$$

Here, $\partial\Omega$ denotes the (sufficiently smooth) boundary of the set Ω, and L is a given positive constant determining the perimeter, resp. surface.

In all these problem statements, we are searching for a minimizer or maximizer being an element of an infinite-dimensional (huge) "something" where the criterion which is to be optimized depends on those objects. In addition, restrictions must be obeyed. Using an appropriate interpretation, all these problems can be considered to be the mother problems of important fields of continuous optimization: the classical Calculus of Variations, a playground of such mathematical heros like Euler, Lagrange, Legendre, Jacobi, Weierstrass, Hilbert, and Carathéodory, and the modern theories of optimal control (Fig. 2), an offspring of the Cold War [Pesch and Plail (2012)], and the rather current fields shape, resp. topology optimization.

This first so-called isoperimetric problem of Jacob Bernoulli is known as Dido's problem in the mathematical literature. This points to an antique origin even far before the turn from the 17th to the 18th century, far before the times of those mathematical pioneers Leibniz, Newton, and the Bernoulli brothers. Hence this problem, at least a simplified version of it, must be solvable by geometric means, too.

2 Florescence in mathematics in antiquity

Indeed, the first isoperimetric problem, more complicated than Euclid's earlier theorem[6] saying that the rectangle of maximum area with given perimeter is

[6] Little is known about Euclid's life, but we have more of his writings than of any other ancient mathematician. Euclid was living in Alexandria about 300 B.C.E. based on a passage in Proclus' Commentary on the First Book of Euclid's Elements; cp. `http://aleph0.clarku.edu/~djoyce/java/elements/Euclid.html`.

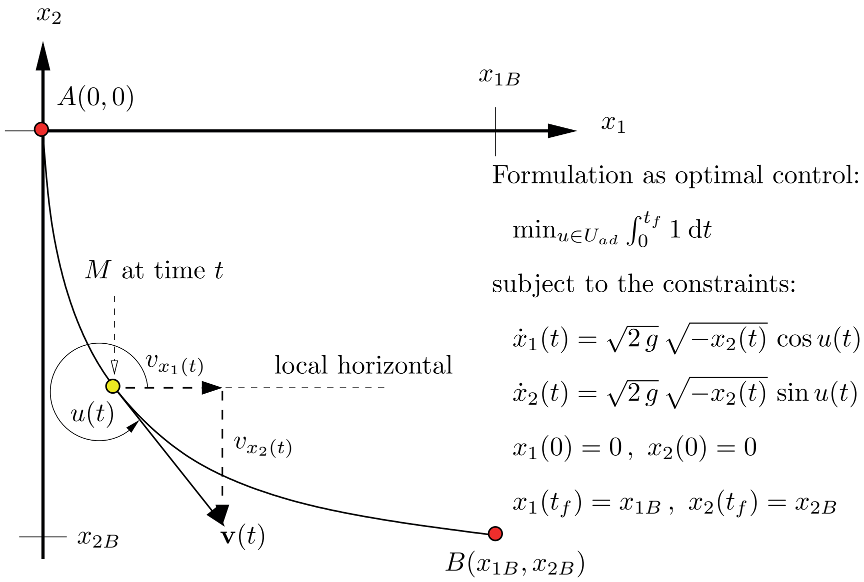

Figure 2: The Brachistochrone problem formulated as problem of optimal control with the class U_{ad} of admissible controls (slopes) defined by $U_{\mathrm{ad}} := \{u \colon [0, t_f] \to (0, 2\,\pi) \,:\, \text{u continuous}\}$. The optimal control u^* is determined by the minimum principle: $u^*(t) = \arg\min_{u \in U_{\mathrm{ad}}} H(\mathbf{x}(t), \mathbf{p}(t), u)$ with the state vector $\mathbf{x} := (x_1, x_2)^\top$ and the adjoint state vector $\mathbf{p} := (p_1, p_2)^\top$. Hereby, the Hamiltonian is defined by $H(\mathbf{x}, \mathbf{p}, u) := 1 + \sqrt{2\,g}\,\sqrt{-x_2}\,(p_1 \cos u + p_2 \sin u)$ and the adjoint state vector $\mathbf{p}$ must satisfy the canonical equation $\dot{\mathbf{p}} = -H_{\mathbf{x}}$.

the square, came down to us in written form by Theon Alexandreus[7] in his commentaries on Klaudios Ptolemaios'[8] *Mathematical Syntaxis*, a handbook of astronomy called *Almagest*.[9] In this syntaxis one can find a theorem which is

[7]Theon Alexandreus: ∗ about 335 C.E. probably in Alexandria, † ca. 405 C.E.; see. e.g., `http://www-groups.dcs.st-andrews.ac.uk/~history/Mathematicians/Theon.html`.
He edited Euclid's *Elements*, published around 364 C.E., authoritative into the 19th century. His daughter Hypatia (∗ about 351 C.E., about † ca. 370 C.E.; cf. Fig. 7) also won fame as the first historically noted women in mathematics. She was murdered by a Christian mob after being accused of witchcraft.
For more see `http://www-history.mcs.st-and.ac.uk/Biographies/Hypatia.html`.

[8]Klaudios Ptolemaios: ∗ about. 85–100 C.E. in Upper Egypt, † about 165–180 C.E. in Alexandria; see, e.g., `http://www-groups.dcs.st-andrews.ac.uk/~history/Mathematicians/Ptolemy.html`. In contrast to Aristarchos of Samos and Seleukos of Seleukia, who both already pleaded the heliocentric system, Ptolemaios held on the geocentric system.

[9]See `http://en.wikipedia.org/wiki/Almagest`.

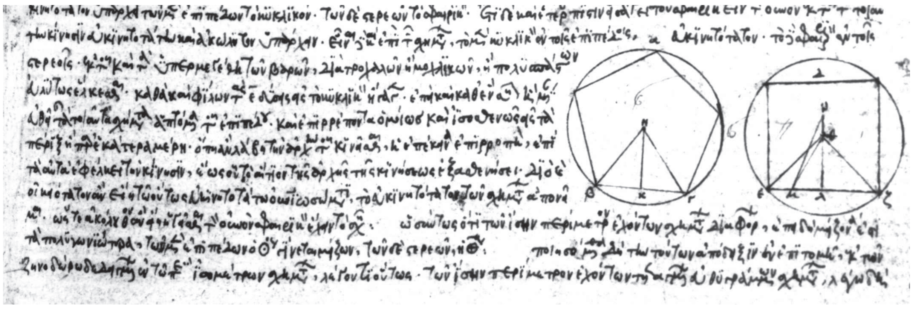

Figure 3: Zenodoros' theorem in a fourteenth century manuscript of the city library of Nuremberg (Cod. Nür. Cent. V App. 8, detail of p. 58ʳ).

accredited to Zenodoros, but may be even older.[10] See also Heath (1981).

It is cited there from Zenodoros' treatise "Περὶ ἰσομέτρων σχημάτων" (On isometric figures) of about the 2nd century B.C.E.:

In the middle of the antepenultimate line of Fig. 3, we can read:

> Ὡσαύτως δ' ὅτι τῶν ἴσην περίμετρον ἐχόντων σχημάτων διαφόρων, ἐπειδὴ μείζονά ἐστι τὰ πολυγωνιώτερα, τῶν μὲν ἐπιπέδων ὁ κύκλος [ligature ⊙] γίνεται μείζων, τῶν δὲ στερεῶν ἡ σφαῖρα [ligature ⊕]. Ποιησόμεθα δὴ τὴν τούτων ἀπόδειξιν ἐν ἐπιτομῇ ἐκ τῶν Ζηνοδώρῳ δεδειγμένων ἐν τῷ 'Περὶ ἰσομέτρων σχημάτων'.
>
> Just as well, since those of different figures which have the same contour are larger which have more angles, the circle is larger than the (other) plane figures and the sphere than the (other) solids. We are going to present the proof for this in an extract of the arguments as has been given by Zenodoros in his work 'On isometric figures'.

Figure 4 shows the entire page No. 58ʳ with Zenodoros' theorem in a fourteenth century manuscript of the city library of Nuremberg. The reverse side shows his proof whose elegance was praised by Carathéodory.[11] For Zenodoros' proof in modern mathematical language and other proofs of his theorem, it is referred to Blåsjö (2005). This ancient problem also still inspires mathematicians until today; see, e.g., Almeida et. al. (2012) for a very recent contribution.

This codex was in possession of the Lower-Franconian mathematician and astronomer Johannes Müller better known as Regiomontanus,[12] who received

[10] Zenodoros: ∗ about 200 B.C.E. in Athen, † about 140 B. C. in Greece; see. e.g., `http://www-groups.dcs.st-andrews.ac.uk/~history/Mathematicians/Zenodorus.html`.

[11] Carathéodory, C.: *Basel und der Beginn der Variationsrechnung*, publication in honor of the sixtieth birthday of Professor Dr. Andreas Speiser, Zürich, Switzerland, 1945; see also Carathéodory, C.: *Gesammelte Mathematische Schriften* 2, C. H. Beck'sche Verlagsbuchhandlung, Munich, Germany, pp. 108–128, 1955.

[12] Johannes Müller (Regiomontanus): ∗ 1436 in Königsberg in Bavaria, † 1476 in

Figure 4: Zenodoros' theorem in a fourteenth century manuscript of the city library of Nuremberg (Cod. Nür. Cent. V App. 8, p. 58^{r}), entire page.

it as a gift from his patron Cardinal Johannes Bessarion, titular patriarch of Constantinople. The codex served as the original printing copy for the *editio princeps* of 1538 published in Basel.

Already hundreds of years before Zenodoros' theorem was proven, "engineering intuition" brought the Phoenician princess Elissa (Roman: Dido) of Tyros, today Sur, Lebanon, to take advantage of it. According to a legend,[13] Dido was on the run from her power-hungry brother Pygmalion, who already had ordered the murder of her husband Acerbas and strived for her life and wealth. Dido with her abiders came, on a sail boat, to the shores of North Africa in the region of today's Tunis, Tunesia at around 800 B.C.E. The local habitans were friendly, but did not want to have the armed strangers in their vicinity

Rome; see, e.g., `http://www-groups.dcs.st-andrews.ac.uk/~history/Mathematicians/Regiomontanus.html`.
Regiomontanus is the Latin word for his native town Königsberg (it is not the famous Königsberg in Prussia, today's Kaliningrad, Russia, which gave Euler's Problem of the Seven Bridges of Königsberg its name.

[13]The legend seems to be apocryphal and may be fictious, but very appropriately invented for the Ionian-Greek word ἡ βῦρσα meaning oxhide.

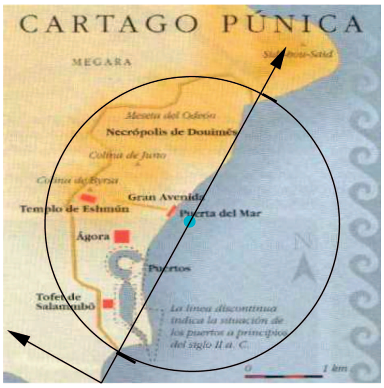

Figure 5: The Punic Carthage and Zenodoros' optimal solution as realized by Dido. Surely Dido has chosen a piece of land by the coast so as to exploit the shore as part of the perimeter

permanently. Therefore the resourceful princess asked the local king Iarbas for a small amount of grassland for their livestock, only so small that it can be covered by an oxhide. Iarbas laughed and could not refuse Dido's modest request. Dido then cut the hide into thin strips (Fig. 6), encircled a large area (Fig. 5) on which their fellows erected the new city *Qart-Hadašt* (Phoenician for new city) with the citadel *Byrsa*, from which the ancient superpower Carthage later developed.

So far the first part of the legend. We will omit here the tragic lovestory between Dido and the Trojan hero Aeneas, who came to Dido's adopted home after his fly from Troja. He left her by command of Jupiter whereupon Dido threw herself into the flames of the fire by which she burned all things that Aeneas and his compagnions left behind. This curse is said to be the source for the later enmity between Rome and Carthage.

The legend of the founding of Carthage was sung by Publius Vergilius Maro[14] in his famous Aeneid (book one, verses 365–368):

devenere locos, ubi nunc ingentia cernis
moenia surgentemque novae Karthaginis arcem,
mercatique solum, facti de nomine Byrsam,
taurino quantum possent circumdare tergo.

and in English verses, perpetuating the hexameter, from the translation of the famous English poet John Dryden, a contemporary of the Bernoulli Brothers:

At last they landed, where from far your Eyes
May view the Turrets of new Carthage rise:

[14]Publius Vergilius Maro, Roman poet: ∗ 70 B.C.E. in Andes (Pietole?) near Mantua, † 19 B.C.E. in Brundisium (Brindisi)

Figure 6: Dido purchases land for the foundation of Carthage, engraving by Mathias Merian the elder from Historische Chronica, Frankfurt a. M., 1630.

There bought a space of Ground, which 'Byrsa' call'd
From the Bull's hide, they first inclos'd, and wall'd.

or in an older translation by the sixteenth century authors Thomas Phaer and Thomas Twyne:

Than past they forth and here they came, where now thou shalt espie
The hugy walles of new Carthage that now they rere so hie.
They bought the soile and Birsa it cald whan first they did begin,
As much as with a bull hide cut they could inclose within.

3 Florescence in mathematics today

Back to present and future: What is the optimal shape of a very fast aircraft, say which is able to fly at supersonic speed with minimal drag? Indeed, that is a modern version of Dido's problem. Figure 8 shows effects of aerodynamic drag minimizing on airfoil and body of a supersonic cruise transporter due to Brezillon and Gauger (2004).

More challenges are waiting such as fuel optimization of aircraft using laminar flow airfoils with blowing and sucking devices or using morphing shape airfoils with smart materials and adaptive structures built-in. Figure 9 shows the, in this sense, non-optimized flow around the Airbus A 380 computed by numerical simulation. Optimization with those respects may be next steps for which infinite-dimensional optimization in various specifications must be employed: optimal control of ordinary and partial differential equations as well as

Figure 7: Hypathia, detail of The Schooll of Athens' by Raphael

shape and topology optimization. Their roots can be traced, tongue-in-cheek, to the renaissance of mathematics with the invention of calculus and even as far as to the geometricians in antiquity.

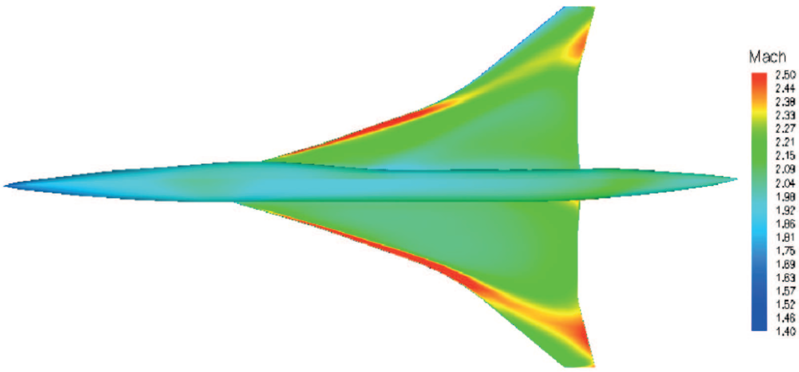

Figure 8: Drag minimization for the EUROSUP SCT (supersonic cruise transporter) at Mach number 2: Optimized shape geometry (upper wing) versus initial design (lower wing) with local flow Mach number distribution. The strong shock on the wing could be reduced. [Brezillon, Gauger (2004)] (Copyright: Prof. Dr. Nicolas Gauger, Head of Computational Mathematics Group, Department of Mathematics and Center for Computational Engineering Science, RWTH Aachen University, Aachen, Germany)

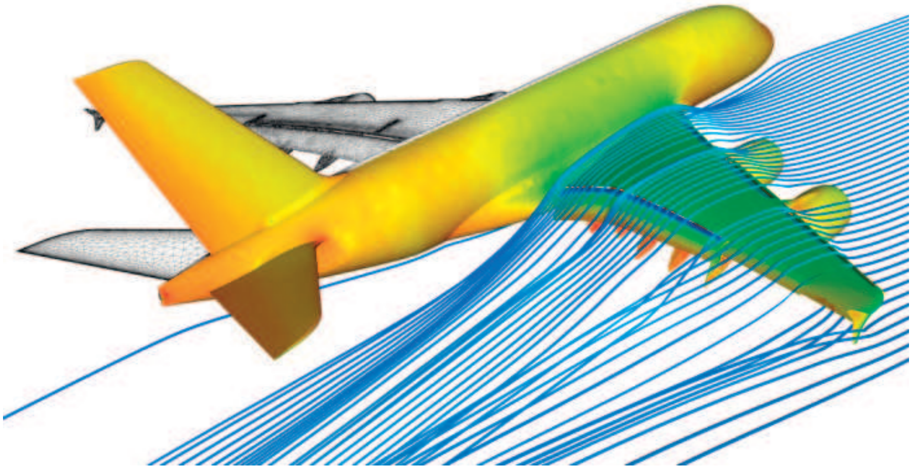

Figure 9: Numerical flow simulation for the Airbus A380 (picture credit: Airbus. Copyright: Dr. Klaus Becker, Senior Manager Aerodynamic Strategies, EGAA, Airbus, Bremen, Germany)

Mathematical optimization has become and will continue to be an important tool in modern high technology. Mathematics in total has even become a key technology by itself.

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